

GALVIN'S PROPERTY AT LARGE CARDINALS AND AN APPLICATION TO PARTITION CALCULUS

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ABSTRACT. In the first part of this paper we explore the possibility for a very large cardinal κ to carry a κ -complete ultrafilter without Galvin's property. In this context we prove the consistency of every ground model κ -complete ultrafilter extends to a non-Galvin one. Oppositely, it is also consistent that every ground model κ -complete ultrafilter extends to a P -point ultrafilter, hence to another one satisfying Galvin's property. Finally, we apply this property to obtain consistently new instances of the classical problem in partition calculus $\lambda \rightarrow (\lambda, \omega + 1)^2$.

1. INTRODUCTION

Let \mathcal{F} be a filter over a regular uncountable cardinal κ . We say that *Galvin's property* holds for \mathcal{F} (in symbols, $\text{Gal}(\mathcal{F}, \kappa, \kappa^+)$) if every family $\langle C_\gamma \mid \gamma < \kappa^+ \rangle \subseteq \mathcal{F}^1$ admits a subfamily $\langle C_{\gamma_i} \mid i < \kappa \rangle$ with the property that $\bigcap \{C_{\gamma_i} \mid i < \kappa\} \in \mathcal{F}$. In the 1970's, Galvin proved that if $\kappa = \kappa^{<\kappa} > \aleph_0$ then $\text{Gal}(\mathcal{F}, \kappa, \kappa^+)$ is true whenever \mathcal{F} is normal. The statement and the proof were published in a paper by Baumgartner, Hajnal and Mate [4].

The motivation of this paper came from an open problem which appeared in [7]. In that work it is shown that, consistently, there is a κ -complete ultrafilter over a measurable cardinal κ which fails to satisfy the Galvin property. One should keep in mind the fact that if κ is measurable then every normal filter \mathcal{F} satisfies the Galvin property $\text{Gal}(\mathcal{F}, \kappa, \kappa^+)$. Thus, the main result of [7] shows that κ -completeness differs from normality in terms of implying Galvin's property. On the other hand, it is consistent that κ is measurable and every κ -complete ultrafilter is Galvin. This can be demonstrated in Solovay's inner model $L[\mathcal{U}]$, as shown in [8]. However, inner models are limited with their tolerance to large cardinals. It was asked in [7] whether it is consistent for a supercompact cardinal κ that every κ -complete ultrafilter \mathcal{U} over κ satisfies $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$. In the first part of this paper we investigate the possibility of very large cardinals carrying κ -complete filters (ultrafilters) that fail to satisfy Galvin's property. In §2.1 we exhibit a generic extension where κ is supercompact and every κ -complete ground model ultrafilter \mathcal{U} over κ extends to an ultrafilter \mathcal{U}^*

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¹Note that, in the absence of AC, it is important to emphasize that a family is a *sequence* in which we allow repetitions.

for which $\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$ fails (see Proposition 2.1). Shortly after, we show that this construction is amenable to preserve even stronger large cardinals, such as $C^{(n)}$ -extendibles and Vopěnka's Principle (Proposition 2.3). Continuing in this vein, we present a result of an opposite nature. Namely, in Theorem 2.5 we construct a generic extension where κ is supercompact and every κ -complete ground model ultrafilter \mathcal{U} extends to a κ -complete ultrafilter \mathcal{U}^* that is *Rudin-Keisler* equivalent to a normal one. In particular, all of these ultrafilters \mathcal{U}^* do satisfy Galvin's property (Proposition 2.4). The reader may have noticed that this is perhaps a too harsh way to convert an arbitrary κ -complete ultrafilter into a Galvin one. During the rest of the section we present alternative strategies to achieve the same configuration without such dramatic changes. Our first attempt takes place in Theorem 2.12 where we employ iterations of *Generalized Mathias forcing* to show that every κ -complete filter \mathcal{U} extends to a κ -complete filter \mathcal{U}^* for which $\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$ holds. Section 2.1 is then culminated with our main result, which builds upon previous work of Gitik and Shelah [20]. More specifically, in Theorem 2.20 we replace the previous iteration by a more sophisticated one also involving Generalized Mathias forcing. This iteration was devised by Gitik and Shelah and here it is adapted to our current purposes. As an outcome we obtain the consistency of a supercompact cardinal κ with every ground model κ -complete ultrafilter \mathcal{U} extending to a κ -complete ultrafilter \mathcal{U}^* which is a P -point. In particular, $\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$ holds (see Proposition 2.19).

The study of Galvin's property led us to the area of partition calculus. In §3 of the paper we address a classical question about ordinary partition relations. An exquisite theorem of Shelah establishes that if $\lambda > \text{cf}(\lambda) = \kappa > \aleph_0$ and $2^\kappa < \lambda$ then $\lambda \rightarrow (\lambda, \omega + 1)^2$ [31]. We prove that the same partition relation follows upon replacing the cardinal arithmetic assumption by an appropriate instance of Galvin's property. This is true in general, but it is particularly interesting under AD. Indeed, under this assumption Galvin's property holds for a wide class of cardinals (i.e. *Boldface GCH cardinals*). In fact, if κ is measurable and $\kappa = \text{cf}(\lambda) < \lambda$ is a limit of measurable cardinals then $\lambda \rightarrow (\lambda, \omega + 1)^2$. This gives an answer to [16, Question 11.4] in the context of AD. For details, see Theorem 3.10 and the subsequent discussion.

Finally, it must be said that our Galvin-like assumption is trivial when $2^\kappa < \lambda$, and forceable when $2^\kappa > \lambda$. Thus, we get more positive instances of $\lambda \rightarrow (\lambda, \omega + 1)^2$. We believe, however, that the relation $\lambda \rightarrow (\lambda, \omega + 1)^2$ is consistent as well. Actually, our result pinpoints which instances of Galvin's property should be violated in order to force this negative partition relation.

Our notation is mostly standard. If $\kappa = \text{cf}(\kappa) > \aleph_0$ then \mathcal{D}_κ denotes the club filter over κ . If $\kappa = \text{cf}(\kappa) < \lambda$ then $S_\kappa^\lambda = \{\delta \in \lambda \mid \text{cf}(\delta) = \kappa\}$. The arrow symbol $\lambda \rightarrow (\alpha, \beta)^2$ is a shorthand for the following statement: for every $f : [\lambda]^2 \rightarrow 2$ either there is a 0-monochromatic subsets of λ of order type α or a 1-monochromatic subset of λ of order-type β . We say that

$\binom{\alpha}{\beta} \rightarrow \binom{\gamma}{\delta}$ iff for every $c : \alpha \times \beta \rightarrow \{0, 1\}$ there are $A \in [\alpha]^\gamma, B \in [\beta]^\delta$ for which $c \upharpoonright (A \times B)$ is constant. We use Θ to denote

$$\sup\{\alpha \mid \text{There exists a mapping from } {}^\omega\omega \text{ onto } \alpha\}.$$

We employ the Jerusalem forcing notation, thus $p \leq q$ means that q is stronger than p . For background in partition calculus we refer the reader to [16] and [34].

2. GALVIN'S PROPERTY AT VERY LARGE CARDINALS WITH AND WITHOUT CHOICE

2.1. Galvin's property at large cardinals. In [8] the following result is proved: It is consistent that κ is a measurable cardinal and every κ -complete ultrafilter \mathcal{U} over κ satisfies $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$. The proof strategy is based on analyzing Solovay's inner model $L[\mathcal{U}]$, where a complete classification of the σ -complete ultrafilters over κ is available. The key observation is that in this inner model every σ -complete ultrafilter over κ is Rudin-Keisler equivalent to a finite power of the normal measure \mathcal{U} . Since these ultrafilters do satisfy Galvin's property one concludes that $\text{Gal}(\mathcal{V}, \kappa, \kappa^+)$ holds for every κ -complete ultrafilter $\mathcal{V} \in L[\mathcal{U}]$. This phenomenon suggests the following question: How about those (very) large cardinals for which there is no available canonical inner model? The epitome of this is supercompactness.

By work of the first two authors together with S. Shelah [7] it is consistent that a supercompact cardinal κ carries a κ -complete ultrafilter \mathcal{U} which extends the club filter and $\neg\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$. Shortly after, the first author together with M. Gitik [9] improved this result by showing that just a measurable cardinal suffices to obtain such an ultrafilter \mathcal{U} .

The forthcoming proposition is a spin-off of the above-mentioned result in the context of general κ -complete ultrafilters:

Proposition 2.1. *Assume that the GCH holds and that κ is a measurable cardinal. Then the following is true in the generic extension of [9, Theorem 2.6]: Every κ -complete (not necessarily normal) ultrafilter \mathcal{U} of the ground model extends to a κ -complete ultrafilter \mathcal{U}^* such that $\neg\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$.*

In addition, if κ was supercompact then it remains so in the extension.

Proof. The sought model is the generic extension by the Easton support iteration $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta \mid \alpha \leq \kappa + 1, \beta \leq \kappa \rangle$ such that for $\alpha \leq \kappa$, \mathbb{Q}_α is trivial unless α is inaccessible, in which case it is a \mathbb{P}_α -name for $\text{Add}(\alpha, \alpha^+)$. This iteration preserves supercompactness (see e.g. [13, Theorem 11.1]).

Let $\mathcal{U} \in V$ be a κ -complete ultrafilter. Let us verify that we can adjust the argument in [9] to encompass non-normal ultrafilters. We will follow the notation from the original proof, considering the elementarity embeddings

$$j_1 := j_{\mathcal{U}} : V \rightarrow M_{\mathcal{U}} =: M_1, \quad j_2 := j_{\mathcal{U}^2} : V \rightarrow M_{\mathcal{U}^2} =: M_2$$

$$k : M_1 \rightarrow M_2, \quad j_2 = k \circ j_1$$

where k is simply the ultrapower embedding defined in $M_{\mathcal{U}}$ using the ultrafilter $j_1(\mathcal{U})$. Let $G := G_\kappa * g_\kappa$ be V -generic for $\mathbb{P}_\kappa * \mathbb{Q}_\kappa$. The argument that these embeddings can be lifted in $V[G]$ does not require normality and remains unaltered. Thus, we form $j_1 \subseteq j_1^* : V[G] \rightarrow M_1[j_1^*(G)]$, $k \subseteq k^* : M_1[j_1^*(G)] \rightarrow M_2[j_2^*(G)]$ and $j_2 \subseteq j_2^* := k^* \circ j_1^*$ such that:

- (1) for every $\alpha \in j_1$ “ κ^+ ”, $f_{\kappa_2, k(\alpha)}(\kappa_1) = 1$.
- (2) for every $\alpha \in \kappa_1 \setminus j_1$ “ κ^+ ”, $f_{\kappa_2, k(\alpha)}(\kappa_1) = 0$.
- (3) $f_{\kappa_2, \kappa_1}(\kappa_1) = \kappa$.

Since we are just dealing with non-normal ultrafilters we need to alter the values of the generic f_2 at $\delta^* := [\text{id}]_{j_1(\mathcal{U})}$, the generator of the second ultrapower. Also, we need to eliminate the generator of the first ultrapower $\delta := [\text{id}]_{\mathcal{U}}$:

- (1) for every $\alpha \in j_1$ “ κ^+ ”, $f_{\kappa_2, k(\alpha)}(\delta^*) = 1$.
- (2) for every $\alpha \in \kappa_1 \setminus j_1$ “ κ^+ ”, $f_{\kappa_2, k(\alpha)}(\delta^*) = 0$
- (3) $f_{\kappa_2, \delta^*}(\delta^*) = \delta$.

Notice that the amount of coordinates that were altered is small. In particular, the counting/genericity arguments of [9, Lemma 2.7] relying on ZFC still go through. Next, derive in $V[G]$ the ultrafilter generated by j_1^* and $[\text{id}]_{\mathcal{U}}$,

$$\mathcal{U}^* := \{X \subseteq \kappa \mid [\text{id}]_{\mathcal{U}} \in j_1^*(X)\}$$

Note that $\mathcal{U} \subseteq \mathcal{U}^*$. Finally, let

$$\mathcal{W} := \{X \subseteq \kappa \mid [\text{id}]_{j_1(\mathcal{U})} \in j_2^*(X)\} \in V[G]. \quad \square$$

Let us prove that \mathcal{W} witnesses the statement of the theorem:

Claim 2.2. *\mathcal{W} is a κ -complete ultrafilter over κ such that:*

- (1) $\mathcal{U} \subseteq \mathcal{W}$.
- (2) $\neg \text{Gal}(\mathcal{W}, \kappa, \kappa^+)$.

Proof of claim. (1): If $A \in \mathcal{U}$ then $j_1(A) \in j_1(\mathcal{U})$, hence $[\text{id}]_{j_1(\mathcal{U})} \in j_2(A)$ and thus $A \in \mathcal{W}$.

(2): Let us define the witness. For each $\alpha < \kappa^+$ let

$$A_\alpha := \{\nu < \kappa \mid f_{\kappa, \alpha}(\nu) = 1\}$$

then

$$j_2^*(A_\alpha) = \{\beta < \kappa_2 \mid f_{\kappa_2, j_2(\alpha)}(\beta) = 1\}.$$

Since $j_2(\alpha) = k(j_1(\alpha))$, our modifications of the generic give

$$f_{\kappa_2, j_2(\alpha)}([\text{id}]_{j_1(\mathcal{U})}) = 1,$$

hence $[\text{id}]_{j_1(\mathcal{U})} \in j_2^*(A_\alpha)$. Finally $A_\alpha \in \mathcal{W}$ by definition of \mathcal{W} . Before proving the failure of the Galvin property, let us denote by $j_{\mathcal{W}} : V[G] \rightarrow M_{\mathcal{W}}$ the ultrapower embedding by \mathcal{W} and $k_{\mathcal{W}} : M_{\mathcal{W}} \rightarrow M_2^*$ defined by $k_{\mathcal{W}}([f]_{\mathcal{W}}) := j_2^*(f)([\text{id}]_{j_1(\mathcal{U})})$ the factor map satisfying $k_{\mathcal{W}} \circ j_{\mathcal{W}} = j_2^*$.

We show that $k_{\mathcal{W}}$ is onto, hence the identity, and thus $j_2^* = j_{\mathcal{W}}$. In effect, if $A \in M_2[j_2^*(G)]$ then there is a name $\underline{A} \in M_2$ with $A = (\underline{A})_{j_2^*(G)}$.

Since j_2 is the second ultrapower by \mathcal{U} , there is $f : [\kappa]^2 \rightarrow V$ such that $j_2(f)([\text{id}]_{\mathcal{U}}, [\text{id}]_{j_1(\mathcal{U})}) = \underline{A}$. By Löb's theorem, we can assume that $f(\alpha, \beta)$ is a $\mathbb{P}_{\kappa+1}$ -name for every $(\alpha, \beta) \in [\kappa]^2$. In $V[G]$ let $f^*(\alpha) = (f(f_{\kappa, \alpha}(\alpha), \alpha))_G$.

