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# INTERMEDIATE MODELS OF MAGIDOR–RADIN FORCING. I

BY

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ABSTRACT

We continue the work done in [3], [1]. We prove that for every set $A$ in a Magidor–Radin generic extension using a coherent sequence such that $o^{\vec{U}}(\kappa) < \kappa$, there is a subset $C'$ of the Magidor club such that $V[A] = V[C']$. Also we classify all intermediate $ZFC$ transitive models $V \subseteq M \subseteq V[G]$.

## 1. Introduction

In this paper we consider the version of Magidor–Radin forcing for $o^{\vec{U}}(\kappa) \leq \kappa$, but prove results for $o^{\vec{U}}(\kappa) < \kappa$. Section 2, will also be relevant to the forcing in Part II.

Denote by $C_G$, the generic Magidor–Radin club derived from a generic filter $G$. In [1], the authors proved the following:

THEOREM 1.1: *Let $\vec{U}$ be a coherent sequence and $G \subseteq \mathbb{M}[\vec{U}]$ be a $V$-generic filter such that $o^{\vec{U}}(\beta) < \delta_0 := \min\{\alpha \mid 0 < o^{\vec{U}}(\alpha)\}$ for every $\beta \in C_G \cup \{\kappa\}$. Then for every set $A \in V[G]$, there is $C \subseteq C_G$ such that $V[A] = V[C]$.*

In this paper we would like to generalize this result to the case where $o^{\vec{U}}(\kappa) < \kappa$. Formally, we prove this generalization by induction $\kappa$, namely, the inductive hypothesis is that for every $\delta < \kappa$, any coherent sequence $\vec{W}$ with maximal

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measurable $\delta$, and any set $A$ in a generic extension $V[H]$, where $H \subseteq \mathbb{M}[\vec{W}]$, there is $C \subseteq C_H$ such that $V[A] = V[C]$. Here we do not restrict the order of $\delta$'s below $\kappa$. To be precise, the proof given in this paper is the inductive step for the case $o^{\vec{U}}(\kappa) < \kappa$:

Theorem 1.2: *Let $U$ be a coherent sequence with maximal measurable $\kappa$ such that $o^{\vec{U}}(\kappa) < \kappa$. Assume the inductive hypothesis that for every $\delta < \kappa$, any coherent sequence $\vec{W}$ with maximal measurable $\delta$, and any set $A$ in a generic extension $V[H]$ for $H \subseteq \mathbb{M}[\vec{W}]$, there is $C \subseteq C_H$ such that $V[A] = V[C]$. Then for every $V$-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and any set $A \in V[G]$, there is $C \subseteq C_G$ such that $V[A] = V[C]$.*

As a corollary of this, we obtain the main result of this paper:

Theorem 1.3: *Let $\vec{U}$ be a coherent sequence such that $o^{\vec{U}}(\kappa) < \kappa$. Then for every $V$-generic filter $G \subseteq \mathbb{M}[\vec{U}]$, such that $\forall\alpha \in C_G.o^{\vec{U}}(\alpha) < \alpha$ and every $A \in V[G]$, there is $C \subseteq C_G$ such that $V[A] = V[C]$.*

The first problem which rises when we let $o^{\vec{U}}(\kappa) \geq \delta_0$ is that we might lose completness for some of the pairs in a condition $p$. For example, if

$$p = \langle\langle\delta_0, A_0\rangle, \langle\kappa, A_1\rangle\rangle,$$

we will be unable to take into account all the measures on $\kappa$, since there are $\delta_0$ many of them and only $\delta_0$-completness. The idea is to split $\mathbb{M}[\vec{U}]$ to the part below $o^{\vec{U}}(\kappa)$ and above it. The cardinality of the lower part is lower than the the degree of $\leq^*$-closure of the upper part. The upper part is an instance of $\mathbb{M}[\vec{U}]$, where the order of every measurable is below the order of $\kappa$ which is similar to the framework of Theorem 1.1, then some but not all of the arguments of [1] generalize.

Note that the classification we had in [1] for models of the form $V[C']$ does not extend, even if $o^{\vec{U}}(\kappa) = \delta_0$.

*Example 1.4:* Consider $C_G$ such that $C_G(\omega) = \delta_0$ and $o^{\vec{U}}(\kappa) = \delta_0$. Then in $V[G]$ we have the following sequence $C' = \langle C_G(C_G(n)) \mid n < \omega\rangle$ of points of the generic $C_G$ which is determined by the first Prikry sequence at $\delta_0$.

Then $I(C', C_G) = \langle C_G(n) \mid n < \omega\rangle \notin V$, where $I(X, Y)$ is the indices of $X \subseteq Y$ in the increasing enumeration of $Y$.

The forcing $\mathbb{M}_I[\vec{U}]$ which was defined in [1] is no longer defined in $V$ since $I \notin V$.

In this case, we will add points to $C'$, which are simply $\langle C_G(n) \mid n < \omega\rangle$, then the forcing will be a two-step iteration. The first will be to add the Prikry sequence $\langle C_G(n) \mid n < \omega\rangle$, then the second will be a Diagonal Prikry forcing adding points from the measures $\langle U(\kappa, C_G(n)) \mid n < \omega\rangle$, which is of the form $M_I[\vec{U}]$.

Generally, we will define forcing $\mathbb{M}_f[\vec{U}]$, which are not subforcing of $\mathbb{M}[\vec{U}]$, but are a natural diagonal generalization of $\mathbb{M}[\vec{U}]$ and a bit closer to Magidor's original formulation in [5].

The classification of models is given by the following theorem:

THEOREM 1.5: *Assume that for every $\alpha \leq \kappa$, $o^{\vec{U}}(\alpha) < \alpha$. Then for every $V$-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ and every transitive $ZFC$ intermediate model $V \subseteq M \subseteq V[G]$, there is a closed subset $C_{\text{fin}} \subseteq C_G$ such that:*

1. *$M = V[C_{\text{fin}}]$.*
2. *There is a finite iteration $\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_2}}[\vec{U}] * \cdots * \mathbb{M}_{\underset{\sim}{f_n}}[\vec{U}]$, and a $V$-generic $H^*$ filter for $\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_2}}[\vec{U}] * \cdots * \mathbb{M}_{\underset{\sim}{f_n}}[\vec{U}]$ such that*

$$V[H^*] = V[C_{\text{fin}}] = M.$$

## 2. Basic definitions and preliminaries

We will follow the description of Magidor forcing as presented in [2].

Let $\vec{U} = \langle U(\alpha, \beta) \mid \alpha \leq \kappa\ , \beta < o^{\vec{U}}(\alpha)\rangle$ be a coherent sequence. For every $\alpha \leq \kappa$, denote

$$\cap\vec{U}(\alpha) = \bigcap_{i < o^{\vec{U}}(\alpha)} U(\alpha, i).$$

*Definition 2.1:* $\mathbb{M}[\vec{U}]$ consists of elements $p$ of the form $p = \langle t_1, \dots, t_n, \langle \kappa, B\rangle\rangle$. For every $1 \leq i \leq n$, $t_i$ is either an ordinal $\kappa_i$ if $o^{\vec{U}}(\kappa_i) = 0$ or a pair $\langle \kappa_i, B_i\rangle$ if $o^{\vec{U}}(\kappa_i) > 0$.

1. $B \in \cap\vec{U}(\kappa)$, $\min(B) > \kappa_n$.
2. For every $1 \leq i \leq n$,
    - (a) $\langle \kappa_1, \dots, \kappa_n\rangle \in [\kappa]^{<\omega}$ (increasing finite sequence below $\kappa$),
    - (b) $B_i \in \cap\vec{U}(\kappa_i)$,
    - (c) $\min(B_i) > \kappa_{i-1}$ $(i > 1)$.

*Definition 2.2:* For $p = \langle t_1, t_2, \dots, t_n, \langle \kappa, B\rangle\rangle$, $q = \langle s_1, \dots, s_m, \langle \kappa, C\rangle\rangle \in \mathbb{M}[\vec{U}]$, define $p \le q$ ($q$ extends $p$) iff:

(1) $n \le m$.
(2) $B \supseteq C$.
(3) $\exists 1 \le i_1 < \cdots < i_n \le m$ such that for every $1 \le j \le m$:
  (a) If $\exists 1 \le r \le n$ such that $i_r = j$ then $\kappa(t_r) = \kappa(s_{i_r})$ and $C(s_{i_r}) \subseteq B(t_r)$.
  (b) Otherwise $\exists\, 1 \le r \le n+1$ such that $i_{r-1} < j < i_r$ then
    (i) $\kappa(s_j) \in B(t_r)$,
    (ii) $B(s_j) \subseteq B(t_r) \cap \kappa(s_j)$,
    (iii) $o^{\vec{U}}(s_j) < o^{\vec{U}}(t_r)$.

We also use "$p$ directly extends $q$", $p \le^* q$ if:

(1) $p \le q$,
(2) $n = m$.

Let us add some notation: for a pair $t = \langle \alpha, X\rangle$ we denote $\kappa(t) = \alpha$, $B(t) = X$. If $t = \alpha$ is an ordinal then $\kappa(t) = \alpha$ and $B(t) = \emptyset$.

For a condition $p = \langle t_1, \dots, t_n, \langle \kappa, B\rangle\rangle \in \mathbb{M}[\vec{U}]$ we denote $n = l(p)$, $p_i = t_i$, $B_i(p) = B(t_i)$ and $\kappa_i(p) = \kappa(t_i)$ for any $1 \le i \le l(p)$, $t_{l(p)+1} = \langle \kappa, B\rangle$, $t_0 = 0$. Also denote

$$\kappa(p) = \{\kappa_i(p) \mid i \le l(p)\} \quad \text{and} \quad B(p) = \bigcup_{i \le l(p)+1} B_i(p).$$

*Remark 2.3:* Condition 3.b.iii is not essential, since the set

$$\{p \in \mathbb{M}[\vec{U}] \mid \forall i \le l(p)+1. \forall \alpha \in B_i(p). o^{\vec{U}}(\alpha) < o^{\vec{U}}(\kappa_i(p))\}$$

is a dense subset of $\mathbb{M}[\vec{U}]$ and the order between any two elements of this dense subset automatically satisfies 3.b.iii.

*Definition 2.4:* Let $p \in \mathbb{M}[\vec{U}]$. For every $i \le l(p)+1$, and $\alpha \in B_i(p)$ with $o^{\vec{U}}(\alpha) > 0$, define

$$p^\frown\langle\alpha\rangle = \langle p_1, \dots, p_{i-1}, \langle \alpha, B_i(p) \cap \alpha\rangle, \langle \kappa_i(p), B_i(p) \setminus (\alpha+1)\rangle, p_{i+1}, \dots, p_{l(p)+1}\rangle.$$

If $o^{\vec{U}}(\alpha) = 0$, define

$$p^\frown\langle\alpha\rangle = \langle p_1, \dots, p_{i-1}, \alpha, \langle \kappa_i(p), B_i(p) \setminus (\alpha+1)\rangle, \dots, p_{l(p)+1}\rangle.$$

For $\langle \alpha_1, \dots, \alpha_n\rangle \in [\kappa]^{<\omega}$ define recursively,

$$p^\frown\langle \alpha_1, \dots, \alpha_n\rangle = (p^\frown\langle \alpha_1, \dots, \alpha_{n-1}\rangle)^\frown\langle \alpha_n\rangle.$$

Proposition 2.5: *Let $p \in \mathbb{M}[\vec{U}]$. If $p^\frown \vec{\alpha} \in \mathbb{M}[\vec{U}]$, then it is the minimal extension of $p$ with stem*

$$\kappa(p) \cup \{\vec{\alpha}_1, \dots, \vec{\alpha}_{|\vec{\alpha}|}\}$$

*Moreover, $p^\frown \vec{\alpha} \in \mathbb{M}[\vec{U}]$ iff for every $i \leq |\vec{\alpha}|$ there is $j \leq l(p)$ such that:*

(1) $\vec{\alpha}_i \in (\kappa_j(p), \kappa_{j+1}(p))$.
(2) $o^{\vec{U}}(\vec{\alpha}_i) < o^{\vec{U}}(\kappa_{j+1})$.
(3) $B_{j+1}(p) \cap \vec{\alpha}_i \in \cap \vec{U}(\vec{\alpha}_i)$. ∎

Note that if we add a pair of the form $\langle \alpha, B \cap \alpha \rangle$, then in $B \cap \alpha$ there might be many ordinals which are irrelevant to the forcing, namely, ordinals $\beta \in B \cap \alpha$ with $o^{\vec{U}}(\beta) \geq o^{\vec{U}}(\alpha)$; such ordinals cannot be added to the sequence.

*Definition 2.6:* Let $p \in \mathbb{M}[\vec{U}]$. Define for every $i \leq l(p)$

$$p \restriction \kappa_i(p) = \langle p_1, \dots, p_i \rangle \quad \text{and} \quad p \restriction (\kappa_i(p), \kappa) = \langle p_{i+1}, \dots, p_{l(p)+1} \rangle.$$

Also, for $\lambda$ with $o^{\vec{U}}(\lambda) > 0$ define

$$\mathbb{M}[\vec{U}] \restriction \lambda = \{p \restriction \lambda \mid p \in \mathbb{M}[\vec{U}] \text{ and } \lambda \text{ appears in } p\},$$
$$\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa) = \{p \restriction (\lambda, \kappa) \mid p \in \mathbb{M}[\vec{U}] \text{ and } \lambda \text{ appears in } p\}.$$

Note that $\mathbb{M}[\vec{U}] \restriction \lambda$ is just Magidor forcing on $\lambda$ and $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ is a subset of $\mathbb{M}[\vec{U}]$. The following decomposition is straightforward.