Then,

$$\begin{aligned} k_{\mathcal{W}}([f^*]_W) &= j_2^*(f^*)([\text{id}]_{j_1(\mathcal{U})}) = (j_2(f)(f_{\kappa_2, [\text{id}]_{j_1(\mathcal{U})}}([\text{id}]_{j_1(\mathcal{U})}, [\text{id}]_{j_1(\mathcal{U})}))_{j_2^*(G)} \\ &= j_2(f)([\text{id}]_{\mathcal{U}}, [\text{id}]_{j_1(\mathcal{U})})_{j_2^*(G)} = (\underline{A})_{j_2^*(G)} = A \end{aligned}$$

Let $\langle A_{\alpha_i} \mid i < \kappa \rangle$ be any subfamily of length κ and $\kappa \leq \eta < [\text{id}]_{\mathcal{W}} = [\text{id}]_{j_1(\mathcal{U})}$. Denote $j_{\mathcal{W}}(\langle A_{\alpha_i} \mid i < \kappa \rangle) := \langle A'_{\alpha'_i} \mid i < j_{\mathcal{W}}(\kappa) \rangle$.

Pick any $\kappa \leq \eta < [\text{id}]_{\mathcal{W}} < j_2(\kappa)$, then $\eta \notin j_1$ " κ^+ and also $\alpha''_{\eta} \notin j_1$ " κ^+ , where α''_{η} is the first image of the $\{\alpha_i \mid i < \kappa\}$. Moreover $k(\alpha''_{\eta}) = \alpha'_{k(\eta)}$ and by definition, $f_{\kappa_2, \alpha''_{\eta}}([\text{id}]_{j_1(\mathcal{U})}) = 0$ and $[\text{id}]_{j_1(\mathcal{U})} \notin A'_{\eta}$. Hence

$$[\text{id}]_{j_1(\mathcal{U})} \notin \bigcap \{A'_{\alpha'_i} \mid i < \kappa_2\} = j_2^*\left(\bigcap_{i < \kappa} A_{\alpha_i}\right)$$

Hence $\bigcap_{i < \kappa} A_{\alpha_i} \notin \mathcal{W}$. \square

Continuing with our original discussion one may ask if the conclusion of Proposition 2.1 is compatible with large cardinals stronger than supercompactness. As argued in [2, 3], the natural model-theoretic strengthening of supercompactness is $C^{(n)}$ -extendibility. Fix $n < \omega$. A cardinal κ is called $C^{(n)}$ -*extendible* if for every $\lambda > \kappa$ there is $\theta \in \text{Ord}$ and an elementary embedding $j : V_{\lambda} \rightarrow V_{\theta}$ with $\text{crit}(j) = \kappa$, $j(\kappa) > \lambda$ and $V_{j(\kappa)} \prec_{\Sigma_n} V$.² The classical notion of extendibility (see [26, §23]) coincides with $C^{(n)}$ -extendibility whenever $n = 1$. However, when $n \geq 2$ the first $C^{(n)}$ -extendible is far above, and has stronger large-cardinal-properties, than the first extendible cardinal. In addition, $C^{(n)}$ -extendibility do entail a proper hierarchy of cardinals [2]. The culmination of this hierarchy is the category-theoretic axiom known as *Vopěnka's Principle* (VP) [26, p. 335]. In effect, it was shown by Bagaria that VP is equivalent to the existence of a (proper class of) $C^{(n)}$ -extendible, for all $n \geq 1$. We refer the reader to [2] for further details.

Let us come back to the argument of Proposition 2.1. If κ is an extendible cardinal performing our iteration $\mathbb{P}_{\kappa+1}$ will ruin extendibility of κ .³ Nevertheless, if one forces with $\text{Add}(\alpha, \alpha^+)$ at every inaccessible cardinal the situation changes completely. In [3] the authors develop a general theory of preservation of extendible cardinals under class-forcing iterations. Specifically, in [3, §8] it is shown that many classical class-forcing iterations (e.g., Jensen's iteration to force the GCH) do preserve extendible cardinals, as well as $C^{(n)}$ -extendible cardinals and Vopěnka's Principle (VP).

The following proposition is an easy corollary of [3, Theorem 8.4]:

²Recall that $V_{\eta} \prec_{\Sigma_n} V$ is a shorthand for the following statement: for every $\bar{a} \in V_{\eta}^{<\omega}$ and every Σ_n formula $\varphi(\bar{x})$ in the language of set theory, $V_{\eta} \models \varphi(\bar{a})$ iff $V \models \varphi(\bar{a})$.

³Actually, adding a single Cohen subset to κ does it.

Proposition 2.3. *Assume that the GCH holds and that κ is a $C^{(n)}$ -extendible cardinal for some $n \geq 1$. Let \mathbb{P} denote the Easton support class iteration forcing with $\text{Add}(\alpha, \alpha^+)$ at each inaccessible cardinal.*

Then, the following hold in $V^{\mathbb{P}}$:

- (1) κ is $C^{(n)}$ -extendible;
- (2) for every measurable cardinal λ every λ -complete ultrafilter $\mathcal{U} \in V$ extends to a λ -complete ultrafilter \mathcal{U}^* such that $\neg \text{Gal}(\mathcal{U}^*, \lambda, \lambda^+)$.

In addition, if one assumes VP this is preserved in $V^{\mathbb{P}}$.

Proof. Clause (1) is an immediate consequence of [3, Theorem 8.4]. For Clause (2) we argue as follows. If λ_0 stands for the first V -inaccessible cardinal then \mathbb{P} admits a gap at λ_0^{++} .⁴ Thus \mathbb{P} does not create new measurable cardinals in $V^{\mathbb{P}}$ [21, Corollary 2]. Let λ be a V -measurable cardinal and \mathcal{U} a λ -complete ultrafilter in the ground model. By Proposition 2.1, $\mathbb{P}_{\lambda+1}$ forces that there is $\mathcal{U}^* \supseteq \mathcal{U}$ such that $\text{Gal}(\mathcal{U}^*, \lambda, \lambda^+)$ fails. Clearly $\mathbb{P}/\mathbb{P}_{\lambda+1}$ is forced to be λ^{++} -directed closed (actually more), hence it preserves that \mathcal{U}^* is a λ -complete ultrafilter over λ witnessing $\neg \text{Gal}(\mathcal{U}^*, \lambda, \lambda^+)$. \square

Let us now show how to obtain the consistency of the opposite statement “Every κ -complete ultrafilter \mathcal{U} over a supercompact cardinal κ extends to a κ -complete ultrafilter \mathcal{U}^* satisfying $\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$ ”. To this aim we will show how to turn a non (necessarily) Galvin ultrafilter \mathcal{U} into another \mathcal{U}^* that is *Rudin-Keisler* equivalent to a normal one. By virtue of Galvin’s theorem [4] this ensures that \mathcal{U}^* itself does satisfy $\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$.

To show this we need a couple of preliminary observations. First, if \mathcal{U} and \mathcal{W} are κ -complete ultrafilters over κ and $\mathcal{U} \equiv_{\text{RK}} \mathcal{W}$ then $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$ holds if and only if $\text{Gal}(\mathcal{W}, \kappa, \kappa^+)$ holds. Second:

Proposition 2.4. *If \mathcal{U} is a κ -complete ultrafilter over κ with $|\text{id}_{\mathcal{U}}| = \kappa$ then \mathcal{U} is Rudin-Keisler equivalent to a normal κ -complete ultrafilter.*

In particular, under the above conditions, $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$ holds.

Proof. Let \mathcal{U}_0 denote the normal measure generated from $j := j_{\mathcal{U}}$ and κ .

For each $\lambda < \kappa^+$ there is $f_\lambda: \kappa \rightarrow \kappa$ such that $j(f_\lambda)(\kappa) = \lambda$. We prove this by induction on λ . Suppose that $\langle f_\alpha \mid \alpha < \lambda < \kappa^+ \rangle$ are defined and let $\langle \lambda_i \mid i < \text{cf}(\lambda) \rangle$ be cofinal in λ . Define $f_\lambda: \kappa \rightarrow \kappa$ as follows:

$$f_\lambda(\alpha) := \sup_{i < \alpha} f_{\lambda_i}(\alpha).$$

Note that $f_\lambda: \kappa \rightarrow \kappa$ due to the regularity of κ . Next, put

$$j(\langle f_\beta \mid \beta < \lambda \rangle) := \langle f'_\beta \mid \beta < j(\lambda) \rangle, \quad j(\langle \lambda_i \mid i < \text{cf}(\lambda) \rangle) := \langle \lambda'_i \mid i < j(\text{cf}(\lambda)) \rangle.$$

Observe that $f'_{j(\alpha)} = j(f_\alpha)$ and $\lambda'_{j(\alpha)} = j(\lambda_\alpha)$. In particular, $f'_\alpha = j(f_\alpha)$ and $\lambda'_\alpha = j(\lambda_\alpha)$ for every $\alpha < \kappa$. Hence,

$$j(f_\lambda)(\kappa) = \sup_{i < \kappa} f'_{\lambda'_i}(\kappa) = \sup_{i < \kappa} j(f)_{j(\lambda_i)}(\kappa) =$$

⁴I.e., $\mathbb{P} \simeq \mathbb{P}_1 * \mathbb{P}_2$ where $|\mathbb{P}_1| < \lambda_0^{++}$ and $\Vdash_{\mathbb{P}_1} \mathbb{P}_2$ is λ_0^{++} -distributive”.

$$= \sup_{i < \kappa} j(f_{\lambda_i})(\kappa) = \sup_{i < \kappa} \lambda_i = \lambda$$

Thus, there is $f: \kappa \rightarrow \kappa$ such that $j(f)(\kappa) = [\text{id}]_{\mathcal{U}}$, so that $\mathcal{U} \leq_{\text{RK}} \mathcal{U}_0$. Also, it is well-known that normal filters are \leq_{RK} -minimal (see e.g. [5, Proposition 2.6]), hence $\mathcal{U} \equiv_{\text{RK}} \mathcal{U}_0$. For the in particular clause use our comments prior to the statement of the proposition. \square

Theorem 2.5. *Assume that the GCH holds and that κ is a huge cardinal.*

Then, there is an inaccessible cardinal $\mu > \kappa$ and a generic extension of V_μ where the following hold:

- (1) κ is supercompact;
- (2) Every κ -complete ultrafilter $\mathcal{U} \in V$ extends to a κ -complete ultrafilter \mathcal{U}^* that is Rudin-Keisler equivalent to a normal ultrafilter. In particular, $\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$ holds.

Proof. Let $j: V \rightarrow M$ be an elementary embedding witnessing that κ is huge; namely, $\text{crit}(j) = \kappa$ and $M^{j(\kappa)} \subseteq M$. Fix $\langle \mathcal{U}_\alpha \mid \alpha < 2^{2^\kappa} \rangle$ an injective enumeration of the κ -complete ultrafilters over κ . For each $\alpha < 2^{2^\kappa}$ note that $j''\mathcal{U}_\alpha \in M$ and that $j''\mathcal{U}_\alpha \in [j''\mathcal{U}_\alpha]^{< j(\kappa)}$ hence, by $j(\kappa)$ -completeness of $j''\mathcal{U}_\alpha$ in M , we can find $\epsilon_\alpha \in \bigcap j''\mathcal{U}_\alpha$. Clearly, $\epsilon_\alpha < j(\kappa)$ and

$$\mathcal{U}_\alpha \subseteq \mathcal{U}_{\alpha,0}^* := \{X \subseteq \kappa \mid \epsilon_\alpha \in j(X)\}.$$

Let λ and μ be, respectively, the first inaccessible cardinals in the intervals $(\sup_{\alpha < 2^{2^\kappa}} \epsilon_\alpha, j(\kappa))$ and $(\lambda, j(\kappa))$.⁵ Next, let $i: V \rightarrow N$ be the μ -supercompact embedding derived from j ; that is, the ultrapower embedding that arises from the measure $\{X \subseteq \mathcal{P}_\kappa(\mu) \mid j''\mu \in j(X)\}$. Let $k: N \rightarrow M$ be the factor embedding between j and i . Usual arguments show that $\text{crit}(k) > \eta$, hence $\mathcal{U}_{\alpha,0}^* = \{X \subseteq \kappa \mid \epsilon_\alpha \in i(X)\}$ for each $\alpha < 2^{2^\kappa}$.

Now, force over V with *Woodin's fast function forcing* \mathbb{F}_κ . By virtue of Lemma 1.10 in [22] we have that i lifts to a μ -supercompact embedding $i: V[f] \rightarrow M[i(f)]$ such that $i(f)(\kappa) = \mu$. Notice that using the fast function $f: \kappa \rightarrow \kappa$ we can easily represent λ as well: let $f^*: \kappa \rightarrow \kappa$ be defined as $\alpha \mapsto \sup\{\beta < f(\alpha) \mid \beta \text{ is inaccessible}\}$ and note that $i(f^*)(\kappa) = \lambda$.⁶

Next, over $V[f]$, force with the two-step iteration $\mathbb{C} := \mathbb{C}_\kappa * \widetilde{\text{Col}}(\kappa, < \lambda)$ where \mathbb{C}_κ is the Easton-supported iteration defined as follows: for $\alpha < \kappa$, the α^{th} -stage of the iteration is trivial unless α is inaccessible, $f''\alpha \subseteq \alpha$ and $\alpha < f^*(\alpha)$, in which case it forces with $\widetilde{\text{Col}}(\alpha, < f^*(\alpha))$.

Claim 2.6. *After forcing with \mathbb{C} the embedding $i: V[f] \rightarrow N[f]$ lifts to a μ -supercompact embedding $i^*: V[f * \mathbb{C}] \rightarrow N[i(f * \mathbb{C})]$ in $V[f * \mathbb{C}]$.*

Proof of claim. Denote $\bar{V} := V[f]$ and $\bar{N} := N[i(f)]$. Let $C := \mathbb{C}_\kappa * c \subseteq \mathbb{C}$ generic over V . We can lift the embedding after forcing with \mathbb{C}_κ to another $i: \bar{V}[C_\kappa] \rightarrow \bar{N}[C_\kappa * c * h] \subseteq \bar{V}[C]$. There are two points here: first, the

⁵This choice is possible as $j(\kappa)$ is a limit of inaccessibles.

⁶Here we are implicitly assuming that $N[i(f)]$ is μ -closed, hence it thinks that μ is inaccessible and that λ is the first inaccessible below it.