Proposition 2.7: *Let $p \in \mathbb{M}[\vec{U}]$ and $\langle \lambda, B \rangle$ be a pair in $p$. Then*

$$\mathbb{M}[\vec{U}]/p \simeq (\mathbb{M}[\vec{U}] \restriction \lambda)/(p \restriction \lambda) \times (\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa))/(p \restriction (\lambda, \kappa)).$$

*Remark 2.8:* When considering $\vec{U}$ in some model $V \subseteq N \subseteq V[C_G \cap \lambda]$, since we added generic sequences, not all of the measures in $\vec{U}$ remain measures in $N$. However, each measure $U(\xi, i)$ for $\lambda < \xi \leq \kappa$ and $i < o^{\vec{U}}(\xi)$ generates a normal measure $W(\xi, i)$ over $\xi$ such that

$$\vec{W} = \langle W(\xi, i) \mid \lambda < \xi \leq \kappa,\, i < o^{\vec{U}}(\xi) \rangle$$

is a coherent sequence. Since $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ is a dense subset of $\mathbb{M}[\vec{W}]$, forcing over $N$ with $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ is the same as forcing with $\mathbb{M}[\vec{W}]$.

Proposition 2.9: *Let $p \in \mathbb{M}[\vec{U}]$ and $\langle \lambda, B\rangle$ be a pair in $p$. Then the order $\leq^*$ in the forcing $(\mathbb{M}[\vec{U}] \restriction (\lambda,\kappa))/(p \restriction (\lambda,\kappa))$ is $\delta$-directed where*

$$\delta = \min\{\nu > \lambda \mid o^{\vec{U}}(\nu) > 0\},$$

*meaning that for every $X \subseteq \mathbb{M}[\vec{U}] \restriction (\lambda,\kappa)$ such that $|X| < \delta$ and for every $q \in X,\ p \leq^* q$, there is an $\leq^*$-upper bound for $X$.*

Lemma 2.10: *$\mathbb{M}[\vec{U}]$ satisfies $\kappa^+$-c.c.*

The following is known as the Prikry condition:

Lemma 2.11: *$\mathbb{M}[\vec{U}]$ satisfies the Prikry condition, i.e., for any statement in the forcing language $\sigma$ and any $p \in \mathbb{M}[\vec{U}]$ there is $p \leq^* p^*$ such that $p^* || \sigma$, i.e., either $p^* \Vdash \sigma$ or $p \Vdash \neg\sigma$.*

The next lemma can be found in [5]:

Lemma 2.12: *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic and suppose that $A \in V[G]$ is such that $A \subseteq V_\alpha$. Let $p \in G$ and $\langle \lambda, B\rangle$ be a pair in $p$ such that $\alpha < \lambda$. Then $A \in V[G \restriction \lambda]$.*

*Proof.* Consider the decomposition 2.7 $p = \langle q, r\rangle$, where $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and $r \in \mathbb{M}[\vec{U}] \restriction (\lambda,\kappa)$. Work in $V[G \restriction \lambda]$. Let $\underset{\sim}{A}$ be a $\mathbb{M}[\vec{U}] \restriction (\lambda,\kappa)$-name for $A$. For every $x \in V_\alpha$ use the Prikry condition 2.11, to find $r \leq^* r_x$ such that $r_x$ decides the statement $r \in \underset{\sim}{A}$. By definition of $\lambda$ and Proposition 2.15, the forcing $\mathbb{M}[\vec{U}] \restriction (\lambda,\kappa)$ is $|V_\alpha|^+$-directed with the $\leq^*$ order. Hence there is $r \leq^* r^*$ such that $p_x \leq^* p^*$ for every $x \in V_\alpha$. By density, we can find such $r^* \in G \restriction (\lambda,\kappa)$. It follows that $A = \{x \in V_\alpha \mid r^* \Vdash x \in \underset{\sim}{A}\}$ is definable in $V[G \restriction \lambda]$. ∎

Corollary 2.13: *$\mathbb{M}[\vec{U}]$ preserves all cardinals.*

*Definition 2.14:* Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic. Define the **Magidor club**

$$C_G = \{\nu \mid \exists p \in G \exists i \leq l(p) \text{ s.t. } \nu = \kappa_i(p)\}.$$

We will abuse notation by sometimes considering $C_G$ as the canonical enumeration of the set $C_G$. The set $C_G$ is closed and unbounded in $\kappa$, therefore, the order type of $C_G$ determines the cofinality of $\kappa$ in $V[G]$. The next propositions can be found in [2].

Proposition 2.15: *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic. Then $G$ can be reconstructed from $C_G$ as follows:*

$$G = \{p \in \mathbb{M}[\vec{U}] \mid (\kappa(p) \subseteq C_G) \wedge (C_G \setminus \kappa(p) \subseteq B(p))\}.$$

*In particular $V[G] = V[C_G]$.*

Proposition 2.16: *Let $G \subseteq \mathbb{M}[\vec{U}]$ be generic.*

1. *$C_G$ is a club at $\kappa$.*
2. *For every $\delta \in C_G$, $o^{\vec{U}}(\delta) > 0$ iff $\delta \in \mathrm{Lim}(C_G)$.*
3. *For every $\delta \in \mathrm{Lim}(C_G)$, and every $A \in \cap\vec{U}(\delta)$, there is $\xi < \delta$ such that $C_G \cap (\xi, \delta) \subseteq A$.*
4. *If $\langle \delta_i \mid i < \theta \rangle$ is an increasing sequence of elements of $C_G$, let $\delta^* = \sup_{i<\theta} \delta_i$. Then $o^{\vec{U}}(\delta^*) \geq \limsup_{i<\theta} o^{\vec{U}}(\delta_i) + 1$.*[1]
5. *Let $\delta \in \mathrm{Lim}(C_G)$ and let $A$ be a positive set, $A \in (\cap\vec{U}(\delta))^+$, i.e., $\delta \setminus A \notin \cap\vec{U}(\delta)$.*[2] *Then $\sup(A \cap C_G) = \delta$.*
6. *If $A \subseteq V_\alpha$, then $A \in V[C_G \cap \lambda]$, where $\lambda = \max(\mathrm{Lim}(C_G) \cap \alpha + 1)$.*
7. *For every $V$-regular cardinal $\alpha$, if $cf^{V[G]}(\alpha) < \alpha$ then $\alpha \in \mathrm{Lim}(C_G)$.*

*Proof.* (1), (2), (3) can be found in [2].

To see (4), use closure of $C_G$, and find $q \in G$ such that $\delta^*$ appears in $q$. Since there are only finitely many ordinals in $q$, there is some $i < \theta$ such that for every $j > i$, $\delta_j$ does not appear in $q$. By 2.2, since every such $\delta_j$ appears in some $q_j \in G$ which is compatible with $q$, $o^{\vec{U}}(\delta_j) < o^{\vec{U}}(\delta^*)$. Hence

$$\limsup_{j<\theta} o^{\vec{U}}(\delta_j) + 1 \leq \sup_{i<j<\theta} o^{\vec{U}}(\delta_j) + 1 \leq o^{\vec{U}}(\delta^*).$$

For (5), let $\rho < \delta$. Each condition $p$, such that $\delta = \kappa_i(p)$ for some $i \leq l(p)+1$, must satisfy that $\sup(A \cap B_i(p)) = \delta$. Hence we can extend $p$ using an element of $A \cap B_i(p)$ above $\rho$. By density, $\sup(A \cap C_G) \geq \rho$. Since $\rho$ is general, $\sup(A \cap C_G) = \delta$.

(6) is a direct corollary of 2.12. As for (7), assume that $cf^{V[G]}(\alpha) < \alpha$, and let $X \subseteq \alpha$ be a club such that $\mathrm{otp}(X) = cf^{V[G]}(\alpha)$. Then $X \in V[G] \setminus V$. Let $\lambda = \max(\mathrm{Lim}(C_G) \cap \alpha + 1)$, then $\lambda \leq \alpha$. By (6), $X \in V[C_G \cap \lambda]$. Toward a contradiction, assume that $\lambda < \alpha$, then the forcing $\mathbb{M}[\vec{U}] \restriction \lambda$ is $\alpha$-c.c., but $cf^{V[C_G \cap \lambda]}(\alpha) < \alpha$, contradiction. ∎

[1] For a sequence of ordinals $\langle \rho_j \mid j < \gamma \rangle$, $\limsup_{j<\gamma} \rho_j = \min\{\sup_{i<j<\gamma} \rho_j \mid i < \gamma\}$.

[2] Equivalently, if there is some $i < o^{\vec{U}}(\delta)$ such that $A \in U(\delta, i)$.

The Mathias-like criteria for Magidor forcing is due to Mitchell [6]:

THEOREM 2.17: *Let $U$ be a coherent sequence and assume that $c : \alpha \to \kappa$ is an increasing function. Then $c$ is $\mathbb{M}[\vec{U}]$-generic iff:*

(1) *$c$ is continuous.*
(2) *$c \restriction \beta$ is $\mathbb{M}[\vec{U} \restriction \beta]$-generic for every $\beta \in \mathrm{Lim}(\alpha)$.*
(3) *$X \in \cap\vec{U}(\kappa)$ iff $\exists\beta < \kappa, \mathrm{Im}(c) \setminus \beta \subseteq X$.*

An equivalent formulation of the Mathias criteria is to require that for every $\beta \in \mathrm{Lim}(\alpha)$, and for every $X \in \cap\vec{U}(c(\beta))$, there is $\xi < \beta$ such that $c''(\xi, \beta) \subseteq X$.

For an additional proof of 2.17, we refer the reader to the last section, where we define a forcing notion $\mathbb{M}_f[\vec{U}]$, which generalizes $\mathbb{M}[\vec{U}]$, and prove in 5.14 a Mathias-like criteria for it.

PROPOSITION 2.18: *Let $G \subseteq \mathbb{M}[\vec{U}]$ be a $V$-generic filter and $C_G$ the corresponding Magidor sequence. Let $p \in G$, then for every $i \leq l(p) + 1$:*

(1) *If $o^{\vec{U}}(\kappa_i(p)) \leq \kappa_i(p)$, then*
$$\mathrm{otp}([\kappa_{i-1}(p), \kappa_i(p)) \cap C_G) = \omega^{o^{\vec{U}}(\kappa_i(p))}.$$
(2) *If $o^{\vec{U}}(\kappa_i(p)) \geq \kappa_i(p)$, then*
$$\mathrm{otp}([\kappa_{i-1}(p), \kappa_i(p)) \cap C_G) = \kappa_i(p).$$

*Proof.* We prove (1) by induction on $\kappa_i(p)$. If $\kappa_i(p) = 0$, then
$$C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) = \{\kappa_{i-1}(p)\}.$$
Hence
$$\mathrm{otp}(C_G \cap [\kappa_{i-1}(p), \kappa_i(p))) = 1 = \omega^0 = \omega^{o^{\vec{U}}(\kappa_i(p))}.$$

Assume the lemma holds for any $\delta < \kappa_i(p)$. If $o^{\vec{U}}(\kappa_i(p)) = \alpha + 1 \leq \kappa_i(p)$, then
$$X = \{\beta < \kappa_i(p) \mid o^{\vec{U}}(\beta) = \alpha\} \in U(\kappa_i(p), \alpha),$$
hence by Proposition 2.16
$$\sup(X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p))) = \kappa_i(p).$$
We claim that $\mathrm{otp}(X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) = \omega$. Otherwise, let $\rho < \kappa_i(p)$ be such that $\rho$ is a limit point of $X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p))$. Again by Proposition 2.16
$$o^{\vec{U}}(\rho) \geq \limsup(o^{\vec{U}}(\xi) \mid \xi \in X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p))) = \alpha + 1,$$

contradicting Definition 2.2. Let $\langle \delta_n \mid n < \omega \rangle$ be the increasing enumeration of $X \cap C_G \cap [\kappa_{i-1}(p), \kappa_i(p))$. By induction hypothesis, for every $n < \omega$,

$$\mathrm{otp}(C_G \cap [\delta_n, \delta_{n+1})) = \omega^\alpha.$$

Hence

$$\mathrm{otp}(C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) = \omega^{\alpha+1}.$$

For limit $o^{\vec{U}}(\kappa_i(p))$, use Proposition 2.16(5) to see that the sequence

$$\langle \delta_\alpha \mid \alpha < o^{\vec{U}}(\kappa_i(p)) \rangle$$

where

$$\delta_\alpha = \min\{\rho \in C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) \mid o^{\vec{U}}(\rho) = \alpha\}$$

is well defined; $x = \sup(\delta_\alpha \mid \alpha < \theta) \leq \kappa_i(p)$ is an element of $C_G$ and, by Proposition 2.16(4), $o^{\vec{U}}(x) \geq o^{\vec{U}}(\kappa_i(p))$, hence $x = \kappa_i(p)$. For every $\alpha < o^{\vec{U}}(\kappa_i(p))$,

$$\mathrm{otp}(C_G \cap [\kappa_i(p), \delta_\alpha)) = \omega^\alpha,$$

since $p^\frown \langle \delta_\alpha \rangle \in G$ and by induction hypothesis. It follows that

$$\begin{aligned}\mathrm{otp}(C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) &= \sup_{\alpha < o^{\vec{U}}(\kappa_i(p))} (\mathrm{otp}(C_G \cap [\kappa_{i-1}(p), \delta_\alpha)) \\ &= \sup_{\alpha < o^{\vec{U}}(\kappa_i(p))} \omega^\alpha = \omega^{o^{\vec{U}}(\kappa_i(p))}.\end{aligned}$$

For (2), use (1) and the limit stage to conclude that if $o^{\vec{U}}(\kappa_i(p)) = \kappa_i(p)$, then

$$\mathrm{otp}(C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) = \kappa_i(p).$$

If $o^{\vec{U}}(\kappa_i(p)) > \kappa_i(p)$, then $\{\alpha < \kappa_i(p)) \mid o^{\vec{U}}(\alpha) = \alpha\} \in U(\kappa_i(p), \kappa_i(p))$, hence by Proposition 2.16 there are unboundedly many $\alpha \in C_G \cap [\kappa_{i-1}(p), \kappa_i(p)) =: Y$ such that $o^{\vec{U}}(\alpha) = \alpha$. Hence

$$\kappa_i(p) = \sup(Y) = \sup(\mathrm{otp}(C_G \cap [\kappa_{i-1}(p), \alpha) \mid \alpha \in Y) \leq \kappa_i(p),$$

so equality holds. ∎

Proposition 2.18 suggests a connection between the index in $C_G$ of ordinals appearing in $p$ and the Cantor normal form.