κ^{th} -stage of the iteration from the perspective of \bar{N} is $\text{Col}(\kappa, < \lambda)^{\bar{V}}$; ⁷ second, the tail forcing $i(\mathbb{C}_\kappa)/\mathbb{C}$ is trivial in the interval (κ, μ) because $i(f)(\kappa) = \mu$ and so the next closure point of $i(f)$ past κ is $\geq (\mu^+)^{\bar{V}}$.

Finally, one can lift i after forcing with $\text{Col}(\kappa, < \lambda)_{\bar{V}[C_\kappa]}$ to another embedding $i: \bar{V}[C] \rightarrow \bar{N}[C_\kappa * c * h * h']$. For this one uses the fact that $i''c \subseteq \text{Col}(i(\kappa), < i(\lambda))_{\bar{N}[C_\kappa * c * h]}$ is a directed set of conditions in $N[C_\kappa * c * h]$, $|i''c| < i(\kappa)$ and $\text{Col}(i(\kappa), < i(\lambda))_{\bar{N}[C_\kappa * c * h]}$ is μ -directed-closed in the model $N[C_\kappa * c * h]$. Standard arguments show that the resulting embedding witnesses μ -supercompactness of κ . \square

Working in $V[f * C]$, for each $\alpha < (2^{2^\kappa})^V$ define

$$\mathcal{U}_\alpha^* := \{X \subseteq \kappa \mid \epsilon_\alpha \in i^*(X)\}.$$

Clearly, \mathcal{U}_α^* is a κ -complete ultrafilter satisfying $\mathcal{U}_\alpha \subseteq \mathcal{U}_\alpha^*$. The point now is that $|\text{id}|_{\mathcal{U}_\alpha^*}^{V[f * C]} \leq |\epsilon_\alpha|^{V[f * C]} = \kappa$ hence $\text{Gal}(\mathcal{U}_\alpha^*, \kappa, \kappa^+)$ holds in $V[f * C]$.

Let $M := V[f * C]_\mu$. This is certainly a model of ZFC because μ remains inaccessible. Also, M satisfies that κ is supercompact. Finally, note that $M = V_\mu[f * C]$. Since every κ -complete ultrafilter \mathcal{U} over κ from the ground model actually comes from V_μ we obtain Clause (2) of the theorem. \square

In the model of Theorem 2.5 our target cardinal κ cannot be extendible. To make it so, one should perform a class-forcing iteration that is nice enough to carry the previous arguments. This suggests the following question:

Question 2.7. Is the statement of the previous theorem compatible with κ being extendible or, more generally, $C^{(n)}$ -extendible?

In the proof of Theorem 2.5 we showed how to *correct* ultrafilters that do not satisfy Galvin's property. The technique used for this purpose consisted of collapsing the generators of every V -ultrafilter to yet another generator of cardinality κ ; namely, the normal generator. In that manner we accomplished our *correction* of non-Galvin ultrafilters by making them essentially *minimal* (to wit, normal) from the Rudin-Keisler-perspective. This is, certainly, a too harsh way to ensure Galvin's property in the final model.

In the light of this one may wonder whether a similar Galvin-like configuration is possible without *trivializing* the relevant ultrafilters. In what is left we show that this is indeed possible. As a warm up exercise we begin describing how to turn a general κ -complete ultrafilter into a Galvin one by using a generalization of *Mathias forcing*. The classical Mathias forcing dealing with subsets of ω appeared in [29], while the version that we will use here follows the template of [18, Definition 3.1]:

Definition 2.8 (Generalized Mathias forcing). Let κ be a regular cardinal and \mathcal{U} a non-principal κ -complete filter over κ .

⁷Because $j(f)''\kappa \subseteq \kappa$ and $j(f^*)(\kappa) = \lambda > \kappa$.

The forcing notion $\mathbb{M}_{\mathcal{U}}$ consists of pairs (a, A) such that $a \in [\kappa]^{<\kappa}$, $A \in \mathcal{U}$ and $\sup(a) < \min(A)$. For the order, one writes $(a_0, A_0) \leq (a_1, A_1)$ if and only if $a_0 \subseteq a_1$, $A_0 \supseteq A_1$ and $a_1 \setminus a_0 \subseteq A_0$.

The next is a brief account of the main properties of $\mathbb{M}_{\mathcal{U}}$:

Proposition 2.9 (Properties of $\mathbb{M}_{\mathcal{U}}$).

- (1) $\mathbb{M}_{\mathcal{U}}$ is κ^+ -centered, provided $\kappa^{<\kappa} = \kappa$;
- (2) $\mathbb{M}_{\mathcal{U}}$ is κ -directed-closed;
- (3) $\mathbb{M}_{\mathcal{U}}$ is countably parallel closed;⁸
- (4) If $G \subseteq \mathbb{M}_{\mathcal{U}}$ is a V -generic filter then $V[G] = V[a_G]$, where

$$a_G := \bigcup \{a \mid \exists A \in \mathcal{U} ((a, A) \in G)\};$$

- (5) The set a_G diagonalizes \mathcal{U} : i.e., $a_G \subseteq^* A$ for every $A \in \mathcal{U}$.

In particular, if \mathcal{U} is an ultrafilter then either $a_G \subseteq^* A$ or $a_G \subseteq^* \kappa \setminus A$ for all $A \in \mathcal{P}(\kappa)^V$.

The next proposition describes how to turn a κ -complete filter into one satisfying Galvin's property by means of $\mathbb{M}_{\mathcal{U}}$:

Proposition 2.10. Let κ be a regular cardinal, \mathcal{U} a κ -complete filter over κ and $G \subseteq \mathbb{M}_{\mathcal{U}}$ a generic filter. Then the following hold in $V[G]$:

- (1) $\mathcal{U}^* := \{A \subseteq \kappa \mid a_G \subseteq^* A\}$ is a κ -complete filter;
- (2) $\mathcal{U} \subseteq \mathcal{U}^*$;
- (3) $\text{Gal}(\mathcal{U}^*, \kappa, \kappa^+)$ holds.

Proof. \mathcal{U}^* is clearly a filter and $\mathcal{U} \subseteq \mathcal{U}^*$ by virtue of Proposition 2.8(4). The argument for κ -completeness of \mathcal{U}^* is essentially the same as the one for Clause (3): Let $\langle A_\alpha \mid \alpha < \kappa^+ \rangle \subseteq \mathcal{U}^*$ and find $I \in [\kappa^+]^{\kappa^+}$ and $\alpha^* < \kappa$ such that $a_G \setminus \alpha^* \subseteq A_\alpha$ for all $\alpha \in I$. Thus, $\bigcap_{\alpha \in I} A_\alpha \in \mathcal{U}^*$. \square

The above argument repeats the one from [6, Proposition 4.5]: the point is that $\mathbb{M}_{\mathcal{U}}$ creates a *generating sequence* of length 1.

Definition 2.11. A family $\mathcal{A} = \langle x_\alpha \mid \alpha < \lambda \rangle \subseteq \mathcal{U}$ is a *generating sequence* for \mathcal{U} if for every $A \in \mathcal{U}$ there is $\alpha < \lambda$ such that $x_\alpha \subseteq^* A$. In addition, \mathcal{A} is called *strong generating* if it is \subseteq^* -decreasing.

As demonstrated in [6, §4] the analysis of (strong) generating sequences provides an effective way to produce certain Galvin-like configurations. The main obstacle, however, is to ensure that the departing κ -complete ultrafilter \mathcal{U} extends to yet another κ -complete ultrafilter. This will be eventually addressed in Theorem 2.20.

Our next goal will be to iterate Mathias forcing over a given filter (and its extensions along the way) so that it will generate a κ -complete ultrafilter with a strong generating sequence of arbitrary length. This idea traces back

⁸I.e., every two decreasing sequences of conditions $\langle p_n \mid n < \omega \rangle$, $\langle q_n \mid n < \omega \rangle$ with $p_n \parallel q_n$ admit an upper bound. See [14].

to Kunen who employed it to separate the ultrafilter number \mathfrak{u} from 2^ω (see [28, Ch. VII Question (A10)]). A similar argument, yet involving a more complex iteration, was considered in [11]. There the authors separate $\mathfrak{u}(\kappa)$ and 2^κ in a context where κ is supercompact.

The *naïve* approach would consist of iterating Mathias forcing over and over with κ -complete filters. Unfortunately, this strategy is doomed to failure and so an additional structure on the forcing is required. Let us illustrate where the problem arises. Suppose that x_0 is a Mathias set for a κ -complete filter \mathcal{U} . Working in the generic extension $V[x_0]$ let \mathcal{U}_0 be a κ -complete filter extending $\{x_0\} \cup \mathcal{U}$ (e.g., by Proposition 2.10 we can take $\{X \in \mathcal{P}(\kappa)^{V[x_0]} \mid x_0 \subseteq^* X\}$). Next, over $V[x_0]$, force a Mathias set x_1 through \mathcal{U}_0 and working over the resulting extension $V[x_0, x_1]$ let \mathcal{U}_1 a κ -complete filter extending $\{x_1\} \cup \mathcal{U}_0$. One can proceed in this fashion ω -many times. Formally speaking, this is forced by the following full-support iteration $\langle \mathbb{P}_n, \mathbb{Q}_n \mid n < \omega \rangle$: for each $n \geq 1$, \mathbb{Q}_n is a \mathbb{P}_n -name for $\mathbb{M}_{\mathcal{U}_n}$ where \mathcal{U}_n is a \mathbb{P}_n -name for a κ -complete ultrafilter extending $\{x_n\} \cup \mathcal{U}_{n-1}$.

An essential obstacle arises at stage $\omega + 1$. Here one needs to find a κ -complete filter which includes all the Mathias sets $\langle x_n \mid n < \omega \rangle$ constructed so far. However, notice that $\mathcal{W} := \{X \subseteq \omega \mid \exists n < \omega (x_n \subseteq^* X)\}$ (i.e., the filter generated by the Mathias sets) is not σ -complete: for if $\bigcap_{n < \omega} x_n \in \mathcal{W}$ then there would be some $n^* < \omega$ such that $x_{n^*} \subseteq^* \bigcap_{n < \omega} x_n$, hence x_{n^*} would be \subseteq^* -included in x_{n^*+1} . This latter is certainly impossible in that x_{n^*+1} is a Mathias set for a filter including x_{n^*} .

For the moment, and as a warm up for Theorem 2.20, we show how to produce κ -complete filters with arbitrarily long strong generating sequences using Mathias forcing.

Theorem 2.12. *Let \mathcal{U} be a κ -complete filter over a Mahlo cardinal κ .*

Then, for every $\lambda \in \text{Ord}$ there is a κ -directed-closed and κ^+ -cc poset $\mathbb{P}(\lambda)$ forcing that \mathcal{U} can be extended to a κ -complete filter \mathcal{U}^ with a strong generating sequence $\langle x_\alpha \mid \alpha < \lambda \rangle$.*

Proof. Let \mathcal{U} and λ be as above. Define a $<\kappa$ -supported iteration $\mathbb{P}(\lambda)$, $\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta \mid \beta < \alpha \leq \lambda \rangle$, as follows. Suppose that \mathbb{P}_α is defined for $\alpha < \lambda$. In $V^{\mathbb{P}_\alpha}$ we will define a filter \mathcal{U}_α^* over κ which we will prove to be κ -complete. Bearing this in mind, we shall let \mathbb{Q}_α be a \mathbb{P}_α -name for $\mathbb{M}_{\mathcal{U}_\alpha^*}$ and denote by $x_\alpha := a_{G_{\mathbb{Q}_\alpha}}$ the generic Mathias set added after forcing with \mathbb{Q}_α .

For $\alpha = 0$, we let $\mathcal{U}_0 := \mathcal{U}$. At successor $\alpha + 1$, in $V^{\mathbb{P}_{\alpha+1}}$ we have x_α , then we let $\mathcal{U}_{\alpha+1}$ be the $\mathbb{P}_{\alpha+1}$ -name for the filter generated by x_α . By proposition 2.10, we have that $0_{\mathbb{P}_{\alpha+1}} \Vdash \mathcal{U}_\alpha \subseteq \mathcal{U}_{\alpha+1}$ and $\mathcal{U}_{\alpha+1}$ is κ -complete. As for the limit stages, let us split into cases

Claim 2.13. *Suppose that $\alpha < \lambda$ is such that $\text{cf}(\alpha) \geq \kappa$. Then $\langle x_\beta \mid \beta < \alpha \rangle$ generates a κ -complete filter \mathcal{U}_α^* in $V^{\mathbb{P}_\alpha}$ that extends \mathcal{U}_β^* for every $\beta < \alpha$.*

Proof. Let $\{X_i \mid i < \mu < \kappa\} \subseteq \mathcal{U}_\alpha^*$. For each $i < \mu$ there is $\beta_i < \alpha$ such that $x_{\beta_i} \subseteq^* X_i$. Since the cofinality of α is at least κ , $\sup_{i < \mu} \beta_i =: \beta^* < \alpha$.

Hence, $x_{\beta^*} \subseteq^* x_{\beta_i} \subseteq^* X_i$ for every $i < \mu$. It follows that for some $\epsilon_i < \kappa$, $x_{\beta^*} \setminus \epsilon_i \subseteq X_i$. Take $\epsilon^* = \sup_{i < \mu} \epsilon_i < \kappa$, then $x_{\beta^*} \setminus \epsilon^* \subseteq \bigcap_{i < \mu} X_i$, by definition, $\bigcap_{i < \mu} X_i \in \mathcal{U}_\alpha^*$. Also, for every $\beta < \alpha$, $\mathcal{U}_\beta^* \subseteq \mathcal{U}_{\beta+1}^*$ and $\mathcal{U}_{\beta+1}^*$ is by definition the filter generated x_β which is clearly a subset of \mathcal{U}_α^* . \square

Claim 2.14. *Suppose that $\alpha < \lambda$ is limit such that $cf(\alpha) < \kappa$, and let*

$$\alpha = \kappa^{\delta_1} \gamma_1 + \dots + \kappa^{\delta_n} \gamma_n$$

be the Cantor normal form of α . Consider the following cofinal subset of α : $I_\alpha := \{\kappa^{\delta_1} \gamma_1 + \dots + \kappa^{\delta_n} \gamma \mid \gamma < \gamma_m\}$. Then $x := \bigcap_{i \in I_\alpha} x_i \in V^{\mathbb{P}_\alpha}$ is unbounded in κ . In particular, $x \subseteq^ x_\beta$ for every $\beta < \alpha$, and the filter \mathcal{U}_α^* generated by x , is a κ -complete filter in $V^{\mathbb{P}_\alpha}$ which extends \mathcal{U}_β^* for every $\beta < \alpha$.*

Proof. Let $p \in \mathbb{P}_\alpha$ and $\alpha_0 < \kappa$. We shall now proceed with a density argument to prove that there are $\alpha_0 < \gamma^* < \kappa$ and $p \leq p_{fin}$ such that $p_{fin} \Vdash \gamma^* \in \bigcap_{i \in I_\alpha} \mathcal{U}_i$. Construct two sequences $\langle M_\rho \mid \rho < \kappa \rangle$ and $\langle q_\rho \mid \rho < \kappa \rangle$ such that:

- (1) $M_\rho \prec H(\theta)$ for θ regular and sufficiently large.
- (2) M_ρ is increasing and continuous.
- (3) $|M_\rho| := \gamma_\rho < \kappa$. γ_0 is regular and also $\gamma_{\rho+1}$.
- (4) If γ_ρ is regular then $M_\rho^{< \gamma_\rho} \subseteq M_\rho$.
- (5) $\alpha, p, \mathbb{P}_\alpha, \kappa \in M_\rho$.
- (6) If $\beta \in M_\rho \cap \alpha + 1$ has cofinality less than κ , then $cf(\beta) \cup I_\beta \subseteq M_{\rho+1}$.

For the condition q_α we require that:

- (1) q_ρ is increasing and continuous.⁹
- (2) q_ρ is M_ρ -generic for \mathbb{P}_α , namely, for every dense open $D \subseteq \mathbb{P}_\alpha$, $D \in M_\rho, q_\rho \in D$.
- (3) $supp(q_\rho) = M_\rho \cap \alpha$.
- (4) $q_\rho \in M_{\rho+1}$

For this construction we need that κ is Mahlo. Such condition exists by κ -closure of \mathbb{P}_α and by standard construction of an increasing sequence of conditions in M . Let $\mu^* < \kappa$ be regular such that $|M_{\mu^*}| = \mu^*$. Denote $M^* = M_{\mu^*}$ and $p^* = q_{\mu^*} = \sup_{i < \mu^*} q_i$.

Claim 2.15. *For every $\beta \in Supp(p^*)$ the following hold:*

- (1) *There is δ_β such that $p^* \restriction \beta \Vdash \mathcal{A}_\beta^{p^*} \subseteq \delta_\beta$.*
- (2) *If $\beta = \beta_0 + 1$ is successor, then there is ϵ_β such that $p^* \restriction \beta \Vdash \mathcal{X}_{\beta_0} \setminus \epsilon_\beta \subseteq \mathcal{A}_{\beta_0}^{p^*}$.*
- (3) *If $\beta \in Supp(p^*)$ is of cofinality less than κ then there is ϵ_β such that $p^* \restriction \beta \Vdash \bigcap_{i \in I_\beta} \mathcal{X}_i \setminus \epsilon_\beta \subseteq \mathcal{A}_\beta^{p^*}$.*
- (4) *If $\beta \in Supp(p^*)$ is of cofinality at least κ , then there $i_\beta < \beta$ and ϵ_β such that $p^* \restriction \beta \Vdash \mathcal{X}_{i_\beta} \setminus \epsilon_\beta \subseteq \mathcal{A}_{i_\beta}^{p^*}$.*

⁹i.e. for limit ρ , we let $q_\rho(\gamma) = \langle \bigcup_{\rho' < \rho} \mathcal{A}_{\gamma}^{q_{\rho'}}, \bigcap_{\rho' < \rho} \mathcal{A}_{\gamma}^{q_{\rho'}} \rangle$.

Proof. To see (1), since $\beta \in \text{Supp}(p^*) = M^* \cap \alpha$, find $\xi_0 < \mu^*$ such that $\beta \in M_{\xi_0} \cap \alpha$. In M_{ξ_0+1} we can define the dense open set

$$D_{\xi_0} := \{q \in \mathbb{P}_\alpha \mid \exists \delta < \kappa. q \upharpoonright \beta \Vdash \underline{a}_{\beta}^{q_{\xi_0}} \subseteq \delta\}$$

Since q_{ξ_0+1} is M_{ξ_0} -generic, $q_{\xi_0+1} \in D_{\xi_0}$. For every $\xi_0 \leq i < \mu^*$ we have that $q_i \in M_{i+1}$, hence we can define in M_{i+1} the dense open set

$$D_{i+1} := \{q \in \mathbb{P}_\alpha \mid \exists \delta < \kappa. q \upharpoonright \beta \Vdash \underline{a}_{\beta}^{q_i} \subseteq \delta\}$$

By genericity, $q_{i+1} \in D_{i+1}$. For every such i , pick $\delta^{(i)}$ witnessing $q_{i+1} \in D_{i+1}$. Let $\delta_\beta = \sup_{i < \mu^*} \delta^{(i)} < \kappa$, by continuity, $p^* \upharpoonright \beta \Vdash \underline{a}_{\beta}^{p^*} \subseteq \delta_\beta$.

The proof of (3), (4) is similar to (1). Just note that by definition of \mathcal{U}_β , it is the filter generated by \underline{x}_{β_0} and replace the dense D_{i+1} by

$$E_{i+1} := \{q \in \mathbb{P}_\alpha \mid \exists \epsilon. q \upharpoonright \beta \Vdash \underline{x}_{\beta_0} \setminus \epsilon \subseteq \underline{A}_{\beta}^{q_i}\}$$

Finally, to see (4), follow a similar path by defining

$$F_{i+1} := \{q \in \mathbb{P}_\alpha \mid \exists j_\beta < \beta. \exists \epsilon. q \upharpoonright \beta \Vdash \underline{x}_{j_\beta} \setminus \epsilon \subseteq \underline{A}_{\beta}^{q_i}\}$$

Pick for every $i < \mu^*$, $j_{\beta,i} \in M_{p^*} \cap \beta$ witnessing $q^* \in F_{i+1}$ and since the cofinality of β is at least κ we can take the sup to find a single i_β . Now choose the epsilons as before. Since $j_{\beta,i} \in M^*$ and unbounded in i_β , there is $i_0 < \mu^*$ such that for every $i_0 \leq i < \mu^*$, the cantor normal form of $j_{\beta,i}$ is a continuation of the on of i_β . Hence the δ'_i 's belong to M^* \square

By (1), for every $\beta \in \text{Supp}(p^*)$ we have $\delta_\beta < \kappa$, pick $\delta^* = \sup \delta_\beta$. By (2), (3) and (4) we choose ϵ_β and set $\epsilon^* = \sup \epsilon_\beta < \kappa$. Pick $\gamma^* \in A_0^{p^*}$ above ϵ^* , δ^* , α_0 and define p_{fin} . Define the support of p_{fin} to be $\text{Supp}(p^*) \cup \{i_\beta \mid \beta \in \text{Supp}(p^*), cf(\beta) \geq \kappa\}$. Define

$$p_{fin}(\gamma) := \begin{cases} \langle \underline{a}_{\gamma}^{p^*} \cup \{\gamma^*\}, \underline{A}_{\gamma}^{p^*} \setminus \gamma^* \rangle, & \gamma \in \text{Supp}(p^*), \\ \langle \{\gamma^*\}, \kappa \setminus \gamma^* + 1 \rangle, & \text{otherwise.} \end{cases}$$

Clearly $p_{fin} \Vdash \gamma^* \in \bigcap_{i \in I_\alpha} \underline{x}_i$. It remains to argue that p_{fin} is an extension of p^* : Indeed $p_{fin}(0) \geq p_0^*$ since $\gamma^* \in A_0$ and $\gamma^* > \delta_0 > \sup(a_0^{p^*})$. Suppose that $\beta \in \text{Supp}(p^*)$ and that $p_{fin} \upharpoonright \beta \geq p^* \upharpoonright \beta$. If $\beta = \beta_0 + 1$ is successor then $p_{fin} \upharpoonright \beta \Vdash \gamma^* \in \underline{x}_{\beta_0} \setminus \epsilon^* \subseteq \underline{A}_{\beta}^{p^*}$, hence $p_{fin} \upharpoonright \beta \Vdash p_{fin}(\beta) \geq p^*(\beta)$. If β is limit of cofinality less than κ , then since $\beta \in M^*$, we have that $I_\beta \subseteq M^* \cap \beta$, hence by induction $p_{fin} \upharpoonright \beta \Vdash \gamma^* \in \bigcap_{i \in I_\beta} \underline{x}_i \setminus \epsilon^* \setminus \underline{A}_{\beta}^{p^*}$. Finally, if the cofinality of β is at least κ , then there is $i_\beta < \kappa$ such that $p^* \upharpoonright \beta \Vdash \gamma^* \in x_{i_\beta}^* \setminus \epsilon^* \subseteq \underline{A}_{\beta}^{p^*}$. \square

This completes the proof of the theorem. \square

Let us briefly describe a natural, yet unfruitful, strategy to make \mathcal{U}^* become a κ -complete ultrafilter. Given a Laver-indestructible supercompact cardinal κ and $\alpha < \lambda$ force over $V^{\mathbb{P}_\alpha}$ with $\mathbb{M}_{\mathcal{V}_\alpha}$, where \mathcal{V}_α is an extension of \mathcal{U}_α^* (in $V^{\mathbb{P}_\alpha}$) to a κ -complete ultrafilter. Note that since \mathbb{P}_α is κ -directed-closed, κ is still supercompact in the corresponding extension and thus the

choice of \mathcal{V}_α is available. In addition, if the length of the iteration is $\lambda = \kappa^+$ then the union of the (tower of) κ -complete ultrafilters generated along the way will be also an ultrafilter, \mathcal{U}_∞ . The problem with this approach is that we lose control upon κ -completeness of \mathcal{U}_∞ . Indeed, even the union of the first ω -many ultrafilters generated might not be κ -complete, as we argued in the discussion preceding Theorem 2.12.

The approach of Theorem 2.20 is to iterate $\mathbb{M}_{\mathcal{U}}$ more carefully so that we have complete control on the completeness of the ultrafilter \mathcal{U}_∞ .

Definition 2.16. For $f: \kappa \rightarrow \kappa$ and an ultrafilter \mathcal{U} over κ we say that f is *constant mod*(\mathcal{U}) if there is $\gamma < \kappa$ such that $f^{-1}\{\gamma\} \in \mathcal{U}$. Similarly, f is *1-1 mod*(\mathcal{U}) if there is $X \in \mathcal{U}$ such that $|f^{-1}\{\gamma\} \cap X| < \kappa$ for every $\gamma < \kappa$.

Definition 2.17. A κ -complete ultrafilter \mathcal{U} is called a *P-point* if every function $f: \kappa \rightarrow \kappa$ that is not constant mod(\mathcal{U}) is 1-1 mod(\mathcal{U}).

Remark 2.18. \mathcal{U} is a *P-point* if and only if every sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$ of elements in \mathcal{U} has a pseudo intersection in \mathcal{U} ; namely, there is $X \in \mathcal{U}$ such that $X \subseteq^* X_\alpha$ for every $\alpha < \kappa$.

Lemma 2.19. *Every κ -complete ultrafilter \mathcal{U} with a generating sequence of size κ^+ is a P-point. In particular, $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$.*

Proof. Let $\mathcal{A} = \langle A_\alpha \mid \alpha < \kappa^+ \rangle$ be a generating sequence of \mathcal{U} . To see it is *P-point* suppose that $\langle X_\alpha \mid \alpha < \kappa \rangle$ is any κ -sequence of members of \mathcal{U} . By definition of generating sequence, for each $\alpha < \kappa$ there is $\beta_\alpha < \kappa^+$ such that $A_{\beta_\alpha} \subseteq^* X_\alpha$. Consider $\beta^* = \sup_{\alpha < \kappa} \beta_\alpha < \kappa^+$. Then A_{β^*} is a pseudo intersection of the sequence $\langle X_\alpha \mid \alpha < \kappa \rangle$. For the in particular claim use the fact that every *P-point* ultrafilter \mathcal{U} satisfies $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$ (see [8, Proposition 5.13]). \square

In analogy to Theorem 2.5, next we show that every ground model κ -complete ultrafilter can be extended to a *well-behaved* one: to wit, to a *P-point*. By virtue of the above lemma, this gives an alternative (and less severe way) to transmute an arbitrary κ -complete ultrafilter into a Galvin one. Unlike Theorem 2.5, the models we produce this time have the extra feature that 2^κ can be made arbitrarily large. Our construction owes much to previous work of Gitik and Shelah [20]. Recall that κ is *almost huge with target* λ if there is $j: V \rightarrow M$ such that $\text{crit}(j) = \kappa$, $j(\kappa) = \lambda$ and $M^{<\lambda} \subseteq M$.

Theorem 2.20. *Assume that the GCH holds and suppose that κ is an almost huge cardinal with measurable target λ . Then for every $\delta < \lambda$ there is a forcing extension $V[G_\delta]$ where $2^\kappa = \delta$ and every ground model κ -complete ultrafilter extends to a *P-point* ultrafilter in $V[G_\delta]$. In addition, $V[G_\delta]_\lambda$ models the same configuration and κ is supercompact there.*

Proof. Let $j: V \rightarrow M$ be such that $\text{crit}(j) = \kappa$, $M^{<\lambda} \subseteq M$ and $j(\kappa) = \lambda$. Recall that λ is assumed to be measurable in the ground model, hence we can let \mathcal{U} be a measure on λ . Let us define an Easton support iteration

$\langle \mathbb{P}_\alpha, \mathbb{Q}_\beta \mid \beta \leq \kappa, \alpha \leq \kappa + 1 \rangle$, where \mathbb{Q}_α is trivial unless α is measurable in $V^{\mathbb{P}_\alpha}$, in which case \mathbb{Q}_α is a \mathbb{P}_α -name for the two-step iteration $\mathbb{Q}_{\alpha,0} * \mathbb{Q}_{\alpha,1}$ defined as follows: $\mathbb{Q}_{\alpha,0}$ is the atomic forcing choosing some ordinal $F(\alpha) < \kappa$ followed by $\mathbb{Q}_{\alpha,0}$, an $<\alpha$ -supported iteration $\langle \mathbb{R}_\beta^\alpha, \mathbb{S}_\gamma^\alpha \mid \beta \leq F(\alpha), \gamma < F(\alpha) \rangle$ defined as follows: At each step $\beta \leq F(\alpha)$, \mathbb{S}_β^α is trivial unless $\Vdash_{\mathbb{P}_\alpha * \mathbb{R}_\beta^\alpha}$ “ α is measurable”, in which case \mathbb{S}_β^α is forced to be the $<\alpha$ -supported product $\prod_{\mathcal{U}} \mathbb{M}_{\mathcal{U}}$, where \mathcal{U} ranges over all $\mathbb{P}_\alpha * \mathbb{R}_\beta^\alpha$ -names for an α -complete ultrafilter over α .