*Definition 2.19:* Let $p \in G$. For each $i \leq l(p)$ define

$$\gamma_i(p) = \sum_{j=1}^{i} \omega^{o^{\vec{U}}(\kappa_j(p))}.$$

Also for an ordinal $\alpha$, denote $o_L(\alpha) = \gamma_n$ where $\alpha = \sum_{i=1}^{n} \omega^{\gamma_i} \cdot m_i$ is the Cantor normal form and $\gamma_1 > \gamma_2 > \cdots > \gamma_n$.

Corollary 2.20: *Let $G \subseteq \mathbb{M}[\vec{U}]$ be $V$-generic and $C_G$ the corresponding Magidor sequence.*

(1) *If $p \in G$, then for every $1 \leq i \leq l(p)$,*

$$p \Vdash \underset{\sim}{C}_G(\gamma_i(p)) = \kappa_i(p).$$

(2) *For every $\alpha < \mathrm{otp}(C_G)$,*

$$o^{\vec{U}}(C_G(\alpha)) = o_L(\alpha).$$

*Proof.* This is directly from 2.18. ∎

For more details and basic properties of Magidor forcing see [5], [2] or [1].

We are going to handle subsequences of the generic club; the following simple definition will turn out to be useful.

*Definition 2.21:* Let $X, X'$ be sets of ordinals such that $X' \subseteq X \subseteq On$. Let $\alpha = \mathrm{otp}(X, \in)$ be the order type of $X$ and $\phi : \alpha \to X$ be the order isomorphism witnessing it. The indices of $X'$ in $X$ are

$$I(X', X) = \phi^{-1''} X' = \{\beta < \alpha \mid \phi(\beta) \in X'\}.$$

In the last part of the proof we will need the definition of quotient forcing.

*Definition 2.22:* Let $\underset{\sim}{C}'$ be a $\mathbb{M}[\vec{U}]$-name for a subset of $C_G$, and let $C' \subseteq C_G$ such that $\underset{\sim}{C}'_G = C'$. Define $\mathbb{P}_{\underset{\sim}{C}'}$, the complete subalgebra of $\langle RO(\mathbb{M}[\vec{U}]), \leq_B \rangle$[3] generated by the conditions $X = \{||\alpha \in \underset{\sim}{C}'|| \mid \alpha < \kappa\}$.

By [4, 15.42], $V[C'] = V[H]$ for some $V$-generic filter $H$ of $\mathbb{P}_{\underset{\sim}{C}'}$. In fact,

$$C' = \{\alpha < \kappa \mid ||\alpha \in \underset{\sim}{C}'|| \in X \cap H\}.$$

[3] $RO(\mathbb{M}[\vec{U}])$ is the set of all regular open cuts of $\mathbb{M}[\vec{U}]$(see for example [4, Thm. 14.10]), as usual we identify $\mathbb{M}[\vec{U}]$ as a dense subset of $RO(\mathbb{M}[\vec{U}])$. The order $\leq_B$ is in the standard position of Boolean algebras orders i.e., $p \leq_B q$ means $p \Vdash q \in \hat{G}$.

*Definition 2.23:* Define the function $\pi : \mathbb{M}[\vec{U}] \to \mathbb{P}_{\underset{\sim}{C}'}$ by

$$\pi(p) = \inf(b \in \mathbb{P}_{\underset{\sim}{C}'} \mid p \leq_B b).$$

It not hard to check that $\pi$ is a projection, i.e.,

(1) $\pi$ is order preserving,
(2) $\forall p \in \mathbb{M}[\vec{U}].\forall q \leq_B \pi(p).\exists p' \geq p.\pi(p') \leq_B q$,
(3) $\mathrm{Im}(\pi)$ is dense in $\mathbb{P}_{\underset{\sim}{C}'}$.

*Definition 2.24:* Let $\pi : \mathbb{P} \to \mathbb{Q}$ be any projection, let $H \subseteq \mathbb{Q}$ be $V$-generic, and define

$$\mathbb{P}/H = \pi^{-1''} H.$$

We abuse notation by defining $\mathbb{M}[\vec{U}]/C' = \mathbb{M}[\vec{U}]/H$, where $H$ is some generic for $\mathbb{P}_{\underset{\sim}{C}'}$ such that $V[H] = V[C']$. It is known that if $G$ is $V[C']$-generic for $\mathbb{M}[\vec{U}]/C'$, then $G$ is $V$-generic for $\mathbb{M}[\vec{U}]$ and $\pi'' G = H$, hence $V[G] = V[C'][G]$.

## 3. Magidor forcing with $o^{\vec{U}}(\kappa) \leq \kappa$

Assume that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a $V$-generic filter, and let $p \in G$. By Proposition 2.18, $\mathrm{otp}(C_G \cap (\kappa_{l(p)}(p), \kappa)) = \omega^{o^{\vec{U}}(\kappa)}$. Hence,

$$cf^{V[G]}(\kappa) = cf^{V[G]}(\omega^{o^{\vec{U}}(\kappa)}) \tag{3.1}$$

Corollary 3.1: (1) *If $o^{\vec{U}}(\kappa) < \kappa$, then $\kappa$ is singular in $V[G]$.*
(2) *If $o^{\vec{U}}(\kappa) = \kappa$, then $cf^{V[G]}(\kappa) = \omega$.*

*Proof.* (1) follows directly from equation (3.1). For (2), the set

$$E = \{\alpha < \kappa \mid o^{\vec{U}}(\alpha) < \alpha\} \in \cap \vec{U}(\kappa).$$

Hence, by proposition 2.16 find $\rho < \kappa$ such that $C_G \setminus \rho \subseteq E$. In $V[G]$ consider the sequence: $\alpha_0 = \min(C_G \setminus \rho)$, then $\alpha_{n+1} = C_G(\alpha_n)$. This is a well defined sequence of ordinals below $\kappa$ since $\mathrm{otp}(C_G) = \kappa$. Also, since $\{\alpha < \kappa \mid \omega^\alpha = \alpha\} \in \cap \vec{U}(\kappa)$, there is $n < \omega$ such that for every $m \geq n$,

$$o^{\vec{U}}(\alpha_{m+1}) = \alpha_m.$$

To see that $\alpha^* := \sup_{n<\omega} \alpha_n = \kappa$, assume otherwise, then by closure of $C_G$, $\alpha^* \in C_G$. Also $\alpha^* > \rho$, hence $o^{\vec{U}}(\alpha^*) < \alpha^*$. By proposition 2.16(4),

$$o^{\vec{U}}(\alpha^*) \geq \limsup_{n<\omega} o^{\vec{U}}(\alpha_n) + 1 = \sup_{n<\omega} \alpha_n = \alpha^*,$$

a contradiction. ∎

If $o^{\vec{U}}(\kappa) \leq \kappa$ we can decompose every set $A \in \cap\vec{U}(\kappa)$ in a very canonical way:

PROPOSITION 3.2: *Assume that $o^{\vec{U}}(\kappa) \leq \kappa$. Let $A \in \cap\vec{U}(\kappa)$.*

(1) *For every $i < \kappa$ define $A_i = \{\nu \in A \mid o^{\vec{U}}(\nu) = i\}$. Then $A = \biguplus_{i<\kappa} A_i$ and $A_i \in U(\kappa, i)$.*
(2) *There exists $A^* \subseteq A$ such that:*
   (a) *$A^* \in \cap\vec{U}(\kappa)$.*
   (b) *For every $0 < j < o^{\vec{U}}(\kappa)$ and $\alpha \in A^*_j$, $A^* \cap \alpha \in \cap\vec{U}(\alpha)$.*

*Proof.* (1) Note that

$$X_i := \{\nu < \kappa \mid o^{\vec{U}}(\nu) = i\} \in U(\kappa, i)$$

and

$$A_i = X_i \cap A \in U(\kappa, i).$$

Moreover, every $\alpha < \kappa$, $o^{\vec{U}}(\alpha) < \kappa$, since there are at most $2^{2^\alpha} < \kappa$ measures over $\alpha$.

(2) For any $i < o^{\vec{U}}(\kappa)$,

$$\text{Ult}(V, U(\kappa, j)) \models A = j_{U(\kappa,j)}(A) \cap \kappa \in \bigcap_{i<j} U(\kappa, i).$$

Coherency of the sequence implies that

$$A' := \{\alpha < \kappa \mid A \cap \alpha \in \cap\vec{U}(\alpha)\} \in U(\kappa, j);$$

this is for every $j < o^{\vec{U}}(\kappa)$.

Define inductively $A^{(0)} = A$, $A^{(n+1)} = A'^{(n)}$. By definition, $\forall\alpha \in A^{(n+1)}_j$, $A^{(n)} \cap \alpha \in \cap\vec{U}(\alpha)$. Define $A^* = \bigcap_{n<\omega} A^{(n)} \in \cap\vec{U}(\kappa)$; this set has the required property. ∎

3.1. EXTENSION TYPES. By convention, for a set of ordinals $B$, $[B]^{<\alpha}$ is the set of increasing sequences of length less than $\alpha$ of ordinals in $B$, $[B]^{[<\alpha]}$ is the set of not necessarily increasing sequences of length less than $\alpha$ of ordinals in $B$. For sets of ordinals $B_i$ for $1 \leq i \leq n$, let $\prod_{i=1}^n B_i$ be the set of increasing sequence $\langle\alpha_1, \ldots, \alpha_n\rangle$ such that $\alpha_i \in B_i$. For double indexed sets $B_{i,j}$ for $1 \leq i \leq n$, $1 \leq j \leq m$, the set $\prod_{i=1}^n \prod_{j=1}^n B_{i,j}$ is viewed as a product of single indexed sets using the left lexicographical order.

*Definition 3.3:* Let $p \in \mathbb{M}[\vec{U}]$. Define the following:

(1) For every $i \leq l(p)+1$, let
$$B_{i,\alpha}(p) = B_i(p) \cap X_\alpha,$$
where $X_\alpha := \{\beta < \kappa \mid o^{\vec{U}}(\beta) = \alpha\}$ are the sets defined in Proposition 3.2.

(2) $\mathrm{Ex}(p) = \prod_{i=1}^{l(p)+1} [o^{\vec{U}}(\kappa_i(p))]^{[<\omega]}$.

(3) If $X \in \mathrm{Ex}(p)$, then $X$ is of the form $\langle X_1, \ldots, X_{n+1}\rangle$. Denote $x_{i,j}$, the $j$-th element of $X_i$, for $1 \leq j \leq |X_i|$ and $mc(X)$ is the last element of $X$ and $l(X) = \sum_{i=1}^{n+1} |X_i|$.

(4) Let $X \in \mathrm{Ex}(p)$; then
$$\vec{\alpha} = \langle \vec{\alpha_1}, \ldots, \vec{\alpha_{l(p)+1}}\rangle \in \prod_{i=1}^{l(p)+1} \prod_{j=1}^{|X_i|} B_{i,x_{i,j}}(p) =: X(p).$$
Call $X$ an **extension-type** of $p$ and $\vec{\alpha}$ is of **type** $X$; note that $\vec{\alpha}$ is an increasing sequence of ordinals.

The idea of extension-types is simply to classify extensions of $p$ according to the measures from which the ordinals added to the stem of $p$ are chosen. Note that if $o^{\vec{U}}(\kappa) = \lambda < \kappa$, then there is a bound on the number of extension-types,
$$|\mathrm{Ex}(p)| < \min\{\nu > \lambda \mid o^{\vec{U}}(\nu) > 0\}.$$

By Proposition 3.2 any $p \in \mathbb{M}[\vec{U}]$ can be extended to $p \leq^* p^*$ such that for every $X \in \mathrm{Ex}(p)$ and any $\vec{\alpha} \in X(p)$, $p^\frown\vec{\alpha} \in \mathbb{M}[\vec{U}]$. Let us move to this dense subset of $\mathbb{M}[\vec{U}]$.

Proposition 3.4: *Let $p \in \mathbb{M}[\vec{U}]$ be any condition and $p \leq q \in \mathbb{M}[\vec{U}]$. Then there exists unique $X \in \mathrm{Ex}(p)$ and $\vec{\alpha} \in X(p)$ such that $p^\frown\vec{\alpha} \leq^* q$. Moreover, for every $X \in \mathrm{Ex}(p)$ the set $\{p^\frown\vec{\alpha} \mid \vec{\alpha} \in X(p)\}$ forms a maximal antichain above $p$.*

*Proof.* The first part is trivial. We will prove that $\{p^\frown\vec{\alpha} \mid \vec{\alpha} \in X(p)\}$ forms an antichain above $p$, by induction on $l(X)$. For $l(X) = 1$, we merely have some $X(p) = B_{i,\xi}(p) \in U(\kappa_i(p), \xi)$. To see it is an antichain, let $\beta_1 < \beta_2$ be in $X(p)$. Toward a contradiction, assume that $p^\frown\beta_1, p^\frown\beta_2 \leq q$. Then $\beta_1$ appears in a pair in $q$ and is added between $\kappa_{i-1}(p)$ and $\beta_2$, so by Definition 2.2 it must be that $\xi = o^{\vec{U}}(\beta_1) < o^{\vec{U}}(\beta_2) = \xi$, a contradiction.

To see it is maximal, fix $q \geq p$ and let $\vec{\alpha}$ be such that $p^\frown\vec{\alpha} \leq^* q$. Consider the type of $\vec{\alpha}$,

$$Y \in \mathrm{Ex}(p);$$

then $\vec{\alpha} \in Y(p)$. In $Y_i$ let $j$ be the minimal such that $y_{i,j} \geq \xi$. If $y_{i,j} = \xi$ then $q \geq p^\frown\langle\alpha_{i,j}\rangle \in X(p)$ and we are done. Otherwise, $y_{i,j} > \xi$, in which case one of the pairs in $q$ is of the form $\langle\alpha_{i,j}, B\rangle$ where $B \in \cap\vec{U}(\alpha_{i,j})$ and $B \subseteq B_i(p)$. Any $\alpha \in B \cap B_{i,\xi}(p)$ will satisfy that $p^\frown\langle\alpha\rangle \in X(p)$ and $p^\frown\langle\alpha\rangle, q \leq q^\frown\langle\alpha\rangle$.