Since $\mathbb{Q}_{\alpha,1}$ is an $<\alpha$ -supported iteration of α^+ -stationary c.c., $<\alpha$ -closed and countably-parallel closed forcing (cf. Proposition 2.9) it follows from [14, Theorem 1.2] that $\mathbb{Q}_{\alpha,1}$ is α^+ -cc. Also, \mathbb{P}_κ is κ -cc because κ is Mahlo and \mathbb{P}_κ is Easton-supported. Let $G_\kappa \subseteq \mathbb{P}_\kappa$ be a V -generic filter. Then, due to closure of M under $<\lambda$ -sequences and κ -ccness of \mathbb{P}_κ , $V[G_\kappa]$ and $M[G_\kappa]$ agree up to λ (see, e.g., [13, Proposition 8.4]). Consider $j(\mathbb{P}_\kappa) := \mathbb{P}'_{j(\kappa)}$.

Claim 2.21. *For every $\rho < \lambda$, $M[G_\kappa * \{\rho\}]^{\mathbb{Q}_{1,\kappa}} \models$ “ κ is measurable”.*

*In fact, κ is $<\lambda$ -supercompact in $M[G_\kappa * \{\rho\}]^{\mathbb{Q}_{1,\kappa}}$, hence also in $V[G_\kappa * \{\rho\}]^{\mathbb{Q}_{1,\kappa}}$, and thus κ is fully supercompact in $(V[G_\kappa * \{\rho\}]^{\mathbb{Q}_{1,\kappa}})_\lambda$.*

Proof. In the ground model V , let $\max(\rho, 2^{2^\kappa}) \leq \theta < \lambda$. Let U be a fine normal measure over $P_\kappa(\theta)$ and let $j_U: V \rightarrow M$ be the corresponding elementary embedding. Then ${}^\theta M \subseteq M$. Put $\mathbb{P}'_{j_U(\kappa)} := j_U(\mathbb{P}_\kappa)$ and $\mathbb{Q}'_{j_U(\kappa)} := j_U(\mathbb{Q}_\kappa)$. Let $G(\mathbb{Q}_{1,\kappa})$ a $V[G_\kappa][\{\rho\}]$ -generic, and let us lift j_U to the model $V[G_\kappa * \{\rho\} * G(\mathbb{Q}_{1,\kappa})]$. Note that in $M[G_\kappa * \{\rho\} * G(\mathbb{Q}_{1,\kappa})]$ we have $2^\kappa \geq \rho$, since at each step of the iteration $\mathbb{Q}_{1,\kappa}$ we add a new subset to κ . Hence, by closure under ρ -sequences, the degree of closure of the forcing

$$\mathbb{P}'_{j_U(\kappa)} / [G_\kappa * \{\rho\} * G(\mathbb{Q}_{1,\kappa})]$$

is at least ρ^+ , even in $V[G_\kappa * \{\rho\} * G(\mathbb{Q}_{1,\kappa})]$. Also note that every dense set in $\mathbb{P}'_{j_U(\kappa)} / G_\kappa * \{\rho\} * G(\mathbb{Q}_{1,\kappa})$ is represented by a function $f: \mathcal{P}_\kappa(\rho) \rightarrow \mathcal{P}(\mathbb{P}_\kappa)$. Since there are in total $(2^\kappa)^\rho = \rho^+$ of such functions we can construct a master sequence which induces an $M[G_\kappa * \{\rho\} * G(\mathbb{Q}_{1,\kappa})]$ -generic filter $G_{<\lambda}$. Next, the top-most forcing $\mathbb{Q}_{0,\kappa} * \mathbb{Q}_{1,\kappa}$ is κ^+ -cc and when its length is restricted to some ρ its size becomes $\rho \cdot 2^{2^\kappa}$. Hence every maximal antichain in $M[G_{<\lambda} * \{j(\rho)\}]$ for $j(\mathbb{Q}_{1,\kappa})$ is represented by a function $F: \mathcal{P}_\kappa(\rho) \rightarrow \mathcal{P}_\kappa((\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\rho\}}) \in V[G_\kappa * \{\rho\}]$ and since $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\rho\}}$ is of size $\rho \cdot 2^{2^\kappa} = \theta$ there are $\theta^\theta = \theta^+$ many such functions. The remaining argument is standard. \square

For each $\rho < \lambda$, $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\rho\}}$ is κ^+ -cc and of size less than λ . Thus, there are less than λ -many nice names for subsets of κ . Denote these names by $\langle \mathcal{A}_\tau^\rho \mid \tau < \theta_\rho \rangle$. We can find in $V[G_\kappa]$ an enumeration $\langle \mathcal{A}_\tau \mid \tau < \lambda \rangle$ of subsets of κ such that for every $\tau_1 \leq \tau_2$, there are δ_1, δ_2 such that $\mathcal{A}_{\tau_1} = \mathcal{A}_{\delta_1}^{\rho(\tau_1)}$ and $\mathcal{A}_{\tau_2} = \mathcal{A}_{\delta_2}^{\rho(\tau_2)}$, and $\rho(\tau_1) \leq \rho(\tau_2)$. For each $\tau < \lambda$ for which $\rho(\tau) < \lambda$ has been defined, let C' be the club of closure points of $\rho(\tau)$. Since λ is

measurable there is $S \in \mathcal{U}$ concentrating on inaccessibles. Hence, there for \mathcal{U} -many $\delta \in S$ such that $\rho \text{``}\delta \subseteq \delta$. In particular, the sequence $\langle \mathcal{A}_\tau \mid \tau < \delta \rangle$ codes all the $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\delta\}}$ -names for subsets of κ .¹⁰

Let $\epsilon < \lambda$ be an ordinal above the generators of all κ -complete ultrafilters. More precisely, for each κ -complete ultrafilter $U \in V$ over κ let $\varepsilon_U \in \lambda \cap \bigcap j \text{``}U$ and define $\epsilon := \sup_U \varepsilon_U$. Note that $\epsilon < \lambda$ as λ is inaccessible in V .

For every $\delta' < \delta$ and every $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\delta'\}}$ -name (i.e., a $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\delta\}} \upharpoonright \delta'$ -name) \underline{U} for a κ -complete ultrafilter over κ , let us define $r_{\mathcal{U}} \in \mathbb{M}_{j(\mathcal{U})}$ as:

$$r_{\mathcal{U}} = \langle a_{\mathcal{U}} \cup (A_{\mathcal{U}} \cap \epsilon), A_{\mathcal{U}} \setminus (\epsilon + 1) \rangle,$$

where

- $a_{\mathcal{U}}$ is the standard $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\delta\}}$ -name for the Mathias set for $\mathbb{M}_{\mathcal{U}}$;
- $A_{\mathcal{U}}$ is a name for the set $\bigcap j \text{``}\underline{U}$.

Note that $r_{\mathcal{U}}$ is a condition in $\mathbb{M}_{j(\mathcal{U})}$: First, the trivial condition of $\mathbb{M}_{j(\mathcal{U})}$ forces “ $\min A_{\mathcal{U}} = \kappa$ ”, hence $a_{\mathcal{U}} \cup (A_{\mathcal{U}} \cap \epsilon)$ is a legitimate stem. Second, this condition also forces $j(\underline{U})$ to be $j(\kappa)$ -complete, hence $\bigcap j \text{``}\underline{U} \in j(\underline{U})$.

The key feature of $r_{\mathcal{U}}$ is that $0_{j(\mathbb{M}_{\mathcal{U}})}$ forces $a_{j(\mathcal{U})}$ to contain all generators ε_W , $W \in V$, that are forced to belong to $A_{\mathcal{U}}$.

Let q_δ be a condition in $\mathbb{Q}'_{j_U(\kappa)}$ with support $j \text{``}\delta$ (hence of cardinality $< j(\kappa)$) be the following condition: For every $\delta' < \delta$ and a $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\delta'\}}$ -name for a κ -complete ultrafilter \underline{U} as above,

$$q_\delta \upharpoonright j(\delta') \Vdash q_\delta(j(\delta'))(j(\underline{U})) = r_{\mathcal{U}};$$

for other coordinates (i.e., names for *ghost ultrafilters*) \underline{W} we require that

$$q_\delta \upharpoonright j(\delta') \Vdash \text{“}q_\delta(j(\delta'))(\underline{W}) \text{ is the trivial condition in } \mathbb{M}_{\underline{W}}\text{”}.$$

A moment's reflection makes clear that q_δ is a *master condition* for $G(\mathbb{Q}_{1,\kappa})$: namely, $j(p) \leq q_\delta$ for every $p \in G(\mathbb{Q}_{1,\kappa})$. In addition, the conditions q_δ are defined in a coherent way: namely, if $\rho \leq \delta$ are both in C then $q_\delta \upharpoonright \rho = q_\rho$.

Fix $\rho \in C \setminus (\epsilon + 1)$. For every $\tau, \zeta < \rho$ let $\underline{D}_{\tau,\zeta} \in M[G_\kappa * \{\rho\}]$ be a $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\rho\}}$ -name for the following dense open set

$$\{p \in \mathbb{P}'_{(\kappa, j_U(\kappa))} \mid \exists s_\kappa \in \mathbb{Q}_{1,\kappa} \exists i \in 2 (\langle s_\kappa, p \rangle \Vdash_{\mathbb{P}'_{[\kappa, j_U(\kappa)]}}^i \zeta \in j(\mathcal{A}_\tau))\}.$$

Since the trivial condition of $\mathbb{Q}_{1,\kappa}$ forces $\mathbb{P}'_{(\kappa, j_U(\kappa))}$ to be ρ^+ -closed it also forces that $\bigcap_{\tau,\zeta} \underline{D}_{\tau,\zeta}$ is a name for a dense open set. In particular, there is some $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\rho\}}$ -name p_ρ for a condition in $\bigcap_{\tau,\zeta} \underline{D}_{\tau,\zeta}$ such that $p_\rho \upharpoonright j(\kappa) \Vdash p_\rho(j(\kappa)) \geq q_\rho$. Notice that p_ρ has the property that for every $\tau, \zeta < \rho$ there is $s \in G_\kappa * \{\rho\}$ and $s_\kappa \in (\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\rho\}}$ such that $\langle s, s_\kappa, p_\rho \rangle \Vdash_{\mathbb{P}'_{[\kappa, j_U(\kappa)]}} \zeta \in j(\mathcal{A}_\tau)$.

For each $(s, s_\kappa) \in G_\kappa * \mathbb{Q}_\kappa$, ordinals $\zeta < \epsilon$ and $\tau < \lambda$, and $i \in 2$ define

$$A_{(s, s_\kappa, \zeta, \tau)}^i := \{\rho < \lambda \mid \langle s, s_\kappa, p_\rho \rangle \Vdash^i \zeta \in j(\mathcal{A}_\tau)\}.$$

¹⁰Specifically, if σ is a $(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\delta\}}$ -name for a subset of κ then there is $\tau < \delta$ such that $0 \Vdash_{(\mathbb{Q}_{1,\kappa})_{G_\kappa * \{\delta\}}} \sigma = \mathcal{A}_\tau$.

Denote by $A_{(s, s_\kappa, \zeta, \tau)}^2$ the complement of the union of the above two sets. For each such quadruple $(s, s_\kappa, \zeta, \tau)$, let $i_{(s, s_\kappa, \zeta, \tau)} \in \mathfrak{I}$ be the unique index i for which $A_{(s, s_\kappa, \zeta, \tau)}^i \in \mathcal{U}$, the λ -complete measure on λ . Now let

$$A := \{\rho < \lambda \mid (\langle s, s_\kappa \rangle \in G * \mathbb{Q}_\kappa \upharpoonright \rho \wedge \max(\zeta, \tau) < \rho) \Rightarrow \rho \in A_{(s, s_\kappa, \zeta, \tau)}^{i_{(s, s_\kappa, \zeta, \tau)}}\}.$$

We claim that $A \in \mathcal{U}$: In effect, for every $\langle s, s_\kappa \rangle \in G * j(\mathbb{Q}_\kappa) \upharpoonright \lambda = G * \mathbb{Q}_\kappa$ and $\zeta, \tau < \lambda$, $A_{(s, s_\kappa, \zeta, \tau)}^{i_{(s, s_\kappa, \zeta, \tau)}} \in \mathcal{U}$. Hence, $\lambda \in j\mathcal{U}(A_{(s, s_\kappa, \zeta, \tau)}^{i_{(s, s_\kappa, \zeta, \tau)}})$, and thus $\lambda \in j\mathcal{U}(A)$.

Put $C^* := A \cap C$. Let $\rho, \rho' \in C^*$ with $\rho < \rho'$, $\langle s, s_\kappa \rangle \in G * (\mathbb{Q}_\kappa \upharpoonright \rho)$ and $\zeta, \tau < \rho$. By definition of A , $\rho, \rho' \in A_{(s, s_\kappa, \zeta, \tau)}^{i_{(s, s_\kappa, \zeta, \tau)}}$, hence

$$\langle s, s_\kappa, p_\rho \rangle \Vdash^i \zeta \in j(\underline{A}_\tau) \text{ iff } \langle s, s_\kappa, p_{\rho'} \rangle \Vdash^i \zeta \in j(\underline{A}_\tau),$$

and also

$$\langle s, s_\kappa, p_\rho \rangle \Vdash \zeta \in j(\underline{A}_\tau) \text{ iff } \langle s, s_\kappa, p_{\rho'} \rangle \Vdash \zeta \in j(\underline{A}_\tau).$$

Next, for all $\rho \in C^*$ and $\zeta < \epsilon$ consider

$$\mathcal{U}_{\rho, \zeta} := \{(\underline{A}_\tau)_{G_\kappa * \{\rho\} * G(\mathbb{Q}_{\kappa, 1})} \subseteq \kappa \mid \exists \langle s, s_\kappa \rangle \in G * \{\rho\} * G(\mathbb{Q}_{\kappa, 1}) \langle s, s_\kappa, p_\rho \rangle \Vdash \zeta \in j(\underline{A}_\tau)\}.$$

Since $\langle \underline{A}_\tau \mid \tau < \rho \rangle$ is an enumeration of the $(\mathbb{Q}_{\kappa, 1})_{G_\kappa * \{\rho\}}$ -names, it is not hard to show that $\mathcal{U}_{\rho, \zeta}$ is a κ -complete ultrafilter in $V[G_\kappa * \{\rho\} * G(\mathbb{Q}_{\kappa, 1})]$.