Assume that the claim holds for $l(X) = n$, and let $X \in \mathrm{Ex}(p)$ be such that $l(X) = n+1$. Let $\vec{\alpha}, \vec{\beta} \in X(p)$ be distinct. If for some $x_{i,j} \neq mc(X)$ we have $\alpha_{i,j} \neq \beta_{i,j}$, apply the induction to $X \setminus mc(X)$ to see that $p^\frown\vec{\alpha}\setminus\alpha^*, p^\frown\vec{\beta}\setminus\beta^*$ are incompatible, hence $p^\frown\vec{\alpha}, p^\frown\vec{\beta}$ are incompatible. If $\vec{\alpha}\setminus\alpha^* = \vec{\beta}\setminus\beta^*$, then $\alpha^* \neq \beta^*$ and by the case $n = 1$ we are done. To see it is maximal, let $q \geq p$ apply the induction to $X'$ which is the extension-type obtained from $X$ by removing $mc(X)$ to find $\vec{\alpha} \in X'(p)$ such that $p^\frown\vec{\alpha}$ is compatible with $q$ and let $q'$ be a common extension. Again by the case $n = 1$, there is $\langle\alpha\rangle \in mc(X)(p^\frown\vec{\alpha})$ such that $p^\frown\vec{\alpha}^\frown\langle\alpha\rangle$ and $q'$ are compatible. ∎

*Definition 3.5:* Let $U_1, \ldots, U_n$ be ultrafilters on $\kappa_1 \leq \cdots \leq \kappa_n$ respectively, and define recursively the ultrafilter $\prod_{i=1}^n U_i$ over $\prod_{i=1}^n \kappa_i$, as follows: for $B \subseteq \prod_{i=1}^n \kappa_i$

$$B \in \prod_{i=1}^n U_i \leftrightarrow \left\{\alpha_1 < \kappa_1 \mid B_{\alpha_1} \in \prod_{i=2}^n U_i\right\} \in U_1$$

where $B_\alpha = B \cap (\{\alpha\} \times \prod_{i=2}^n \kappa_i)$.

Proposition 3.6: *If $U_1, \ldots, U_n$ are normal ultrafilters, then $\prod_{i=1}^n U_i$ is generated by sets of the form $A_1 \times \cdots \times A_n$ such that $A_i \in U_i$.*

*Proof.* By induction of $n$, for $n = 1$ there is nothing to prove. Assume that the proposition holds for $n-1$, and let $B \in \prod_{i=1}^n U_i$. By definition, $A_1 = \{\alpha_1 < \kappa_1 \mid B_{\alpha_1} \in \prod_{i=2}^n U_i\} \in U_1$, and by the induction hypothesis each $B_{\alpha_1}$ contains a set of the form $A_{2,\alpha_1} \times \cdots \times A_{n,\alpha_1}$. By normality, $A_i := \Delta_{\alpha\in A_1} A_{i,\alpha} \in U_i$. Consider $\langle\alpha_1, \ldots, \alpha_n\rangle \in A_1 \times \cdots \times A_n$, by convention, for each $2 \leq i \leq n$, $\alpha_1 \leq \alpha_i$, and by definition of diagonal intersection, $\alpha_i \in A_{i,\alpha_1}$, hence $\langle\alpha_2, \ldots, \alpha_n\rangle \in A_{2,\alpha_1} \times \cdots \times A_{n,\alpha_1} \subseteq B_{\alpha_1}$. It follows by the definition of $B_{\alpha_1}$ that $\langle\alpha_1, \ldots, \alpha_n\rangle \in B$, hence $A_1 \times \cdots \times A_n \subseteq B$. ∎

Every $X \in \mathrm{Ex}(p)$ defines an ultrafilter

$$\vec{U}(X,p) = \prod_{i=1}^{n+1} \prod_{j=1}^{|X_i|} U(\kappa_i(p), x_{i,j}).$$

Note that $X(p) \in \vec{U}(X,p)$ by the definition of the product. Fix an extension-type $X$ of $p$; every extension of $p$ of type $X$ corresponds to some element in the set $X(p)$ which is just a product of large sets.

Let us state here some combinatorial properties; the proof can be found in [1].

Lemma 3.7: *Let $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ be a non-descending finite sequence of measurable cardinals and let $U_1, \ldots, U_n$ be normal measures*[4] *over them respectively. Assume $F : \prod_{i=1}^n A_i \longrightarrow \nu$ where $\nu < \kappa_1$ and $A_i \in U_i$. Then there exists $H_i \subseteq A_i$, $H_i \in U_i$ such that $\prod_{i=1}^n H_i$ is homogeneous for $F$, i.e., $|\operatorname{Im}(F \restriction \prod_{i=1}^n H_i)| = 1$.* ∎

Let $F : \prod_{i=1}^n A_i \to X$ be a function, and $I \subseteq \{1, \ldots, n\}$. Let

$$\left(\prod_{i=1}^n A_i\right)_I = \left\{\vec{\alpha} \restriction I \mid \vec{\alpha} \in \prod_{i=1}^n A_i\right\}.$$

For $\vec{\alpha}' \in (\prod_{i=1}^n A_i)_I$, define $F_I(\vec{\alpha}') = F(\vec{\alpha})$ where $\vec{\alpha} \restriction I = \vec{\alpha}'$. With no further assumption, $F_I$ is not a well defined function.

Lemma 3.8: *Let $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ be a non descending finite sequence of measurable cardinals and let $U_1, \ldots, U_n$ be normal measures over them, respectively. Assume $F : \prod_{i=1}^n A_i \longrightarrow B$ where $B$ is any set, and $A_i \in U_i$. Then there exist $H_i \subseteq A_i$, $H_i \in U_i$ and set a $I \subseteq \{1, \ldots, n\}$ such that $F_I \restriction (\prod_{i=1}^n H_i)_I : (\prod_{i=1}^n H_i)_I \to B$ is well defined and injective.*

*Definition 3.9:* Let $F : \prod_{i=1}^n A_i \to X$ be a function. An **important coordinate** is an index $r \in \{1, \ldots, n\}$, such that for every $\vec{\alpha}, \vec{\beta} \in \prod_{i=1}^n A_i$,

$$F(\vec{\alpha}) = F(\vec{\beta}) \to \vec{\alpha}(r) = \vec{\beta}(r).$$

Lemma 3.8 ensures the existence of a set $I$ of important coordinates, such that $I$ is ideal in the sense of removing any coordinate defect definition of $F_I$ as a function, and any coordinate outside of $I$ is redundant.

We will need here another property that does not appear in [1].

[4] A measure over a measurable cardinal $\lambda$ is a $\lambda$-complete nonprincipal ultrafilter over $\lambda$.

Lemma 3.10: *Let $\kappa_1 \leq \kappa_2 \leq \cdots \leq \kappa_n$ and $\theta_1 \leq \theta_2 \leq \cdots \leq \theta_m$ be non-descending finite sequences of measurable cardinals with corresponding normal measures $U_1, \ldots, U_n, W_1, \ldots, W_m$. Let*

$$F : \prod_{i=1}^{n} A_i \to X, \quad G : \prod_{j=1}^{m} B_j \to X$$

*be functions such that $X$ is any set, $A_i \in U_i$ and $B_j \in W_j$. Assume that $I \subseteq \{1, \ldots, n\}$ and $J \subseteq \{1, \ldots, m\}$ are sets of important coordinates for $F, G$ respectively obtained by lemma 3.8. Then there exist $A'_i \in U_i$ and $B'_j \in W_j$ such that one of the following holds:*

(1) $\mathrm{Im}(F \restriction \prod_{i=1}^{n} A'_i) \cap \mathrm{Im}(G \restriction \prod_{j=1}^{m} B'_j) = \emptyset$.
(2) $(\prod_{i=1}^{n} A'_j)_I = (\prod_{j=1}^{m} B'_j)_J$ *and* $F_I \restriction (\prod_{i=1}^{n} A'_i)_I = G_J \restriction (\prod_{j=1}^{m} B'_j)_J$.

*Proof.* Fix $F, G$. Let us first deal with some trivial cases: If $I = J = \emptyset$, i.e., $F, G$ are constantly $d_F, d_G$, respectively, either $d_1 \neq d_2$ and (1) holds, or $d_1 = d_2$ and (2) holds. If $I = \emptyset$ and $j_0 \in J \neq \emptyset$, then $F$ is constantly $d_F$. If $d_F \notin \mathrm{Im}(G)$ then (1) holds, otherwise, there is $\vec{\beta}$ such that $G(\vec{\beta}) = d_F$; remove $\vec{\beta}_{j_0}$ from $B_{j_0}$, then . If $\vec{\beta}' \in B_1 \times \cdots \times B_{j_0} \setminus \{\vec{\beta}_{j_0}\} \times \cdots \times B_m$, then $G(\vec{\beta}') \neq d_F$, otherwise, $\vec{\beta}' \restriction J = \vec{\beta} \restriction J$ and in particular $\vec{\beta}_{j_0} = \vec{\beta}'_{j_0}$, a contradiction. Similarly, if $J = \emptyset$ and $I \neq \emptyset$ then we can ensure (1). We assume that $I, J \neq \emptyset$; also, without loss of generality, assume that $\kappa_1 \leq \theta_1$. If $\kappa_1 < \theta_1$, shrink the sets so that $\min(B_1) > \kappa_1$. We proceed by induction on $\langle n, m \rangle \in \mathbb{N}^2_+$ with respect to the lexicographical order.

Case 1: Assume that $n = m = 1$. Assume that $I, J \neq \emptyset$. Define

$$H_1 : A_1 \times B_1 \to \{0, 1\}, \quad H_1(\alpha, \beta) = 1 \Leftrightarrow F(\alpha) = G(\beta).$$

By Lemma 3.7, shrink $A_1, B_1$ to $A'_1, B'_1$ so that $H_1$ is constant with colors $c_1$. If $c_1 = 1$, by fixing $\alpha$ we see that $G$ is constant on $B'_1$ with some value $\gamma$. It follows that $J = \emptyset$, a contradiction. Assume that $c_1 = 0$; then for every $\alpha \in A_1, \beta \in B_1$ if $\alpha < \beta$ we have $H_1(\alpha, \beta) = 0$, which implies $F(\alpha) \neq G(\beta)$. This suffices for the case $\kappa_1 < \theta_1$. If $\kappa_1 = \theta_1$, then it is possible that $\beta < \alpha$, so define

$$H_2 : B_1 \times A_1 \to \{0, 1\} \quad H_2(\beta, \alpha) = 1 \Leftrightarrow F(\alpha) = G(\beta).$$

Again shrink the sets so that $H_2$ is constantly $c_2 \in \{0, 1\}$. In case $c_2 = 1$ we reach a similar contradiction to $c_1 = 1$. Assume that $c_2 = 0$, together with $c_1 = 0$; it follows that if $\beta \neq \alpha$ then $F(\alpha) \neq G(\beta)$. If $U_1 \neq W_1$, then we can avoid the situation where $\alpha = \beta$ by separating $A'_1, B'_1$ and conclude that

$$\operatorname{Im}(F \restriction A'_1) \cap \operatorname{Im}(G \restriction B'_1) = \emptyset.$$

If $U_1 = W_1$ then define

$$H_3 : A'_1 \cap B'_1 \to \{0,1\}, \quad H_3(\alpha) = 1 \Leftrightarrow F(\alpha) = G(\alpha).$$

Again by 3.7 we can assume that $H_3$ is constant on $A^*$. If that constant is 1 then we have

$$F \restriction A^* = G \restriction A^*$$

(in particular, $I = J = \{1\}$ and $F_I \restriction (A^*)_I = G_J \restriction (A^*)_J$), otherwise

$$\operatorname{Im}(F \restriction A^*) \cap \operatorname{Im}(G \restriction A^*) = \emptyset.$$

Case 2a: Assume $n = 1$ and $m > 1$. By the assumption that $I, J \neq \emptyset$, $I = \{1\}$. Define

$$H_1 : A_1 \times \prod_{j=1}^{m} B_j \to \{0,1\}, \quad H_1(\alpha, \vec{\beta}) = 1 \Leftrightarrow F(\alpha) = G(\vec{\beta}).$$

Shrink the sets so that $H_1$ is constantly $c_1$. As before, if $c_1 = 1$ then $F, G$ are constant which is a contradiction. Assume that $c_1 = 0$, which means that whenever $\alpha < \beta_1$, then $F(\alpha) \neq G(\vec{\beta})$. As before, if $\kappa_1 < \theta_1$ then we are done. If $\kappa_1 = \theta_1$, for each $\beta \in B_1$, consider the function

$$G_\beta : \prod_{j=2}^{m} B_j \setminus (\beta + 1) \to X, \ G_\beta(\vec{\beta}) = G(\beta ^\frown \vec{\beta}).$$

Apply induction to $F$ and $G_\beta$, $\{1\}, J \setminus \{1\}$ to find

$$A_1^\beta \in U_1, \quad B_j^\beta \in W_j \quad \text{for } 2 \leq j \leq m$$

such that one of the following holds:

(1) $A_1^\beta = (\prod_{j=1}^{m} B_j^\beta)_{J\setminus\{1\}}$, and $F \restriction A_1^\beta = (G_\beta)_{J\setminus\{1\}} \restriction (\prod_{j=2}^{m} B_j^\beta)_{J\setminus\{1\}}$.

(2) $\operatorname{Im}(F \restriction A_1^\beta) \cap \operatorname{Im}(G_\beta \restriction \prod_{j=2}^{m} B_j^\beta) = \emptyset$.

Denote by $j_\beta \in \{1,2\}$ the relevant case. There is $B'_1 \subseteq B_1$, $B'_1 \in W_1$, and $j^* \in \{1,2\}$ such that for every $\beta \in B'_1$, $j_\beta = j^*$. Let

$$A'_1 = \Delta_{\beta \in B'_1} A_1^\beta, \quad B'_j = \Delta_{\beta \in B'_1} B_j^\beta$$

(since $\theta_1 = \kappa_1$ we can take the diagonal intersection).

If $j^*=1$, then since $A_1^\beta=(\prod_{j=1}^m B_j^\beta)_{J\setminus\{1\}}$, it follows that $J=\{j_0\}$ and $A_1^\beta=B_{j_0}^\beta$, thus $A_1'=B_{j_0}'$. Also for $\beta_1,\beta_1'$, and some $\beta_1,\beta_1'<\beta_2,\dots,\beta_m$ in the product,

$$\begin{aligned}G(\langle\beta_1,\dots,\beta_m\rangle)&=(mbnG_{\beta_1})_{j_0}(\beta_{j_0})\\&=F(\beta_{j_0})=(G_{\beta_1'})_{j_0}(\beta_{j_0})\\&=G(\langle\beta_1',\dots,\beta_n\rangle).\end{aligned}$$

Hence $1\notin J$, $A_1'=B_{j_0}'=(\prod_{j=1}^m B_j')_J$ and $F_1\restriction A_1'=G_{j_0}\restriction B_{j_0}'$.

If $j^*=2$, for every $\langle\beta_1,\dots,\beta_m\rangle\in\prod_{j=1}^m B_j'$,

$$G(\langle\beta_1,\dots,\beta_m\rangle)\in\operatorname{Im}\left(G_{\beta_1}\restriction\prod_{j=1}^m B_j^\beta\right).$$

Now if $\beta_1<\alpha\in A_1'$ then by definition of diagonal intersection $\alpha\in A_1^{\beta_1}$ and therefore $F(\alpha)\in\operatorname{Im}(F\restriction A_1^{\beta_1})$ and we are done. Together with the assumption that $c_1=0$, we conclude that if $\alpha\neq\beta_1$ then $F(\alpha)\neq G(\vec{\beta})$. As before, we can avoid this situation if $U_1\neq W_1$. Assume that $U_1=W_1$, and assume that $A_1'=B_1'$. Let

$$T_1: A_1'\times\prod_{j=2}^m B_j'\to\{0,1\},\quad T_1(\alpha,\vec{\beta})=1\Leftrightarrow F(\alpha)=G(\alpha,\vec{\beta}).$$

We shrink $A_1'$ and $B_j'$ so that $T_1$ is constantly $d_1$. If $d_1=0$ then we have eliminated the possibility of $\alpha=\beta$, and again we conclude that

$$\operatorname{Im}\left(F\restriction\prod_{i=1}^n A_i'\right)\cap\operatorname{Im}\left(G\restriction\prod_{j=1}^m B_j'\right)=\emptyset.$$

If $d_1=1$ then $G$ only depends on $B_1'$, i.e., $J=\{1\}$, hence

$$\left(\prod_{j=1}^m B_j'\right)_{\{1\}}=A_1'\quad\text{and}\quad F\restriction A_1'=G_{\{1\}}\restriction A_1'.$$

Case 2b: Assume $n>1$ and $m=1$. Then by the assumption that $I,J\neq\emptyset$ it follows that $J=\{1\}$. For $\alpha\in A_1$ define the functions

$$F_\alpha:\prod_{i=2}^n A_i\setminus(\alpha+1)\to X,\quad F_\alpha(\vec{\alpha})=F(\alpha,\vec{\alpha}).$$

By the induction hypothesis applied to $F_\alpha, G$ and $I\setminus\{1\},\{1\}$, we obtain

$$A_i^\alpha\in U_i\quad\text{for }2\le i\le n,\quad B_j^\alpha\in W_j\quad\text{for }1\le j\le m$$

such that one of the following holds:

(1) $(\prod_{i=2}^n A_i^\alpha)_{I\setminus\{1\}} = B_1^\alpha$ and $(F_\alpha)_{I\setminus\{1\}} \restriction (\prod_{i=2}^n A_i^\alpha)_{I\setminus\{1\}} = G \restriction B_1^\alpha$.

(2) $\mathrm{Im}(F_\alpha \restriction \prod_{i=2}^n A_i^\alpha) \cap \mathrm{Im}(G \restriction B_1^\alpha) = \emptyset$.

Denote by $i_\alpha \in \{1,2\}$ the relevant case. There is $A_1' \subseteq A_1$, $A_1' \in U_1$, and $i^* \in \{1,2\}$ such that for every $\alpha \in A_1'$, $i_\alpha = i^*$. Let

$$A_i' = \Delta_{\alpha\in A_1} A_i^\alpha, \quad B_1' = \Delta_{\alpha\in A_1} B_1^\alpha$$
$$\text{(since } \theta_1 \geq \kappa_1 \text{ we can take the diagonal intersection).}$$

If $i^* = 1$, then $(\prod_{i=2}^n A_i^\alpha)_{I\setminus\{1\}} = B_1^\alpha$, hence $I = \{i_0\}$. Note that $A_{i_0}^\alpha = B_1^\beta$ and in turn it follows that $A_{i_0}' = B_1' \in U_{i_0} \cap W_1$.

Let $\alpha, \alpha' \in A_1'$, and $\alpha_1, \alpha_1' < \alpha_2 < \cdots < \alpha_n$ in the product. Then

$$F(\langle \alpha_1 \cdots \alpha_n\rangle) = (F_{\alpha_1})_{\{i_0\}}(\alpha_{i_0}) = G(\alpha_{i_0}) = (F_{\alpha_1'})_{\{i_0\}}(\alpha_{i_0}) = F(\langle \alpha_1' \cdots \alpha_n\rangle).$$

From this it follows that $1 \notin I$, $B_1' = A_{i_0}' = (\prod_{i=1}^n A_i')_I$ and $F_I \restriction A_{i_0}' = G \restriction B_1'$. Assume $i^* = 2$, which means that for every $\langle \alpha_1, \ldots, \alpha_n\rangle \in \prod_{i=1}^n A_1'$, by definition of diagonal intersection, $\langle \alpha_2, \ldots, \alpha_n\rangle \in \prod_{i=2}^n A_i^{\alpha_1}$ hence

$$F(\langle \alpha_1, \ldots, \alpha_n\rangle) = F_{\alpha_1}(\langle \alpha_2, \ldots, \alpha_n\rangle) \in \mathrm{Im}\left(F_{\alpha_1} \restriction \prod_{i=2}^n A_i^{\alpha_1}\right).$$

If $\beta \in B_1'$, we cannot conclude automatically that $\beta \in B_1^{\alpha_1}$, since it is possible that $\beta_1 \leq \alpha_1$. If $\kappa_1 < \theta_1$, then $\beta_1 \leq \alpha_1$ is impossible, thus, $\beta \in B_1^{\alpha_1}$ and $G(\beta_1) \in \mathrm{Im}(G \restriction B_1^{\alpha_1})$. Since $i_{\alpha_1} = i^* = 2$, it follows that

$$F(\langle \alpha_1, \ldots, \alpha_n\rangle) \neq G(\beta_1)$$

which implies

$$\mathrm{Im}\left(F \restriction \prod_{i=1}^n A_i'\right) \cap \mathrm{Im}(G \restriction B_1') = \emptyset.$$

If $\theta_1 = \kappa_1$, then we define

$$H_2 : B_1 \times \prod_{i=1}^n A_i \to \{0,1\}, \quad H_2(\beta, \vec{\alpha}) = 1 \Leftrightarrow F(\vec{\alpha}) = G(\beta).$$

Shrink the sets so that $H_2$ is constantly $c_1$. As before, if $c_1 = 1$ then $F, G$ are constant which is a contradiction. Assume that $c_1 = 0$, which means that whenever $\beta < \alpha_1$, then $F(\vec{\alpha}) \neq G(\beta)$. So we are left with the case $\alpha_1 = \beta$. If $U_1 \neq W_1$

then we can eliminate such an example, and if $U_1 = W_1$ consider $A_1^* = A_1' \cap B_1'$:

$$T_2 : A_1^* \times \prod_{i=2}^{n} A_i' \to \{0,1\}, \quad T_2(\alpha, \vec{\alpha}) = 1 \Leftrightarrow G(\alpha) = F(\alpha, \vec{\alpha}).$$

We shrink $A_1^*$ and $A_i'$ so that $T_2$ is constantly $d_1$. If $d_1 = 0$ then we have eliminated the possibility of $\alpha = \beta$, and again we conclude that

$$\mathrm{Im}\left(F \restriction \prod_{i=1}^{n} A_i'\right) \cap \mathrm{Im}(G \restriction A_1^*) = \emptyset.$$

If $d_1 = 1$ then $F$ only depends on $A_1^*$, i.e., $I = \{1\}$, hence

$$\left(A_1^* \times \prod_{i=2}^{n} A_i'\right)_{\{1\}} = A_1^* \quad \text{and} \quad G \restriction A_1^* = G_{\{1\}} \restriction A_1^*.$$

Case 3: Assume $n, m > 1$. For $\alpha \in A_1$ define the functions

$$F_\alpha : \prod_{i=2}^{n} A_i \setminus (\alpha + 1) \to X, \quad F_\alpha(\vec{\alpha}) = F(\alpha, \vec{\alpha}).$$

By the induction hypothesis applied to $F_\alpha, G$ and $I \setminus \{1\}, J$, we obtain

$$A_i^\alpha \in U_i \quad \text{for } 2 \le i \le n, \quad B_j^\alpha \in W_j \quad \text{for } 1 \le j \le m$$

such that one of the following holds:

(1) $(\prod_{i=2}^{n} A_i^\alpha)_{I\setminus\{1\}} = (\prod_{j=1}^{m} B_j^\alpha)_J$, and

$$(F_\alpha)_{I\setminus\{1\}} \restriction \left(\prod_{i=2}^{n} A_i^\alpha\right)_{I\setminus\{1\}} = G_J \restriction \left(\prod_{j=1}^{m} B_j^\alpha\right)_J.$$

(2) $\mathrm{Im}(F_\alpha \restriction \prod_{i=2}^{n} A_i^\alpha) \cap \mathrm{Im}(G \restriction \prod_{j=1}^{m} B_j^\alpha) = \emptyset$.

Denote by $i_\alpha \in \{1,2\}$ the relevant case. There is $A_1' \subseteq A_1$, $A_1' \in U_1$, and $i^* \in \{1,2\}$ such that for every $\alpha \in A_1'$, $i_\alpha = i^*$. Let

$$A_i' = \Delta_{\alpha \in A_1} A_i^\alpha, \quad B_j' = \Delta_{\alpha \in A_1} B_j^\alpha$$

(Since $\theta_1 \ge \kappa_1$ we can take the diagonal intersection).

If $i^* = 1$, then

$$\left(\prod_{i=2}^{n} A_i^\alpha\right)_{I\setminus\{1\}} = \left(\prod_{j=1}^{m} B_j^\alpha\right)_J.$$

Denote $I \setminus \{1\} = \{i_1, \ldots, i_k\}$, $J = \{j_1, \ldots, j_k\}$. Note that for every $1 \leq r \leq k$, $A^{\alpha}_{i_r} = B^{\beta}_{j_r}$, thus $A'_{i_r} = B'_{j_r} \in U_{i_r} \cap W_{j_r}$. It follows that

$$\left(\prod_{i=1}^{n} A'_i\right)_{I\setminus\{1\}} = \left(\prod_{j=1}^{m} B'_j\right)_J.$$

Let $\alpha, \alpha' \in A'_1$, $\vec{\alpha} \in \prod_{i=2}^{n} A'_i$ with $\min(\vec{\alpha}) > \alpha, \alpha'$. Then

$$F_\alpha(\vec{\alpha}) = (F_\alpha)_{I\setminus\{1\}}(\vec{\alpha} \restriction I) = G_J(\vec{\alpha} \restriction I) = (F_{\alpha'})_{I\setminus\{1\}}(\vec{\alpha} \restriction I) = F_{\alpha'}(\vec{\alpha}).$$

From this it follows that $1 \notin I$ and $F_I = F_{I\setminus\{1\}} = G_J$. Assume $i^* = 2$, which means that for every $\langle \alpha_1, \ldots, \alpha_n \rangle \in \prod_{i=1}^{n} A'_1$, by definition of diagonal intersection, $\langle \alpha_2, \ldots, \alpha_n \rangle \in \prod_{i=2}^{n} A^{\alpha_1}_i$, hence

$$F(\langle \alpha_1, \ldots, \alpha_n \rangle) = F_{\alpha_1}(\langle \alpha_2, \ldots, \alpha_n \rangle) \in \mathrm{Im}\left(F_{\alpha_1} \restriction \prod_{i=2}^{n} A^{\alpha_1}_i\right).$$

If $\vec{\beta} \in \prod_{j=1}^{m} B'_j$, we cannot conclude automatically that $\vec{\beta} \in \prod_{j=1}^{m} B^{\alpha_1}_j$, since it is possible that $\beta_1 \leq \alpha_1$. If $\kappa_1 < \theta_1$, then $\beta_1 \leq \alpha_1$ is impossible, thus,

$$\vec{\beta} \in \prod_{j=1}^{m} B^{\alpha_1}_j \quad \text{and} \quad G(\langle \beta_1, \ldots, \beta_n \rangle) \in \mathrm{Im}(G \restriction \prod_{j=1}^{n} B^{\alpha_1}_j).$$

Since $i_{\alpha_1} = i^* = 2$, it follows that $F(\langle \alpha_1, \ldots, \alpha_n \rangle) \neq G(\langle \beta_1, \ldots, \beta_n \rangle)$, which implies

$$\mathrm{Im}\left(F \restriction \prod_{i=1}^{n} A'_i\right) \cap \mathrm{Im}\left(G \restriction \prod_{j=1}^{n} B'_j\right) = \emptyset.$$

If $\theta_1 = \kappa_1$, we repeat the same process. We use $G_\beta$ and fix $F$, denoting $j_\beta$ the relevant case, and shrink the sets so that $j^*$ is constant. In case $j^* = 1$ the proof is the same as $i^* = 1$. So we assume that $i^* = j^* = 2$, meaning that for every $\langle \alpha \rangle^\frown \vec{\alpha} \in \prod_{i=1}^{n} A'_i$ and every $\langle \beta \rangle^\frown \vec{\beta} \in \prod_{j=1}^{m} B'_j$

$$\alpha \neq \beta \to F(\alpha, \vec{\alpha}) \neq G(\beta, \vec{\beta}).$$

We are left with the case $\alpha = \beta$.