Also, for each $\zeta < \epsilon$, $\langle \mathcal{U}_{\rho, \zeta} \mid \rho \in C^* \rangle$ defines a tower of ultrafilters: Suppose $\rho < \rho' \in C^*$ and let $A \in \mathcal{U}_{\rho, \zeta}$. Then, there is a pair $\langle s, s_\kappa \rangle$ such that $\langle s, s_\kappa, p_\rho \rangle \Vdash \zeta \in j(\underline{A}_\tau)$. By our definition of C^* this is also true when replacing p_ρ by $p_{\rho'}$. Thus, $(\underline{A}_\tau)_{G_\kappa * \{\rho\} * G(\mathbb{Q}_{\kappa, 1})} \in \mathcal{U}_{\rho', \zeta}$.

Let δ be the limit of some sequence $\langle \rho_\alpha \mid \alpha < \kappa^+ \rangle \subseteq C^*$. From our previous comments, $\langle \mathcal{U}_{\rho_\alpha, \zeta} \mid \alpha < \kappa^+ \rangle$ defines a tower of measures. Now, define $\mathcal{V}_{\delta, \zeta} := \bigcup_{\alpha < \kappa^+} \mathcal{U}_{\rho_\alpha, \zeta}$. Since $V[G_\kappa * \{\rho_\alpha\} * G(\mathbb{Q}_{\kappa, 1}) \upharpoonright \rho_\alpha]$ is a submodel of $V[G_\kappa * \{\delta\} * G(\mathbb{Q}_{\kappa, 1})]$ and $(\mathbb{Q}_{1, \kappa})_{G_\kappa * \{\delta\}}$ is κ^+ -cc it is immediate that $\mathcal{V}_{\delta, \zeta}$ is a κ -complete ultrafilter.

Claim 2.22. $\mathcal{V}_{\delta, \zeta}$ admits a strong generating sequence of size κ^+ .

Proof. For each $\alpha < \kappa^+$, $\mathcal{U}_{\rho_\alpha, \zeta} \in V[G_\kappa * \{\delta\} * G(\mathbb{Q}_{\kappa, 1}) \upharpoonright \rho_\alpha]$ hence the iteration $\mathbb{Q}_{\kappa, 1}$ at stage $\rho_\alpha + 1$ shoots a Mathias set $x_\alpha \subseteq \kappa$ (over the model $V[G_\kappa * \{\delta\} * G(\mathbb{Q}_{\kappa, 1}) \upharpoonright \rho_\alpha]$) for the measure $\mathcal{U}_{\rho_\alpha, \zeta}$. Namely, x_α is almost included in every $A \in \mathcal{U}_{\rho_\alpha, \zeta}$.

We claim that $\langle x_\alpha \mid \alpha < \kappa^+ \rangle$ is the sought strong generating sequence. First, for each $A \in \mathcal{V}_{\delta, \zeta}$ there is $\alpha < \kappa^+$ such that $A \in \mathcal{U}_{\rho_\alpha, \zeta}$ and so $x_\alpha \subseteq^* A$. Second, $\langle x_\alpha \mid \alpha < \kappa^+ \rangle$ is \subseteq^* -decreasing: Fix $\alpha < \beta < \kappa^+$. We would like to show that $x_\alpha \subseteq \mathcal{U}_{\rho_\beta, \zeta}$. Recall that $x_\alpha = (a_{\mathcal{U}_{\rho_\alpha, \zeta}})_{G_\kappa * \{\rho_\beta\} * G(\mathbb{Q}_{\kappa, 1}) \upharpoonright \rho_\beta}$, hence we should check that for some $\langle s, s_\kappa \rangle \in G_\kappa * \{\rho_\beta\} * G(\mathbb{Q}_{\kappa, 1}) \upharpoonright \rho_\beta$ we have that:

$$\langle s, s_\kappa, p_{\rho_\beta} \rangle \Vdash \zeta \in j(a_{\mathcal{U}_{\rho_\alpha, \zeta}})$$

By elementarity of j , it follows that $j(a_{\mathcal{U}_{\rho_\alpha, \zeta}}) = a_{j(\mathcal{U}_{\rho_\alpha, \zeta})}$ is the canonical $\mathbb{P}'_{j(\kappa)} * \{j(\rho_\beta)\} * \mathbb{Q}'_{j(\kappa)}$ -name for the Mathias generic of $\mathbb{M}_{j(\mathcal{U}_{\rho_\alpha, \zeta})}$. By the

definition of p_{ρ_β} (which extends q_{ρ_β} , and the definition of $r_{\mathcal{U}_{\rho_\alpha, \zeta}}$),

$$\langle 0, p_{\rho_\beta} \rangle \Vdash a_{j(\mathcal{U}_{\rho_\alpha, \zeta})} \cap (\epsilon + 1) \supseteq \bigcap j^{\mathcal{U}_{\rho_\alpha, \zeta}} \cap (\epsilon + 1).$$

Working in $M[G_\kappa * \{\rho_\alpha\} * G(\mathbb{Q}_{\kappa, 1})]$, we have that for every $A \in \mathcal{U}_{\rho_\alpha, \zeta}$, there is a name A_τ for A such that $p_{\rho_\alpha} \Vdash \zeta \in j(A_\tau)$. Hence $p_{\rho_\alpha} \Vdash \zeta \in \bigcap j^{\mathcal{U}_{\rho_\alpha, \zeta}}$. Hence there is $\langle s, s_\kappa \rangle \in G_\kappa * \{\rho_\alpha\} * G(\mathbb{Q}_{\kappa, 1}) \upharpoonright \rho_\alpha$ such that

$$\langle s, s_\kappa, p_{\rho_\alpha} \rangle \Vdash \zeta \in j(a_{\mathcal{U}_{\rho_\alpha, \zeta}})$$

Since $\rho_\alpha, \rho_\beta \in C^*$, this means that $\langle s, s_\kappa, p_{\rho_\beta} \rangle \Vdash \zeta \in j(a_{\mathcal{U}_{\rho_\alpha, \zeta}})$ which by definition implies that $x_\alpha \in \mathcal{U}_{\rho_\beta, \zeta}$. \square

Claim 2.23. *Every κ -complete ultrafilter \mathcal{U} from the ground model is extended by $\mathcal{V}_{\delta, \zeta}$, for some $\zeta < \epsilon$.*

Proof of claim. Let $\zeta < \epsilon$ be such that $\mathcal{U} = \{X \subseteq \kappa \mid \zeta \in j(X)\}$. Clearly, $\mathcal{U} \subseteq \mathcal{U}_{\rho_\alpha, \zeta}$ for all $\alpha < \kappa^+$, hence $\mathcal{U} \subseteq \mathcal{V}_{\delta, \zeta}$. \square

This completes the proof of the theorem. \square

Remark 2.24. Note that every measurable cardinal κ always carries a κ -complete ultrafilter which is not a P -point. To see this, take any κ -complete ultrafilter \mathcal{U} over κ , and a bijection $\phi : [\kappa]^2 \rightarrow \kappa$ and define $\mathcal{W} := \phi_*(\mathcal{U} \times \mathcal{U})$. One can check that \mathcal{W} is a κ -complete non- P -point ultrafilter. As witnessed by $L[\mathcal{W}]$, it is consistent that every ultrafilter is a finite power of a normal one (hence of a P -point), and such ultrafilters are always Galvin (see [8, Corollary 5.29]). If κ is κ -compact¹¹ then there is a κ -complete ultrafilter which is not a finite power of a P -point (see, e.g., [25, §3.9]).

Question 2.25. Is it consistent to have a measurable cardinal carrying a κ -complete ultrafilter \mathcal{U} such that $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$ but it is not Rudin-Keisler equivalent to a finite power of P -points?

Question 2.26. Is it ZFC-provable that a supercompact cardinal always admits a κ -complete non Galvin ultrafilter?

2.2. Galvin's property in the choiceless context. Another way to examine Galvin's property at very large cardinals is to consider relatively small cardinals in ZF. A typical example is \aleph_1 under AD. Indeed, Solovay proved that both \aleph_1 and \aleph_2 are measurable under AD and that \mathcal{D}_{\aleph_1} is a normal ultrafilter (see [24, Theorem 33.12]). In fact, \aleph_1 is \aleph_2 -supercompact under AD, by a result of Martin (see [26, p. 401]). Moreover, under $\text{AD}_{\mathbb{R}}$, \aleph_1 is γ -supercompact for all $\gamma < \Theta$ [26, Theorem 28.22]. However, the classical proof of Galvin's theorem employs the Axiom of Choice in a crucial way.

In private communication [12], W. Chan informed us that it is possible to get Galvin's property at many cardinals even in the absence of AC. The argument involves Steel's notion of *boldface GCH cardinal* [33, p. 1678]:

¹¹A cardinal $\kappa \geq \aleph_1$ is called κ -compact if every κ -complete filter over κ extends to a κ -complete ultrafilter.

Definition 2.27. A cardinal κ is called *boldface GCH* if there is no injection $F: \kappa^+ \rightarrow \mathcal{P}(\kappa)$.

In [33, Theorem 8.26] Steel shows that under “ $V = L(\mathbb{R}) + \text{AD}$ ” every cardinal $\kappa < \Theta$ is boldface GCH. Moreover, Chan indicated that using *Moschovakis coding lemma* one can show that AD alone implies that every $\kappa < \Theta^{L(\mathbb{R})}$ is boldface GCH.

Proposition 2.28. *If κ is boldface GCH and κ^+ is regular then $\text{Gal}(\mathcal{F}, \kappa^+, \kappa^+)$ holds for every κ -complete filter \mathcal{F} over κ .*

Proof. Note that if κ is boldface GCH and κ^+ is regular then every function $F: \kappa^+ \rightarrow \mathcal{P}(\kappa)$ admits some $B \in \text{Im}(F)$ such that $|(F^{-1})^{\ast}B| = \kappa^+$.

Suppose $\langle A_\alpha \mid \alpha < \kappa^+ \rangle$ is a sequence of elements of a κ -complete filter \mathcal{F} over κ . By the previous observation there is a κ^+ -length subsequence which is constant. Hence the corresponding intersection is this constant value, which belongs to \mathcal{F} . Thereby, $\text{Gal}(\mathcal{F}, \kappa^+, \kappa^+)$ holds. \square

Remark 2.29. Even ditching the regularity assumption on κ^+ one can show that $\text{Gal}(\mathcal{F}, \kappa, \kappa^+)$ holds for every κ -complete ultrafilter \mathcal{F} over κ . This is because every $F: \kappa^+ \rightarrow \mathcal{P}(\kappa)$ admits a $B \in \text{Im}(F)$ such that $|(F^{-1})^{\ast}B| \geq \kappa$.

Anticipating the results in the next section we consider a natural generalization of Proposition 2.28. The crucial point is that both the family of sets with large intersection and the corresponding large set contained in all of them are explicitly constructed. We will apply the claim to many pairs simultaneously, and since these objects are explicitly definable we do not need the axiom of choice in order to pick them.

Definition 2.30. We say that $\text{Gal}_{\text{Def}}(\mathcal{U}, \lambda, \lambda)$ holds if for every sequence $\mathcal{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$, there is a subsequence $\langle C_{\rho_j} \mid j < \lambda \rangle$ definable from \mathcal{C} such that $\bigcap_{j < \lambda} C_{\rho_j} \in \mathcal{U}$.

Proposition 2.31. *Suppose that κ is measurable, $\kappa < \lambda$ is regular and there is no injection $f: \lambda \rightarrow \mathcal{P}(\kappa)$. Then, $\text{Gal}_{\text{Def}}(\mathcal{U}, \lambda, \lambda)$ holds for every κ -complete ultrafilter \mathcal{U} over κ .*

If one seeks for a parallel of the above in ZFC then one may consider real-valued measurable cardinals, which can be described as measurables without the cardinal arithmetic. A cardinal κ is real-valued measurable if there exists a non-trivial κ -additive measure over κ . The size of such cardinals, if exist at all, is at most 2^{\aleph_0} . Solovay proved that if there is a measurable cardinal κ and one forces a κ -product of random reals then one obtains $2^{\aleph_0} = \kappa$ in the generic extension and κ is real-valued measurable. It is tempting to try an amalgamation between this theorem and the forcing construction of Abraham-Shelah [1]. The framework will be similar, but rather than Cohen reals (added in the Abraham-Shelah model) one can try random reals. The most difficult part is to verify that the *Main lemma* [1, Lemma 1.7] still holds true when replacing the Cohen’s by the Random

reals. This is in principle not evident for the original argument of [1] relied upon some specific properties of Cohen reals. Once this is accomplished, the failure of Galvin's property at \aleph_1 follows. However, there is the additional caveat of ensuring that the Abraham-Shelah poset preserves the real-valued measurability of κ . All of this suggests the following question:

Question 2.32. Is it consistent that κ is a real-valued measurable cardinal and $\text{Gal}(\mathcal{D}_{\aleph_1}, \aleph_1, \kappa)$ fails?

3. AN APPLICATION TO ORDINARY PARTITION RELATIONS

Ramsey's theorem from [30] says that $\omega \rightarrow (\omega, \omega)^2$. That is, for every $c : [\omega]^2 \rightarrow 2$ there exists an infinite monochromatic subset $A \subseteq \omega$. A natural generalization is obtained by replacing ω with some cardinal $\kappa > \aleph_0$. The resulting relation is $\kappa \rightarrow (\kappa, \kappa)^2$ and implies that κ is weakly compact.

There is yet another possible way to generalize Ramsey's theorem to uncountable cardinals and in this way one obtains a positive relation at small cardinals as well. Recall that $\lambda \rightarrow (\kappa, \theta)^2$ means that for every $c : [\lambda]^2 \rightarrow 2$, either there is $A \subseteq \lambda, |A| = \kappa$ such that $c \upharpoonright [A]^2 = \{0\}$ or $B \subseteq \lambda, |B| = \theta$ such that $c \upharpoonright [B]^2 = \{1\}$. Ramsey's theorem generalizes to the statement $\kappa \rightarrow (\kappa, \omega)^2$ in which we increase the first component only. A theorem of Erdős, Dushnik and Miller says, indeed, that this relation holds at every infinite cardinal κ , see [15] and [16]. In terms of graph theory this means that if G is a complete graph of size κ then either G contains an edge-free subset of size κ or an infinite clique.

Can one improve the positive relation $\lambda \rightarrow (\lambda, \omega)^2$? The lightest possibility would be $\lambda \rightarrow (\lambda, \omega + 1)^2$, but this relation does not hold at every infinite cardinal λ anymore. Suppose that $\lambda > \text{cf}(\lambda) = \omega$ and let $\lambda = \bigcup_{n < \omega} \Delta_n$ where $m \neq n \Rightarrow \Delta_m \cap \Delta_n = \emptyset$ and $|\Delta_n| < \lambda$ for every $n < \omega$. Define $c : [\lambda]^2 \rightarrow 2$ by letting $c(\alpha, \beta) = 0$ iff there exists $n < \omega$ for which $\{\alpha, \beta\} \subseteq \Delta_n$. A 0-monochromatic set A must be contained in some Δ_n , so there is no such a set of size λ . A 1-monochromatic set B satisfies $|B \cap \Delta_n| \leq 1$ for every $n < \omega$, so there is no such a set of order-type $\omega + 1$. Hence there is a class of cardinals which fail to satisfy $\lambda \rightarrow (\lambda, \omega + 1)^2$. On the other hand, if $\lambda = \text{cf}(\lambda) > \aleph_0$ then $\lambda \rightarrow (\lambda, \omega + 1)^2$, see [16, Theorem 11.3]. In fact, one can prove a slightly stronger statement.

Proposition 3.1. *Suppose that $\lambda = \text{cf}(\lambda) > \aleph_0$. For every $c : [\lambda]^2 \rightarrow 2$, either there is a stationary set $T \subseteq \lambda$ such that $c \upharpoonright [T]^2$ is 0-monochromatic or there is $B \subseteq \lambda, \text{otp}(B) = \omega + 1$ such that $c \upharpoonright [B]^2$ is 1-monochromatic.*

Proof. Suppose that $c : [\lambda]^2 \rightarrow 2$. If there exists $B \subseteq \lambda$ of order type $\omega + 1$ such that $c \upharpoonright [B]^2 = \{1\}$ then we are done. Suppose that there is no such B , and let S be the set of limit ordinals of λ . For every $\delta \in S$ choose a sequence $\bar{\alpha}_\delta = \langle \alpha_0^\delta, \dots, \alpha_{n-1}^\delta \rangle$ such that $\bar{\alpha}_\delta \widehat{\langle \delta \rangle}$ is 1-monochromatic and if $\max(\bar{\alpha}_\delta) < \xi < \delta$ then $\bar{\alpha}_\delta \widehat{\langle \xi, \delta \rangle}$ is not 1-monochromatic. The choice is

possible by our assumption that there is no 1-monochromatic sequence of length $\omega + 1$.

By shrinking S if needed, we may assume that $\ell g(\bar{\alpha}_\delta) = n$ for some fixed $n < \omega$ and every $\delta \in S$. We remark that in this shrinking process we retain the fact that S is stationary. Let ξ_δ be the top-element of $\bar{\alpha}_\delta$ for every $\delta \in S$. The function $h(\delta) = \xi_\delta$ is regressive on S , so by Fodor's lemma there is a stationary $T_0 \subseteq S$ and a fixed ordinal $\xi < \lambda$ such that $\delta \in T_0 \Rightarrow \xi_\delta = \xi$. By repeating this process n -many times we obtain a stationary set T_n and a fixed sequence $\bar{\alpha}$ such that $\bar{\alpha} \restriction \delta$ is 1-monochromatic and $\bar{\alpha} \restriction \langle \zeta, \delta \rangle$ is not 1-monochromatic whenever $\delta \in T_n$ and $\max(\bar{\alpha}) < \zeta < \delta$.

In particular, if $T = T_n \setminus (\max(\bar{\alpha}) + 1)$ then T is a stationary subset of λ and if $\zeta, \delta \in T, \zeta < \delta$ then necessarily $c(\zeta, \delta) = 0$. Otherwise, $\bar{\alpha} \restriction \langle \zeta \rangle$ will contradict the conclusion of the previous paragraph. Thus, $c''[T]^2 = \{0\}$ and we are done. \square

As mentioned above, the statement of the proposition gives a bit more than just $\lambda \rightarrow (\lambda, \omega + 1)^2$ since it yields a *stationary* 0-monochromatic set. Notice that the argument applies to singular cardinals of uncountable cofinality for which the concepts of club and stationary subsets are sound. However, if T is a stationary subset of a singular cardinal λ then it is possible that $|T| < \lambda$. Therefore, one may wonder about $\lambda \rightarrow (\lambda, \omega + 1)^2$ in such cardinals. The following is [16, Question 11.4]:

Question 3.2. Does the relation $\lambda \rightarrow (\lambda, \omega + 1)^2$ hold for $\lambda > \text{cf}(\lambda) > \omega$?

Actually, the question is phrased in [16] with respect to $\lambda = \aleph_{\omega_1}$, the first relevant instance. We indicate that in [16] there appears a partial answer, based on canonization theorems of Shelah, which apply to strong limit singular cardinals. Namely, if $\lambda > \text{cf}(\lambda) > \omega$ and λ is a strong limit cardinal then $\lambda \rightarrow (\lambda, \omega + 1)^2$. In some sense, this result gives many ZFC instances since for every $\kappa = \text{cf}(\kappa) > \aleph_0$ there is a class of singular cardinals which are strong limit of cofinality κ . On the other hand, for every specific $\lambda > \text{cf}(\lambda) = \kappa$ one can choose $\theta < \lambda$ and force $2^\theta > \lambda$, thus locally it is not a theorem of ZFC.

A substantial progress with regard to the above question was made by Shelah in [31]. Using methods of pcf theory, Shelah proved that if $\lambda > \text{cf}(\lambda) = \kappa > \aleph_0$ and $2^\kappa < \lambda$ then $\lambda \rightarrow (\lambda, \omega + 1)^2$. The assumption $2^\kappa < \lambda$ is a weakening of the assumption that λ is a strong limit cardinal, but the overall question remains open if one wishes to eliminate any further assumption.

In this section we would like to replace $2^\kappa < \lambda$ by a version of Galvin's property. Our assumption on the Galvin property is also easily forced in the ZFC context if $\kappa = \text{cf}(\lambda)$ is measurable. This gives a slight improvement to Shelah's result, but it seems that the real importance of the ZFC result is that it guides us towards the (tentative) direction of forcing the negative relation $\lambda \nrightarrow (\lambda, \omega + 1)^2$. In effect, our approach indicates that one must kill all the pertinent instances of the Galvin property to obtain $\lambda \nrightarrow (\lambda, \omega + 1)^2$.

As observed in the previous section, AD yields many instances of Galvin's property; consequently, it also gives several instances of $\lambda \rightarrow (\lambda, \omega + 1)^2$.

To this end, we must render the proof of [31] by removing any use of choice apart from AC_ω . Let us begin with models of ZF. Our first mission is to prove that $\kappa \rightarrow (\kappa, \omega + 1)^2$ holds for many regular cardinals. The proof of Theorem 3.1 seems to make use of the axiom of choice in two places. Firstly, when one chooses the maximal green sequence $\bar{\alpha}_\delta$ below δ for every $\delta \in S$. Secondly, when one employs Fodor's lemma (finitely many times). The first issue is not a real obstacle, since finite sequences are well ordered even without choice. The second issue is more substantial, but can be settled if one works with normal filters. In this respect, we note that the club filter might not be normal, as proved by A. Karagila [27]. Measurability does not suffice, either. In effect, in [10] Bilinsky and Gitik constructed a model of ZF with a measurable cardinal which does not carry any normal ultrafilter.

Proposition 3.3 (ZF). *Suppose that \mathcal{F} is a normal filter over κ whose dual ideal contains $[\kappa]^{<\kappa}$. Then, for every $c: [\kappa]^2 \rightarrow 2$ one can find either $A \in \mathcal{F}^+$, such that $c \upharpoonright [A]^2 = \{0\}$ or $B \subseteq \kappa$ with $\text{otp}(B) = \omega + 1$ such that $c \upharpoonright [B]^2 = \{1\}$. In particular, $\kappa \rightarrow (\kappa, \omega + 1)^2$.*

Proof. Let $c: [\kappa]^2 \rightarrow 2$ be a coloring. We refer to the first color as red and to the second color as green. Repeat the arguments in the proof of Proposition 3.1 with the following changes. Set $S := \{\alpha < \kappa \mid \alpha \text{ is a limit ordinal}\}$. Clearly, $S \in \mathcal{F}^+$. For every $\delta \in S$ let $\bar{\alpha}_\delta$ be the first $<_{\text{lex}}$ -finite sequence which is green with δ and maximal with respect to this property. Now use the normality and apply Fodor's lemma to \mathcal{F} finitely many times to obtain a fixed maximal green sequence $\bar{\alpha}$ with respect to some set $A \in \mathcal{F}^+$. It follows that $c \upharpoonright [A]^2 = \{0\}$. Also, by our assumption on \mathcal{F} , $|A| = \kappa$. \square

Now we come to the main result of this section.

Theorem 3.4 (ZF). *Suppose that $\aleph_0 < \kappa = \text{cf}(\lambda) < \lambda$ and that κ carries a normal measure \mathcal{U} . In addition, suppose that $\langle \lambda_i \mid i < \kappa \rangle$ is a cofinal sequence in λ admitting a family $\langle \mathcal{U}_i \mid i < \kappa \rangle$ such that:*

- \mathcal{U}_i is a normal filter whose dual ideal contains $[\lambda_i]^{<\lambda_i}$;
- $\text{Gal}_{\text{Def}}(\mathcal{U}, \lambda_i, \lambda_i)$ holds.

Then, $\lambda \rightarrow (\lambda, \omega + 1)^2$.

Proof. We may assume that for every $i < \kappa$, $\lambda_i > \kappa$. Given a function $f: \kappa \rightarrow \text{Ord}$ we define a rank function $\neg(f)$ as follows. Set $\neg(f) = \alpha$ iff for every $\beta < \alpha$ one has $\neg(f) \neq \beta$ and $\neg(g) = \beta$ for some $g: \kappa \rightarrow \text{Ord}$ which satisfies $g <_{\mathcal{U}} f$ (i.e., $\{\alpha < \kappa \mid g(\alpha) < f(\alpha)\} \in \mathcal{U}$).

Let $c: [\lambda]^2 \rightarrow 2$ be a coloring and suppose that there is no 1-monochromatic subset of order type $\omega + 1$. Define $\Delta_0 := \lambda_0$ and $\Delta_{1+i} := [\lambda_i, \lambda_{i+1})$ for every $i < \kappa$. By our assumption, there is a full-sized 0-monochromatic subset of Δ_i for every $i < \kappa$. Moreover, this set is explicitly constructible by the proof of Proposition 3.3 (Indeed, for every i we take the $<_{\text{lex}}$ -minimal sequence

$\bar{\alpha}_i^*$ such that the set of $\delta \in \Delta_i$ such that $\bar{\alpha}_\delta = \bar{\alpha}_i^*$ is in \mathcal{U}_i). Hence we may assume without loss of generality that $c[\Delta_i]^2 = \{0\}$ for every $i < \kappa$.

For each $\alpha < \kappa$ let $\eta(\alpha)$ be the unique $i < \kappa$ for which $\alpha \in \Delta_i$. For every $0 < i < \kappa$ let Seq_i be the set $\{\langle \alpha_0, \dots, \alpha_{n-1} \rangle \mid \eta(\alpha_0) < \dots < \eta(\alpha_{n-1}) < i\}$. For $i < \kappa$ and $\zeta \in \Delta_i$ we define a tree \mathcal{T}_ζ as follows. For every $\bar{\alpha} = \langle \alpha_0, \dots, \alpha_{n-1} \rangle \in Seq_i$ and every $\zeta \in \Delta_i$ we let $\bar{\alpha} \in \mathcal{T}_\zeta$ iff $\{\alpha_0, \dots, \alpha_{n-1}, \zeta\}$ is 1-monochromatic under c .

By our assumption each \mathcal{T}_ζ is well-founded with respect to the reversed order. Therefore, one can define a rank function rk_ζ over Seq_i , for every $\zeta \in \Delta_i$, by the following procedure. If $\bar{\alpha} \in Seq_i - \mathcal{T}_\zeta$ then let $rk_\zeta(\bar{\alpha}) = -1$. If $\bar{\alpha} \in \mathcal{T}_\zeta$ then $rk_\zeta(\bar{\alpha}) = \xi$ iff for every $\varepsilon \in \xi$ one has $rk_\zeta(\bar{\alpha}) \neq \varepsilon$ and there exists an ordinal β for which $rk_\zeta(\bar{\alpha} \frown \langle \beta \rangle) = \varepsilon$.

The idea of this rank function is to express the degree of maximality of finite sequences below ζ . For a maximal 1-monochromatic sequence below ζ , rk_ζ assumes the value zero. If there is more room for adding ordinals above $\max(\bar{\alpha})$ and keeping the 1-monochromaticity with ζ , then the rank grows. Notice that for every $i < \kappa$ and every $\zeta \in \Delta_i$ the range of rk_ζ is bounded in λ_{i+1} since it is an initial segment of λ_{i+1} .

Let $\Delta_i^{end} \subseteq \Delta_i$ be an end-segment such that for every $\bar{\alpha} \in Seq_i$ and $\gamma < \lambda_i$, if there is $\zeta \in \Delta_i^{end}$ such that $rk_\zeta(\bar{\alpha}) = \gamma$ then there are unboundedly many such ζ 's in Δ_i^{end} below λ_{i+1} . The set Δ_i^{end} has a concrete definition: this end-segment is obtained by omitting bounded subsets of Δ_i , which amounts in our case to the intersection of their complements. Since λ_{i+1} is regular, we obtain an end-segment of λ_{i+1} .

We define a set of triples K as follows. A triple $(\bar{\alpha}, Z, f)$ belongs to K iff $Z \in \mathcal{U}$, $f: \kappa \rightarrow \text{Ord}$ and for some $0 < i < \kappa$, $\bar{\alpha} \in Seq_i$, $\min(Z) > i$ and for every $j \in Z$ there exists $\zeta \in \Delta_j^{end}$ such that $rk_\zeta(\bar{\alpha}) = f(j)$. It is easy to verify that $K \neq \emptyset$.

Let $(\bar{\alpha}^*, Z^*, f^*)$ be a triple in K for which $\mathfrak{T}(f^*)$ is minimal amongst the elements of K . Without loss of generality, all the elements of Z^* are limit ordinals. For every $j \in Z^*$ we isolate a section $\Sigma_j \subseteq \Delta_j^{end}$ by defining $\Sigma_j = \{\zeta \in \Delta_j^{end} \mid rk_\zeta(\bar{\alpha}^*) = f^*(j)\}$. Notice that $|\Sigma_j| = \lambda_{j+1}$. Fix $j \in Z^*$. We would like to understand what happens between the j th level and upper levels mentioned in Z^* . For every $\ell \in Z^*$ such that $j < \ell$ and every $\zeta \in \Sigma_j$, let $\Sigma_j^\ell(\zeta) = \{\eta \in \Sigma_\ell \mid c(\zeta, \eta) = 1\}$. Let $L_\zeta = \{\ell \in Z^* \mid j < \ell, |\Sigma_j^\ell(\zeta)| = \lambda_{\ell+1}\}$, the set of *large* 1-monochromatic levels above j . Similarly, let $T_\zeta = Z^* - L_\zeta$, the set of *tiny* 1-monochromatic levels above j .