Case 3a: Assume that $U_1 \neq W_1$. Then we can just shrink the sets $A'_1, B'_1$ so that $A'_1 \cap B'_1 = \emptyset$. Together with the construction of case 3, conclude that

$$\mathrm{Im}\left(F \restriction \prod_{i=1}^{n} A'_i\right) \cap \mathrm{Im}\left(G \restriction \prod_{j=1}^{m} B'_j\right) = \emptyset.$$

Case 3b: Assume that $U_1 = W_1$. Then we shrink the sets so that $A'_1 = B'_1$. For every $\alpha \in A'_1$ we apply the induction hypothesis to the functions $F_\alpha, G_\alpha$, this time denoting the cases by $r^*$. If $r^* = 2$, then we have eliminated the possibility of $F(\alpha, \vec{\alpha}) = G(\alpha, \vec{\beta})$; together with $i^* = 2, j^* = 2$ we are done. Finally, assume $r^* = 1$, namely that for

$$I^* := I \setminus \{1\} \subseteq \{2, \dots, n\}, \quad J^* := J \setminus \{1\} \subseteq \{2, \dots, m\}$$

we have

$$\left(\prod_{i=2}^n A'_i\right)_{I^*} = \left(\prod_{j=2}^m B'_j\right)_{J^*} \quad \text{and} \quad (F_\alpha)_{I^*} \restriction \left(\prod_{i=2}^n A'_i\right)_{I^*} = (G_\alpha)_{J^*} \restriction \left(\prod_{j=2}^m B'_j\right)_{J^*}.$$

Since $A'_1 = B'_1$ it follows that

$$\left(\prod_{i=1}^n A'_i\right)_{I^* \cup \{1\}} = \left(\prod_{j=1}^m B'_j\right)_{\in J^* \cup \{1\}}$$

$(*)$ and

$$(F_\alpha)_{I^* \cup \{1\}} \restriction \left(\prod_{i=2}^n A'_i\right)_{I^*} = (G_\alpha)_{J^*} \restriction \left(\prod_{j=2}^m B'_j\right)_{J^* \cup \{1\}}$$

Since if $\langle \alpha \rangle^\frown \vec{\alpha} \in (\prod_{i=1}^n A'_i)_I$,

$$F_{I^* \cup \{1\}}(\alpha, \vec{\alpha}) = (F_\alpha)_{I^*}(\vec{\alpha}) = (G_\alpha)_{J^*}(\vec{\alpha}) = G_{J^* \cup \{1\}}(\alpha, \vec{\alpha}),$$

we claim that $1 \in I$ if and only if $1 \in J$. By symmetry, it suffices to prove one implication. For example, if $1 \in I$, then $I = I^* \cup \{1\}$, take $\vec{\alpha} \restriction I$,

$$\vec{\alpha}' \restriction I \in \left(\prod_{i=1}^n A'_i\right)_I$$

which differs only at the first coordinate, therefore $F(\vec{\alpha}) \neq F(\vec{\alpha}')$. By $(*)$, there are $\vec{\beta}, \vec{\beta}' \in \prod_{i=1}^m B'_i$ such that

$$\vec{\beta} \restriction (J^* \cup \{1\}) = \vec{\alpha} \restriction I \quad \text{and} \quad \vec{\beta}' \restriction (J^* \cup \{1\}) = \vec{\alpha}' \restriction I.$$

It follows from $(*)$ that $G(\vec{\beta}) = F(\vec{\alpha}) \neq F(\vec{\alpha}') = G(\vec{\beta}')$, therefore $1 \in J$.

In any case, $F_I \restriction (\prod_{i=1}^n A'_i)_I = G_J \restriction (\prod_{i=1}^m B'_i)_J$. ∎

## 4. The main result

Let us turn to prove the main result (Theorem 1.3) for Magidor forcing with $o^{\vec{U}}(\kappa) < \kappa$. The proof presented here is based on what was done in [1] and before that in [3]; it is a proof by induction of $\kappa$.

4.1. Short sequences. In this section we prove the theorem for sets $A$ of small cardinality.

Proposition 4.1: *Let $p \in \mathbb{M}[\vec{U}]$ be any condition, $X$ an extension-type of $p$. For every $\vec{\alpha} \in X(p)$ let $p_{\vec{\alpha}} \geq^* p^\frown\vec{\alpha}$. Then there exists $p \leq^* p^*$ such that for every $\vec{\beta} \in X(p^*)$, every $p^{*\frown}\vec{\beta} \leq q$ is compatible with $p_{\vec{\beta}}$ .*

*Proof.* By induction of $l(X)$. If $l(X) = 1$, $X = \langle\xi\rangle$, then $\vec{U}(X, p) = U(\kappa_i(p), \xi)$ and $X(p) = B_{i,\xi}(p)$. For each $\beta \in B_{i,\xi}(p)$

$$p_\beta = \langle\langle\kappa_1(p), A_1^\beta\rangle, \ldots, \langle\kappa_{i-1}(p), A_{i-1}^\beta\rangle, \langle\beta, B_\beta\rangle, \langle\kappa_i(p), A_i^\beta\rangle, \ldots, \langle\kappa, A_\beta\rangle\rangle.$$

For $j > i$ let $A_j^* = \bigcap_{\beta\in B_{i,\xi}(p)} A_j^\beta$. For $j < i$ we can find $A_j^*$ and shrink $B_{i,\xi}(p)$ to $E_\xi$ so that for every $\beta \in E_\xi$ and $j < i$ $A_j^\beta = A_j^*$. For $i$, first let $E = \Delta_{\alpha\in B_{i,\xi}(p)} A_i^\beta$. By ineffability of $\kappa_i(p)$ we can find $A_\xi^* \subseteq E_\xi$ and a set $B^* \subseteq \kappa_i(p)$ such that for every $\beta \in A_\xi^*$, $B^* \cap \beta = B_\beta$. We claim that $B^* \in U(\kappa_i(p), \gamma)$ for every $\gamma < \xi$,

$$\mathrm{Ult}(V, U(\kappa_i(p), \xi)) \models B^* = j_{U(\kappa_i(p),j)}(B^*) \cap \kappa_i(p),$$

and since

$$\{\beta < \kappa \mid B^* \cap \beta \in \cap\vec{U}(\beta)\} \in U(\kappa_i(p), \xi),$$

it follows that $B^* \in \bigcap j_{U(\kappa_i(p),\xi)}(\vec{U})(\kappa_i(p))$. By coherency

$$B^* \in \bigcap_{\gamma<\xi} U(\kappa_i(p), \gamma).$$

Define

$$A_i^* = B^* \uplus A_\xi^* \uplus (\cup_{\xi<i} E_i) \in \cap\vec{U}(\kappa_i(p)).$$

Let $q \geq p^{*\frown}\beta$ and suppose that $q \geq^* (p^{*\frown}\beta)^\frown\vec{\gamma}$. Then every $\gamma \in \vec{\gamma}$ such that $\gamma > \beta$ belongs to some $A_j^* \setminus \beta$ for $j \geq i$, and by the definition of these sets $\gamma \in A_j^\beta$. If $\gamma < \kappa_{i-1}$, then also $\gamma \in A_j^*$ for some $j < i$. Since $\beta \in E_\xi$ it follows that $A_j^\beta = A_j^*$, so $\gamma \in A_j^\beta$. For $\gamma \in (\kappa_{i-1}, \beta)$, by definition of the order we have $o^{\vec{U}}(\gamma) < o^{\vec{U}}(\beta) = \xi$ and therefore $\gamma \in A_{i,\eta}^* \cap \beta$ for some $\eta < \xi$, but

$$A_{i,\eta}^* \cap \beta \subseteq B^* \cap \beta = B_\beta;$$

it follows that $q, p_\beta$ are compatible. For general $X$, fix $\min(\vec{\beta}) = \beta$. Apply the induction hypothesis to $p^\frown\beta$ and $p_{\vec{\beta}}$ to find $p^*_\beta \geq^* p^\frown\beta$. Next apply the case $n = 1$ to $p^*_\beta$ and $p$, find $p^* \geq p$. Let $q \geq p^{*\frown}\vec{\beta}$ and denote $\beta = \min(\vec{\beta})$ then $q$ is compatible with $p^*_\beta$ thus let $q' \geq q, p^*_\beta$. Since $q' \geq p^*_\beta$ and $q' \geq p^{*\frown}\vec{\beta}$ it follows that $q' \geq p^{*\frown}_\beta\vec{\beta}$. Therefore there is $q'' \geq q', p_{\vec{\beta}}$. ∎

Lemma 4.2: *Let $\lambda < \kappa$, $p \in \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$, $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and $X \in \mathrm{Ex}(p)$. Also. let $\underset{\sim}{x}$ be an ordinal $\mathbb{M}[\vec{U}]$-name. There is $p \leq^* p^*$ such that:*

$$\text{If } \exists\vec{\alpha} \in X(p^*)\, \exists p' \geq^* p^{*\frown}\vec{\alpha}\ \langle q, p'\rangle || \ \underset{\sim}{x},$$
$$\text{then } \forall\vec{\alpha} \in X(p^*)\langle q, p^{*\frown}\vec{\alpha}\rangle || \underset{\sim}{x}.$$

*Proof.* Fix $p, \lambda, q, X$ as in the lemma. Consider the set

$$B_0 = \{\vec{\beta} \in X(p) \mid \exists p' \,{}^*{\geq} p^\frown\vec{\beta} \text{ s.t. } \langle q, p'\rangle || \underset{\sim}{x}\}.$$

One and only one of $B_0$ and $X(p) \setminus B_0$ is in $\vec{U}(X, P)$. Denote this set by $A'$. By Proposition 3.6, we can find $A'_{i,j} \in U(\alpha_i, x_{i,j})$ such that

$$\prod_{i=1}^{l(p)+1} \prod_{j=1}^{|X_i|} A'_{i,j} \subseteq A'.$$

Let $p \leq^* p'$ be the condition obtained by shrinking $B_{i,j}(p)$ to $A'_{i,j}$ so that $X(p') = \prod_{i=1}^{n+1} \prod_{j=1}^{|X_i|} A'_{i,j}$. If

$$\exists\vec{\beta} \in X(p')\ \exists p'' \,{}^*{\geq} p'^\frown\vec{\beta}\ \langle q, p''\rangle || \ \underset{\sim}{x},$$

then $\vec{\beta} \in B_0 \cap A'$ and therefore $B_0 = A'$. We conclude that

$$\forall\vec{\beta} \in X(p')\ \exists p_{\vec{\beta}} \,{}^*{\geq} p'^\frown\vec{\beta}\ \langle q, p_{\vec{\beta}}\rangle || \ \underset{\sim}{x}.$$

By Proposition 4.1 we can amalgamate all these $p_{\vec{\beta}}$ to find $p' \leq^* p^*$, such that for every $\vec{\beta} \in X(p^*)$, $p^{*\frown}\vec{\beta}$ decides $\underset{\sim}{x}$; then $p^*$ is as wanted. ∎

Lemma 4.3: *Consider the decomposition of 2.7 at some $\lambda \geq o^{\vec{U}}(\kappa)$ and let $\underset{\sim}{x}$ be a $\mathbb{M}[\vec{U}]$-name for an ordinal. Then for every $p \in \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$, there exists $p \leq^* p^*$ such that for every $X \in \mathrm{Ex}(p)$ and $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ the following holds:*

$$\text{If } \exists\vec{\alpha} \in X(p^*)\ \exists p' \geq^* p^{*\frown}\vec{\alpha}\ \langle q, p'\rangle || \ \underset{\sim}{x},$$
$$\text{then } \forall\vec{\alpha} \in X(p^*)\ \langle q, p^{*\frown}\vec{\alpha}\rangle || \underset{\sim}{x}.$$

*Proof.* Fix $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and $X \in \mathrm{Ex}(p)$. Use Lemma 4.2 to find $p \leq^* p_{q,X}$ such that

$$\text{If } \exists \vec{\alpha} \in X(p_{q,X})\ \exists p' \geq^* (p_{q,X})^\frown \vec{\alpha} \text{ s.t. } \langle q, p' \rangle || \underset{\sim}{x}, \quad \text{then } \forall \vec{\alpha} \in X(p_{q,X})\ \langle q, (p_{q,X})^\frown \vec{\alpha} \rangle || \underset{\sim}{x}.$$

By the definition of $\lambda$, the forcing $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ is $\leq^*$-$\max(|\mathrm{Ex}(p)|^+, |\mathbb{M}[\vec{U}] \restriction \lambda|^+)$-directed. Hence we can find $p \leq^* p^*$ so that for every $X, q$, $p_{q,X} \leq^* p^*$. ∎

Lemma 4.4: *Let $A \in V[G]$ be a set of ordinals such that $|A| < \kappa$. Then there exists $C' \subseteq C_G$ such that $V[A] = V[C']$.*

*Proof.* Assume that $|A| = \lambda' < \kappa$ and let $\delta = \max(\lambda', \mathrm{otp}(C_G)) < \kappa$. Split $\mathbb{M}[\vec{U}]$ as in Proposition 2.7. Find $p \in G$ such that some $\lambda \geq \delta$ appears in $p$. The generic $G$ also splits to $G = G_1 \times G_2$ where $G_1$ is the generic for Magidor forcing below $\lambda$ and, by Remark 2.8, $G_2$ is $V[G_1]$-generic for the upper part of the forcing. Let $\langle \underset{\sim}{a}_i \mid i < \lambda' \rangle$ be a $\mathbb{M}[\vec{U}]$-name for $A$ in $V$ and $p \in \mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$. For every $i < \lambda'$ find $p \leq^* p_i$ as in Lemma 4.3, such that for every $q \in \mathbb{M}[\vec{U}] \restriction \lambda$ and $X \in \mathrm{Ex}(p)$ we have

$$(*) \qquad \text{If } \exists \vec{\alpha} \in X(p_i)\ \exists p_i^\frown \vec{\alpha} \leq^* p'\ \langle q, p' \rangle \ || \ \underset{\sim}{a}_i, \quad \text{then } \forall \vec{\alpha} \in X(p_i)\ \langle q, p_i^\frown \vec{\alpha} \rangle \ || \ \underset{\sim}{a}_i.$$

Since in $\mathbb{M}[\vec{U}] \restriction (\lambda, \kappa)$ we have $\lambda^+$-closure for $\leq^*$, we can find a single $p_i \leq^* p_*$. Next, for every $i < \lambda'$, fix a maximal antichain $Z_i \subseteq \mathbb{M}[\vec{U}] \restriction \lambda$ such that for every $q \in Z_i$ there is an extension-type $X_{q,i}$ for which

$$\forall \vec{\alpha} \in p_*^\frown X_{q,i}\ \langle q, p_*^\frown \vec{\alpha} \rangle \ || \ \underset{\sim}{a}_i;$$

these antichains can be found using $(*)$ and Zorn's lemma. Recall that the sets $X_{q,i}(p_*)$ are a product of large sets. Define $F_{q,i} : X_{q,i}(p_*) \to On$ by

$$F_{q,i}(\vec{\alpha}) = \gamma \quad \Leftrightarrow \quad \langle q, p_*^\frown \vec{\alpha} \rangle \Vdash \underset{\sim}{a}_i = \check{\gamma}.$$

By Lemma 3.8 we can assume that there are important coordinates

$$I_{q,i} \subseteq \{1, \ldots, \mathrm{Dom}(X_{q,i}(p_*))\}.$$

Fix $i < \lambda'$. For every $q, q' \in Z_i$ we apply Lemma 3.10 to the functions $F_{q,i}, F_{q,i'}$ and find $p_* \leq^* p_{q,q'}$ for which one of the following holds:

(1) $\mathrm{Im}(F_{q,i} \restriction A(X_{q,i}, p_{q,q'})) \cap \mathrm{Im}(F_{q',i} \restriction A(X_{q',i}, p_{q,q'})) = \emptyset$.
(2) $(F_{q,i})_{I_{q,i}} \restriction (A(X_{q,i}, p_{q,q'}))_{I_{q,i}} = (F_{q',i})_{I_{q',i}} \restriction (A(X_{q',i}, p_{q,q'}))_{I_{q',i}}$.

Finally find $p^*$ such that for every $q, q'$, $p_{q,q'} \leq^* p^*$. By density, there is such $p^* \in G_2$. We use $F_{q,i}$ to translate information from $C_G$ to $A$ and vice versa, distinguishing from [1] that this translation is made in $V[G_1]$ rather than $V$: For every $i < \lambda'$, $G_1 \cap Z_i = \{q_i\}$. Use Lemma 3.4 to find $D_i \in X_{q_i,i}(p^*)$ such that $p^* {}^\frown D_i \in G_2$, define $C_i = D_i \restriction I_{q_i,i}$ and let

$$C' = \bigcup_{i<\lambda'} C_i.$$

Define, as in Definition 2.21, $I(C_i, C') \in [\mathrm{otp}(C_G)]^{<\omega}$, since

$$\mathrm{otp}(C') \leq \mathrm{otp}(C_G) \leq \lambda,$$

and by Proposition 2.16(6), $G_2$ does not add $\lambda$-sequences of ordinals below $\lambda$ to $V[G_1]$. We conclude that $\langle I(C_i, C') \mid i < \lambda' \rangle \in V[G_1]$. It follows that

$$(V[G_1])[A] = (V[G_1])[\langle C_i \mid i < \lambda' \rangle] = (V[G_1])[C'].$$

In fact let us prove that $\langle C_i \mid i < \lambda' \rangle \in V[A]$. Indeed, define in $V[A]$ the sets

$$M_i = \{q \in Z_i \mid a_i \in \mathrm{Im}(F_{q,i})\}.$$

Then, for any $q, q' \in M_i$ $a_i \in \mathrm{Im}(F_{q_i}) \cap \mathrm{Im}(F_{q',i}) \neq \emptyset$. Hence 2 must hold for $F_{q,i}, F_{q',i}$, i.e.,

$$(F_{q,i})_{I_{q,i}} \restriction (X_{q,i}(p^*))_{I_{q,i}} = (F_{q',i})_{I_{q',i}} \restriction (X_{q',i}(p^*))_{I_{q',i}}.$$

This means that no matter how we pick $q'_i \in M_i$, we will end up with the same function $(F_{q'_i,i})_{I_{q'_i,i}} \restriction (X_{q'_i,i}(p^*))_{I_{q'_i,i}}$. In $V[A]$, choose any $q'_i \in M_i$ and let $D'_i \in F^{-1}_{q'_i,i}(a_i)$, $C'_i = D_i \restriction I_{q'_i,i}$. Since $q_i, q'_i \in M_i$ we have $C_i = C'_i$, hence $\langle C_i \mid i < \lambda' \rangle \in V[A]$. We still have to determine what information $A$ uses in the part of $G_1$, namely, $\{q'_i \mid i < \lambda'\}, \langle I(C_i, C') \mid i < \lambda' \rangle \in V[A]$. This set can be coded as a subset of ordinals below $(2^\lambda)^+$, therefore

$$\{q'_i \mid i < \lambda'\}, \langle I(C_i, C') \mid i < \lambda' \rangle \in V[G_1].$$

By the induction hypothesis applied to $G_1$, we can find $C'' \subseteq C_{G_1}$ such that

$$V[\{q'_i \mid i < \lambda'\}, \langle I(C_i, C') \mid i < \lambda' \rangle] = V[C''].$$

Since all the information needed to restore $A$ is coded in $C' \uplus C''$, it is clear that $V[A] = V[C'' \uplus C']$. ∎

4.2. General subsets of $\kappa$. Assume that $A \in V[G]$ such that $A \subseteq \kappa$. For some $A$'s the proof, similar to the one in [1], works. This proof relies on the following lemma:

Lemma 4.5: *Assume that $o^{\vec{U}}(\kappa) < \kappa$ and let $A \in V[G]$, $\sup(A) = \kappa$. Assume that $\exists C^* \subseteq C_G$ such that*

(1) *$C^* \in V[A]$ and $\forall \alpha < \kappa\ A \cap \alpha \in V[C^*]$.*
(2) *$cf^{V[A]}(\kappa) < \kappa$.*

*Then $\exists C' \subseteq C_G$ such that $V[A] = V[C']$.*

*Proof.* Let $\langle \alpha_i \mid i < \lambda\rangle \in V[A]$ be cofinal in $\kappa$. Since $|C^*| < \kappa$, by Lemma 4.4 we can find $C'' \subseteq C_G$ such that

$$V[C''] = V[C^*, \langle \alpha_i \mid i < \lambda\rangle] \subseteq V[A].$$

In $V[C'']$, choose for every $i$ a bijection

$$\pi_i : 2^{\alpha_i} \to P^{V[C'']}(\alpha_i).$$

Since $A \cap \alpha_i \in V[C'']$ there is $\delta_i$ such that $\pi_i(\delta_i) = A \cap \alpha_i$. Finally let $C' \subseteq C_G$ such that

$$V[C'] = V[C'', \langle \delta_i \mid i < \lambda\rangle].$$

We claim that $V[A] = V[C']$. Obviously, $C' \in V[A]$, for the other direction

$$\langle A \cap \alpha_i \mid i < \lambda\rangle = \langle \pi_i(\delta_i) \mid i < \lambda\rangle \in V[C'].$$

Thus $A \in V[C']$. ∎

*Definition 4.6:* We say that $A \cap \alpha$ **stabilizes** if

$$\exists \alpha^* < \kappa.\ \forall \alpha < \kappa.\ A \cap \alpha \in V[A \cap \alpha^*].$$

First we deal with $A$'s such that $A \cap \alpha$ does not stabilize.

Lemma 4.7: *Assume $o^{\vec{U}}(\kappa) < \kappa$, $A \subseteq \kappa$ unbounded in $\kappa$ such that $A \cap \alpha$ does not stabilizes. Then there is $C' \subseteq C_G$ such that $V[C'] = V[A]$.*

*Proof.* Work in $V[A]$. Define the sequence $\langle \alpha_\xi \mid \xi < \theta\rangle$:

$$\alpha_0 = \min\{\alpha \mid V[A \cap \alpha] \supsetneq V\}.$$

Assume that $\langle \alpha_\xi \mid \xi < \lambda\rangle$ has been defined and for every $\xi$, $\alpha_\xi < \kappa$. If $\lambda = \xi + 1$ then set

$$\alpha_\lambda = \min\{\alpha \mid V[A \cap \alpha] \supsetneq V[A \cap \alpha_\xi]\}.$$

To see that $\alpha_\lambda$ is a well defined ordinal below $\kappa$, note that by the assumption that $A$ does not stabilize, there is $\alpha < \kappa$ such that $A \cap \alpha \notin V[A \cap \alpha_\xi]$, hence

$$V[A \cap \alpha_\xi] \subsetneq V[A \cap \alpha].$$

If $\lambda$ is limit, define

$$\alpha_\lambda = \sup(\alpha_\xi \mid \xi < \lambda);$$

if $\alpha_\lambda = \kappa$ define $\theta = \lambda$ and stop. The sequence $\langle \alpha_\xi \mid \xi < \theta \rangle \in V[A]$ is a continuous, increasing unbounded sequence in $\kappa$. Therefore,

$$cf^{V[A]}(\kappa) = cf^{V[A]}(\theta).$$

Let us argue that $\theta < \kappa$. Work in $V[G]$, for every $\xi < \theta$ pick $C_\xi \subseteq C_G$ such that $V[A \cap \alpha_\xi] = V[C_\xi]$. The map $\xi \mapsto C_\xi$ is injective from $\theta$ to $P(C_G)$, by the definition of $\alpha_\xi$'s. Since $o^{\vec{U}}(\kappa) < \kappa$, $|C_G| < \kappa$, and $\kappa$ stays strong limit in the generic extension. Therefore

$$\theta \leq |P(C_G)| = 2^{|C_G|} < \kappa.$$

Hence $\kappa$ changes cofinality in $V[A]$, according to Lemma 4.5; it remains to find $C^*$. Denote $\lambda = |C_G|$ and work in $V[A]$, for every $\xi < \theta$, $C_\xi \in V[A]$ (although the sequence $\langle C_\xi \mid \xi < \theta \rangle$ may not be in $V[A]$). $C_\xi$ witnesses that

$$\exists d_\xi \subseteq \kappa.\ |d_\xi| \leq \lambda \quad \text{and} \quad V[A \cap \alpha_\xi] = V[d_\xi].$$

Fix $d = \langle d_\xi | \xi < \theta \rangle \in V[A]$. It follows that $d$ can be coded as a subset of $\kappa$ of cardinality $\leq \lambda \cdot \theta < \kappa$. Finally, by Lemma 4.4, there exists $C^* \subseteq C_G$ such that $V[C^*] = V[d] \subseteq V[A]$, so

$$\forall \alpha < \kappa.\ A \cap \alpha \in V[d_\xi] \subseteq V[C^*]. \qquad \blacksquare$$

Next we assume that $A \cap \alpha$ stabilizes on some $\alpha^* < \kappa$. By Lemma 4.4, there exists $C^* \subseteq C_G$ such that $V[A \cap \alpha^*] = V[C^*]$, if $A \in V[C^*]$ then we are done. Assume that $A \notin V[C^*]$. To apply Lemma 4.5, it remains to prove that $cf^{V[A]}(\kappa) < \kappa$. The subsequence $C^*$ must be bounded; denote $\kappa_1 = \sup(C^*) < \kappa$ and $\kappa^* = \max(\kappa_1, \mathrm{otp}(C_G))$. Find $p \in G$ that decides the value of $\kappa^*$ and assume that $\kappa^*$ appears in $p$ (otherwise take some ordinal above it). As in Lemma 2.7 we split

$$\mathbb{M}[\vec{U}]/p \simeq (\mathbb{M}[\vec{U}] \restriction \kappa^*)/(p \restriction \kappa^*) \times (\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa))/(p \restriction (\kappa^*, \kappa))$$

There is a complete subalgebra $\mathbb{P}$ of $RO((\mathbb{M}[\vec{U}] \restriction \kappa^*)/(p \restriction \kappa^*))$ such that $V[C^*] = V[H]$ for some $V$-generic filter $H \subseteq \mathbb{P}$. Let

$$\mathbb{Q} = [(\mathbb{M}[\vec{U}] \restriction \kappa^*)/(p \restriction \kappa^*)]/C^*$$

be the quotient forcing completing $\mathbb{P}$ to $(\mathbb{M}[\vec{U}] \restriction \kappa^*)/(p \restriction \kappa^*)$. Finally note that $G$ is generic over $V[C^*]$ for

$$\mathbb{S} = \mathbb{Q} \times (\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa))/(p \restriction (\kappa^*, \kappa)).$$

Lemma 4.8: $cf^{V[A]}(\kappa) < \kappa$.