Our goal is to garner many ordinals ζ with big T_ζ , since then we will be able to remove the “1” edges (there will be only a few of them) and create a 0-monochromatic union of size λ . We claim, therefore, that $L_\zeta \notin \mathcal{U}$ (and parallelly, $T_\zeta \in \mathcal{U}$) for every $j \in Z^*, \zeta \in \Sigma_j$. For suppose not. Fix $i \in Z^*, \beta \in \Sigma_i$ such that $L_\beta \in \mathcal{U}$. Define $\bar{\alpha}' = \bar{\alpha}^* \frown \langle \beta \rangle$, $Z' = L_\beta$ and for $j \in Z'$ let $f'(j) = \min\{rk_\eta(\bar{\alpha}') \mid \eta \in \Sigma_i^j(\beta)\}$ and $f'(j) = 0$ otherwise. Notice that $(\bar{\alpha}', Z', f') \in K$. We claim that $\mathfrak{T}(f') < \mathfrak{T}(f^*)$. Indeed, for

every $j \in Z' = L_\beta$ one has $f'(j) = rk_\eta(\bar{\alpha}')$ for some $\eta \in \Sigma_i^j(\beta)$, so $f'(j) = rk_\eta(\bar{\alpha}^* \frown \langle \beta \rangle) < rk_\eta(\bar{\alpha}^*) = f^*(j)$. Thus, $\neg(f') < \neg(f^*)$, contradicting the minimality of $\neg(f^*)$.

We conclude, therefore, that $T_\zeta \in \mathcal{U}$ for every $j \in Z^*, \zeta \in \Sigma_j$. Fix now $j \in Z^*$.

Let $\mathcal{A}^j = \langle T_\zeta \mid \zeta \in \Sigma_j \rangle \subseteq \mathcal{U}$, apply $\text{Gal}_{\text{Def}}(\mathcal{U}, \lambda_{j+1}, \lambda_{j+1})$ and find $M_j \subseteq \Sigma_j$ of cardinality λ_{j+1} such that $\bigcap_{\zeta \in M_j} T_\zeta =: Y_j \in \mathcal{U}$. We emphasize that we do not need the axiom of choice in order to define these objects, because of definability. We render this process at every $j \in Z^*$. Finally, let $Y = \Delta\{Y_j \mid j \in Z^*\} \in \mathcal{U}$.

For every $j \in Y$ we wish to define $A_j \subseteq M_j$ such that $|A_j| = \lambda_{j+1}$. The sets A_j will be 0-monochromatic, and our goal is to show that their union is 0-monochromatic as well. We define these sets by induction on $j \in Y$. So fix $j \in Y$ and define $A_j = \{\xi \in M_j \mid \forall i \in Y \cap j, \forall \zeta \in A_i, \xi \notin \Sigma_i^j(\zeta)\}$. We claim that $|A_j| = \lambda_{j+1}$. To see this, observe that if $i \in Y \cap j$ then for every $\zeta \in A_i$ the set $\Sigma_i^j(\zeta)$ is bounded in λ_{j+1} , hence $\lambda_{j+1} - \Sigma_i^j(\zeta)$ contains a final segment of λ_{j+1} . Thus, by regularity of λ_{j+1} , $\bigcap\{\lambda_{j+1} - \Sigma_i^j(\zeta) \mid \zeta \in A_i\}$ contains a final segment. Notice that A_j is obtained by intersecting this set with M_j , hence $|A_j| = \lambda_{j+1}$.

Define $A = \bigcup_{j \in Y} A_j$, so $|A| = \lambda$. By proving that $c^{\text{``}}[A]^2 = \{0\}$ we will be done. Pick $\alpha, \beta \in A$ such that $\alpha < \beta$. If there exists $j \in \omega_1$ such that $\alpha, \beta \in A_j$ then $c(\alpha, \beta) = 0$ since $A_j \subseteq A_0^j \subseteq \Sigma_j$ and $c^{\text{``}}[\Sigma_j]^2 = \{0\}$. If not, then there are $i < j < \omega_1$ such that $\alpha \in A_i, \beta \in A_j$, and $i, j \in Y$. By definition, $\beta \notin \Sigma_i^j(\alpha)$ and hence $c(\alpha, \beta) = 0$ and the proof is accomplished. \square

The above proof also gives the following corollary in ZFC, where the club filter over regular cardinals is always normal.

Corollary 3.5 (ZFC). *Suppose that \mathcal{U} is a normal measure over κ , λ is singular with $\text{cf}(\lambda) = \kappa$ and that $\text{Gal}(\mathcal{U}, \lambda_i, \lambda_i)$ holds for cofinally many regular cardinals below λ . Then, $\lambda \rightarrow (\lambda, \omega + 1)^2$*

The situation described in the previous corollary is a consequence of a small ultrafilter number: Suppose that κ is measurable, and \mathcal{U} is a normal ultrafilter over κ with a base of \mathcal{U} of size κ^+ . One says, in this case, that $\mathfrak{u}_\kappa^{\text{nor}} = \kappa^+$. It is possible to force $\mathfrak{u}_\kappa^{\text{nor}} = \kappa^+$ even if 2^κ is arbitrarily large, see e.g. [20] and [19].

It is easy to verify that if \mathcal{U} witnesses $\mathfrak{u}_\kappa^{\text{nor}} = \kappa^+$ then $\text{Gal}(\mathcal{U}, \mu, \mu)$ holds whenever $\mu = \text{cf}(\mu) > \kappa^+$, see e.g. [6].

Corollary 3.6. *If κ is measurable and $\mathfrak{u}_\kappa^{\text{nor}} = \kappa^+$ then $\lambda \rightarrow (\lambda, \omega + 1)^2$ whenever $\kappa = \text{cf}(\lambda) < \lambda$.*

The corollary shows that the positive relation $\lambda \rightarrow (\lambda, \omega + 1)^2$ may hold even if $2^\kappa > \lambda$. This fact was known already to Shelah and Stanley [32]. It is shown there that if λ is a strong limit singular of uncountable cofinality then many versions of Cohen forcing preserve the positive relation $\lambda \rightarrow (\lambda, \omega + 1)^2$;

in particular, adding many Cohen subsets of κ in a way that $2^\kappa > \lambda$. Our corollary generalizes these results, since any κ -complete forcing notion will preserve the statement $\mathfrak{u}_\kappa^{\text{nor}} = \kappa^+$.

In the above statements we used Galvin's property in order to prove $\lambda \rightarrow (\lambda, \omega + 1)^2$. The connection between ordinary partition relations and the structure of normal filters seems to be helpful in the opposite direction as well. That is, from the assumption that $\lambda \rightarrow (\lambda, \omega + 1)^2$ one can learn something about the Galvin property.

Proposition 3.7. *Let $\kappa = \text{cf}(\kappa) > \aleph_0$ and let \mathcal{F} be a normal filter over κ . Suppose that $\neg_{\text{st}} \text{Gal}(\mathcal{F}, \kappa, \kappa^+)$ is witnessed by $\mathcal{C} = \{C_\alpha \mid \alpha < \kappa^+\}$. Then one can find $a = \{\alpha_n \mid n < \omega\} \subseteq \kappa^+$ such that $\bigcap\{(\kappa \setminus C_{\alpha_n}) \mid n < \omega\} \neq \emptyset$.*

Proof. Assume toward contradiction that $\mathcal{C} = \{C_\alpha \mid \alpha < \kappa^+\}$ witnesses the strong failure of $\text{Gal}(\mathcal{F}, \kappa, \kappa^+)$, yet the complements are not overlapping in the sense that every infinite collection of them has empty intersection. Define a coloring $c : [\kappa^+]^2 \rightarrow 2$ as follows. For $\alpha < \beta < \kappa^+$ let $c(\alpha, \beta) = 0$ iff $\beta \in C_\alpha$.

Notice that there is no 1-monochromatic sequence of length $\omega + 1$ under c . For if $\langle \alpha_n \mid n \leq \omega \rangle$ is such a sequence then $\alpha_\omega \in \bigcap\{(\kappa \setminus C_{\alpha_n}) \mid n < \omega\}$, in contrary to our assumption at the beginning of the proof. Likewise, there is no 0-monochromatic set $A \in [\kappa^+]^{\kappa^+}$. For if $A = A_0 \cup A_1$ with $\text{sup}(A_0) < \text{min}(A_1)$ and $\text{otp}(A_0) = \text{otp}(A_1) = \kappa$, then $A_1 \subseteq \bigcap\{C_\alpha \mid \alpha \in A_0\}$. This is impossible since $\{C_\alpha \mid \alpha \in A_0\} \subseteq \mathcal{C}$. Thus, the coloring c witnesses $\kappa^+ \not\rightarrow (\kappa^+, \omega + 1)^2$, which is an absurd since κ^+ is regular and uncountable. \square

A similar argument becomes more powerful at successors of large cardinals. By [7] one can force a κ -complete ultrafilter \mathcal{U} over a measurable cardinal κ such that $\text{Gal}(\mathcal{U}, \kappa, \kappa^+)$ fails.

Proposition 3.8. *Suppose that κ is measurable and \mathcal{U} is a κ -complete ultrafilter over κ for which $\neg_{\text{st}} \text{Gal}(\mathcal{U}, \kappa, \kappa^+)$ is witnessed by the sequence $\mathcal{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$. Then there is $S \in [\kappa]^\kappa$ such that $\bigcap\{(\kappa \setminus C_\alpha) \mid \alpha \in S\}$ is non-empty.*

Proof. We define $c : [\kappa^+]^2 \rightarrow 2$ as before, by letting $c(\alpha, \beta) = 0$ iff $\beta \in C_\alpha$. Of course, the definition is needed at $\alpha < \beta < \kappa^+$ only, since the coloring is symmetric. Assume toward contradiction that there is no $S \in [\kappa]^\kappa$ such that $\bigcap\{(\kappa \setminus C_\alpha) \mid \alpha \in S\} \neq \emptyset$. By the same argument as in the previous claim, c witnesses the negative relation $\kappa^+ \not\rightarrow (\kappa + \kappa, \kappa + 1)^2$. However, this relation holds if κ is measurable, as proved in [17], a contradiction. \square

The above statements show that there is a limitation on forcing empty intersection over the complements of sets which witness the strong failure of the Galvin property. One can try, however, to force empty intersection only over certain families. The following is tailored to the goal of obtaining a negative relation at singular cardinals.

Given $\lambda > \text{cf}(\lambda) = \kappa > \aleph_0$ and a cofinal sequence $\langle \lambda_i \mid i < \kappa \rangle$ in λ , let $\Delta_0 := [0, \lambda_0)$ and $\Delta_{1+i} := [\lambda_i, \lambda_{i+1})$.

Proposition 3.9. *Suppose that:*

- (\aleph) $\lambda > \text{cf}(\lambda) = \kappa > \aleph_0$ and \mathcal{F} is a normal filter over κ .
- (\beth) $\{C_\alpha \mid \alpha < \lambda\} \subseteq \mathcal{F}$ witnesses $\neg_{\text{st}} \text{Gal}(\mathcal{F}, \kappa, \lambda)$.
- (\beth) $\bigcap \{(\kappa \setminus C_{\alpha_n}) \mid n < \omega\} = \emptyset$ whenever $\alpha_n \in \Delta_{i_n}$ for every $n < \omega$, and $m \neq n \Rightarrow \alpha_m \neq \alpha_n$.

Then $\lambda \rightarrow (\lambda, \omega + 1)^2$.

Proof. For $\alpha < \beta < \lambda$, if there exists $i \in \kappa$ such that $\alpha, \beta \in \Delta_i$ then let $c(\alpha, \beta) = 0$. If $\alpha \in \Delta_i, \beta \in \Delta_j$ and $i < j$, let $c(\alpha, \beta) = 0$ iff $\beta \in C_\alpha$. One can verify that $c : [\lambda]^2 \rightarrow 2$ witnesses $\lambda \rightarrow (\lambda, \omega + 1)^2$, so we are done. \square

On a different path, Theorem 3.10 can be applied also to models of AD, where the distance between regular and measurable is quite short. This distance is even shorter under the extra assumption that $V = L(\mathbb{R})$, where every regular cardinal below Θ is measurable, as proved in [33].

Corollary 3.10 (AD + $V = L(\mathbb{R})$). *Suppose that $\aleph_0 < \kappa = \text{cf}(\lambda) < \lambda$. If λ is a limit of regular cardinals then $\lambda \rightarrow (\lambda, \omega + 1)^2$.*

Proof. For transparency, assume that $\kappa = \omega_1$. Let $\lambda = \bigcup_{i < \omega_1} \lambda_i$, where $\langle \lambda_i \mid i < \omega_1 \rangle$ is increasing and continuous, $\omega_1 < \lambda_0$ is measurable and λ_{i+1} is measurable for every $i < \omega_1$. Let \mathcal{U} denote the club filter over ω_1 , which is a normal ultrafilter under AD. For every $i < \omega_1$ let \mathcal{U}_{i+1} be the filter generated by the unbounded ω -closed subsets of λ_{i+1} . This is a normal ultrafilter over λ_{i+1} under AD+V = $L(\mathbb{R})$, as shown in [33, Theorem 8.27]. Also, $\text{Gal}_{\text{Def}}(\mathcal{U}, \lambda_{i+1}, \lambda_{i+1})$ holds by virtue of Proposition 2.31. Thereby, the assumptions of Theorem 3.10 are met. \square

It turns out that if $\lambda = \aleph_{\omega_1}$ we can do it better:

Corollary 3.11 (AD). *The relation $\aleph_{\omega_1} \rightarrow (\aleph_{\omega_1}, \omega + 1)^2$ holds.*

Proof. For each limit $\alpha < \omega_1$ denote by δ_α^1 the supremum of the lengths of all Δ_α^1 prewellorderings. It follows that, under AD, the sequence $\langle \delta_{\alpha+1}^1 \mid \alpha \in \text{acc}(\omega_1) \rangle$ satisfies the following property: for each $\alpha \in \text{acc}(\omega_1)$ the filter generated by the ω -closed subsets of $\delta_{\alpha+1}^1$ is a normal measure [23, §5.4]. Also $\sup_{\alpha < \omega_1} \delta_{\alpha+1}^1 = \aleph_{\omega_1}$. Now we can argue as in the previous corollary that $\aleph_{\omega_1} \rightarrow (\aleph_{\omega_1}, \omega + 1)^2$. \square

We finalize the current section with a ZFC question:

Question 3.12. Does $\lambda \not\rightarrow (\lambda, \omega + 1)^2$ have any consistency strength?

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