*Proof.* Let $G = G_1 \times G_2$ be the decomposition such that $G_1$ is generic for $\mathbb{Q}$ above $V[C^*]$ and $G_2$ is $\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa)$-generic over $V[C^*][G_1]$. Let $\underset{\sim}{A}$ be an $\mathbb{S}$-name for $A$ in $V[C^*]$, and $\langle q_0, p_0 \rangle \in G$ such that

$$\langle q_0, p_0 \rangle \Vdash \text{“}\forall \alpha < \kappa \ \underset{\sim}{A} \cap \alpha \text{ is old”} \quad \text{(i.e., in } V[C^*]).$$

Proceed by a density argument in $\mathbb{M}[\vec{U}] \restriction (\kappa^*, \kappa))/p \restriction (\kappa^*, \kappa)$; let $p_0 \leq p$. As in Lemma 4.4 find $p \leq^* p^*$ such that for all $q_0 \leq q \in \mathbb{Q}$ and $X \in \text{Ex}(p^*)$:

$$\exists \vec{\alpha}^\frown \langle \alpha \rangle \in X(p^*) \, \exists p' \geq^* p^{*\frown} \vec{\alpha}^\frown \langle \alpha \rangle$$
$$\langle q, p' \rangle || \underset{\sim}{A} \cap \alpha \Rightarrow \forall \vec{\alpha}^\frown \langle \alpha \rangle \in X(p^*). \ \langle q, p^{*\frown} \vec{\alpha}^\frown \langle \alpha \rangle \rangle \ || \ \underset{\sim}{A} \cap \alpha.$$

Denote the consequent result by $(*)_{X,q}$. Since $\underset{\sim}{A} \cap \alpha$ is forced to be old, we will find many $q, X$ for which $(*)_{q,X}$ holds. For such $q, X$, for every $\vec{\alpha}^\frown \langle \alpha \rangle \in X(p^*)$ define the value forced for $\underset{\sim}{A} \cap \alpha$ by $a(q, \vec{\alpha}, \alpha)$. Fix $q, X$ such that $(*)_{q,X}$ holds. Assume that the maximal measure which appears in $X$ is $U(\kappa_i(p), mc(X))$ and fix $\vec{\alpha} \in (X \setminus \{mc(X)\})(p^*)$. For every $\alpha \in B_{i,mc(X)}(p) \setminus \max(\vec{\alpha})$ the set $a(q, \vec{\alpha}, \alpha) \subseteq \alpha$ is defined. By ineffability, we can shrink $B_{i,mc(X)}(p)$ to $A^{q,\vec{\alpha}}_{i,mc(X)}$ and find a set $A(q, \vec{\alpha}) \subseteq \kappa_i(p)$ such that for every $\alpha \in A^{q,\vec{\alpha}}_{i,mc(X)}$,

$$A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha).$$

Define

$$A'_{i,mc(X)} = \Delta_{\vec{\alpha},q} A^{q,\vec{\alpha}}_{i,mc(X)}.$$

Let $p^* \leq^* p'$ be the condition obtained by shrinking to those sets. Then $p'$ has the property that whenever $(*)_{q,X}$ holds for some $q \in \mathbb{Q}$ and $X \in \text{Ex}(p')$, there exist sets $A(q, \vec{\alpha})$ for $\vec{\alpha} \in (X \setminus \{mc(X)\})(p')$ such that for every $\vec{\alpha}^\frown \langle \alpha \rangle \in X(p')$,

$$A(q, \vec{\alpha}) \cap \alpha = a(q, \vec{\alpha}, \alpha).$$

By density there is such $p' \in G_2$.

Work in $V[A]$. For every $\vec{\alpha}$ and $q$, if $A(q,\vec{\alpha})$ is defined, let

$$\eta(q,\vec{\alpha}) = \min(A\Delta A(q,\vec{\alpha})),$$

otherwise $\eta(q,\vec{\alpha}) = 0$. Now $\eta(q,\vec{\alpha})$ is well defined since $A \notin V[C^*]$ and $A(q,\vec{\alpha}) \in V[C^*]$. Also let

$$\eta(\vec{\alpha}) = \sup(\eta(q,\vec{\alpha}) \mid q \in \mathbb{Q}).$$

If $\eta(\vec{\alpha}) = \kappa$ then we are done (since $|\mathbb{Q}| < \kappa$). Define a sequence in $V[A]$: $\alpha_0 = \kappa^*$. Fix $\xi < \mathrm{otp}(C_G)$ and assume that $\langle \alpha_i \mid i < \xi \rangle$ is defined. At limit stages take

$$\alpha_\xi = \sup(\alpha_i \mid i < \xi) + 1.$$

Assume that $\xi = \lambda + 1$ and let

$$\alpha_\xi = \sup(\eta(\vec{\alpha}) + 1 \mid \vec{\alpha} \in [\alpha_\lambda]^{<\omega}).$$

If at some point we reach $\kappa$ we are done. If not, let us prove by induction on $\xi$ that $C_G(\xi) < \alpha_\xi$, which will indicate that the sequence $\alpha_\xi$ is unbounded in $\kappa$. At limit $\xi$ we have $C_G(\xi) = \sup(C_G(\beta) \mid \beta < \xi)$ since the Magidor sequence is a club. By the definition of the sequence $\alpha_\xi$ and the induction hypothesis, $\alpha_\xi > C_G(\xi)$. If $\xi = \lambda + 1$, use Corollary 2.20 to find $\vec{\alpha}^\frown\langle\alpha\rangle$ and $q \in \mathbb{Q}$ such that

$$\langle q, p'^\frown\vec{\alpha}^\frown\langle\alpha\rangle\rangle \Vdash \check{\alpha} = \underset{\sim}{C}_G(\check{\xi}).$$

Fix any $q' \in \mathbb{Q}$ above $q$, and split the forcing at $\alpha$ so that

$$\langle q', p'^\frown\vec{\alpha}^\frown\langle\alpha\rangle\rangle = \langle q', r_1, r_2\rangle,$$

where $r_1 \in \mathbb{M}[\vec{U}] \restriction (\kappa^*, \alpha)$ and $r_2 \in \mathbb{M}[\vec{U}] \restriction (\alpha, \kappa)$. Let $H_1$ be some generic up to $\alpha$ with $\langle q', r_1\rangle \in H_1$ and work in $V[C^*][H_1]$. The name $\underset{\sim}{A}$ has a natural interpretation in $V[C^*][H_1]$ as a $\mathbb{M}[\vec{U}] \restriction (\alpha,\kappa)$-name, $(\underset{\sim}{A})_{H_1}$. Use the fact that $\mathbb{M}[\vec{U}] \restriction (\alpha,\kappa)$ is $\leq^*$-closed and the Prikry condition to find $r_2 \leq^* r_2' \in \mathbb{M}[\vec{U}] \restriction (\alpha,\kappa)$ and $A_0$ such that

$$r_2' \Vdash_{\mathbb{M}[\vec{U}]\restriction(\alpha,\kappa)} (\underset{\sim}{A})_{H_1} \cap \alpha = A_0.$$

Since it is forced that $\underset{\sim}{A}$ is old,

$$A_0 \in V[C^*]$$

and therefore we can find $\langle q'', r_1'\rangle \in \mathbb{Q} \times \mathbb{M}[\vec{U}] \restriction (\kappa^*, \alpha)$ such that

$$\langle q'', r_1'\rangle \geq \langle q', r_1\rangle$$

and

$$\langle q'', r_1'\rangle \Vdash \text{“}r_2' \Vdash \underset{\sim}{A} \cap \alpha = A_0\text{”} \text{ therefore } \langle q'', r_1', r_2'\rangle \Vdash \underset{\sim}{A} \cap \alpha = A_0.$$

Since $r_2 \leq^* r_2'$ and $r_1' \in \mathbb{M}[\vec{U}] \restriction (\kappa^*, \alpha)$, then there is some $\vec{\beta} \in [\alpha]^{<\omega}$ such that

$$\langle r_1', r_2' \rangle^* \geq p'^\frown \vec{\beta}^\frown \langle \alpha \rangle.$$

Let $X$ be the extension-type of $\vec{\beta}^\frown\langle\alpha\rangle$; by definition of $p'$, $(*)_{q'',X}$ holds. Use density to find a condition $q^*$ in the generic of $\mathbb{Q}$ such that for some extension-type $X$ that decides the $\xi$th element of $C_G$, $(*)_{X,q^*}$ holds. The set

$$\{p'^\frown\vec{\gamma} \mid \vec{\gamma} \in X(p')\}$$

is a maximal antichain according to Proposition 3.4, so let $\vec{C}^\frown C_G(\xi)$ be the extension of $p'$ of type $X$ in $C_G$. By the construction of $q^*$ and $p'$ we have that

$$\langle q^*, p'^\frown \vec{C}^\frown C_G(\xi)\rangle \Vdash \underset{\sim}{A} \cap \check{C_G(\xi)} = A(q^*, \vec{C}) \cap \check{C_G(\xi)}.$$

Since $(\underset{\sim}{A})_G = A$, $A(q^*, \vec{C}) \cap C_G(\xi) = A \cap C_G(\xi)$ (otherwise we would have found compatible conditions forcing contradictory information). This implies that

$$\eta(q^*, \vec{C}) \geq C_G(\xi).$$

By the induction hypothesis $\alpha_\lambda > C_G(\lambda)$ and $\vec{C} \subseteq C_G(\lambda)$, thus $\vec{C} \in [\alpha_\lambda]^{<\omega}$ so

$$\alpha_\xi > \sup(\eta(\vec{\alpha}) \mid \vec{\alpha} \in [\alpha_\lambda]^{<\omega}) \geq \eta(\vec{C}) \geq \eta(q^*, \vec{C}) \geq C_G(\xi).$$

This proves that

$$\langle \alpha_\xi \mid \xi < \mathrm{otp}(C_G) < \kappa \rangle \in V[A]$$

is cofinal in $\kappa$ indicating $cf^{V[A]}(\kappa) < \kappa$. ∎

Thus we have proven the result for any subset of $\kappa$.

Corollary 4.9: *Let $A \in V[G]$ be a set of ordinals such that $|A| = \kappa$. Then there is $C' \subseteq C_G$ such that $V[A] = V[C']$.*

*Proof.* By $\kappa^+$-c.c. of $\mathbb{M}[\vec{U}]$, there is $B \in V$, $|B| = \kappa$ such that $A \subseteq B$. Fix in $V$ $\phi : \kappa \to B$ a bijection and let $B' = \phi^{-1''} A$. Then $B' \subseteq \kappa$. By the theorem for subsets of $\kappa$ there is $C' \subseteq C_G$ such that

$$V[C'] = V[B'] = V[A]. \quad ∎$$

4.3. General sets of ordinals. In [1], we gave an explicit formulation of subforcings of $\mathbb{M}[\vec{U}]$ using the indices of subsequences of $C_G$. In the larger framework of this paper, these indices might not be in $V$. By Example 1.4, subforcing of the Magidor forcing can be an iteration of Magidor type forcing.

LEMMA 4.10: *Let $A \in V[G]$ be such that $A \subseteq \kappa^+$. Then there is $C^* \subseteq C_G$ such that:*

(1) $\exists \alpha^* < \kappa^+$ *such that* $C^* \in V[A \cap \alpha^*] \subseteq V[A]$.
(2) $\forall \alpha < \kappa^+$ $A \cap \alpha \in V[C^*]$.

*Proof.* Work in $V[G]$. For every $\alpha < \kappa^+$ find subsequences $C_\alpha \subseteq C_G$ such that

$$V[C_\alpha] = V[A \cap \alpha]$$

using Corollary 4.9. The function $\alpha \mapsto C_\alpha$ has range $P(C_G)$ and domain $\kappa^+$ which is regular in $V[G]$, and since $o^{\vec{U}}(\kappa) < \kappa$ then $|C_G| < \kappa$, and since $\kappa$ is strong limit (even in $V[G]$) $|P(C_G)| < \kappa < \kappa^+$. Therefore there exist $E \subseteq \kappa^+$ unbounded in $\kappa^+$ and $\alpha^* < \kappa^+$ such that for every $\alpha \in E$, $C_\alpha = C_{\alpha^*}$. Set $C^* = C_{\alpha^*}$. Note that for every $\alpha < \kappa$ there is $\beta \in E$ such that $\beta > \alpha$, therefore

$$A \cap \alpha = (A \cap \beta) \cap \alpha \in V[A \cap \beta] = V[C^*]. \qquad \blacksquare$$

LEMMA 4.11: *Let $C^*$ be as in the last lemma. If there is $\alpha < \kappa$ such that $A \in V[C_G \cap \alpha][C^*]$. Then $V[A] = V[C^*]$.*

*Proof.* Consider the quotient forcing $\mathbb{M}[\vec{U}]/C^* \subseteq \mathbb{M}[\vec{U}]$ completing $V[C^*]$ to $V[C^*][G]$. Then the forcing

$$\mathbb{Q} = (\mathbb{M}[\vec{U}]/C^*) \restriction \alpha$$

completes $V[C^*]$ to $V[C^*][C_G \cap \alpha]$ and $|\mathbb{Q}| < \kappa$. By the assumption, $A \in V[C^*][C_G \cap \alpha]$, and for every $\beta < \kappa^+$, $A \cap \beta \in V[C^*]$. Let $\underset{\sim}{A} \in V[C^*]$ be a $\mathbb{Q}$-name for $A$ and $q \in G \restriction \alpha$ be any condition such that

$$q \Vdash \forall \beta < \kappa^+, \underset{\sim}{A} \cap \beta \in V[C^*].$$

In $V[C^*]$, for every $\beta < \kappa^+$ find $q_\beta \geq q$ such that $q_\beta ||_{\mathbb{Q}} \underset{\sim}{A} \cap \beta$. There is $q^* \geq q$ and $E \subseteq \kappa^+$ of cardinality $\kappa^+$ such that for every $\beta \in E$, $q_\beta = q^*$. By density, find such $q^* \in G \restriction \alpha$ in the generic. In $V[C^*]$, consider the set

$$B = \{X \subseteq \kappa^+ \mid \exists \beta \ q^* \Vdash X = \underset{\sim}{A} \cap \beta\}.$$

Let us argue that $\cup B = A$. Let $X \in B$; then there is $\beta < \kappa^+$ such that $q^* \Vdash X = \underset{\sim}{A} \cap \beta$ then $X = A \cap \beta \subseteq A$ thus, $\bigcup B \subseteq A$. Let $\gamma \in A$. There is $\beta \in E$ such that $\gamma < \beta$, by the definition of $E$ there is $X \subseteq \beta$ such that $q^* \Vdash \underset{\sim}{A} \cap \beta = X$; it must be that $X = A \cap \beta$ otherwise we would have found compatible conditions forcing contradictory information. But then $\gamma \in A \cap \beta = X \subseteq \cup B$. We conclude that $A = \cup B \in V[C^*]$. $\blacksquare$

Eventually we will prove that there is $\alpha < \kappa$ such that $A \in V[C_G \cap \alpha][C^*]$ and by the last lemma we will be done.

We would like to change $C^*$ so that it is closed. We can do that above $\alpha_0 := \mathrm{otp}(C_G)$:

Lemma 4.12: $V[C_G \cap \alpha_0][Cl(C^*)] = V[C_G \cap \alpha_0][C^*]$.[5]

*Proof.* Consider $I(C^*, Cl(C^*)) \subseteq \mathrm{otp}(C_G)$. By Proposition 2.16(5),

$$I(C^*, Cl(C^*)) \in V[C_G \cap \alpha_0].$$

Thus $V[C_G \cap \alpha_0][C^*] = V[C_G \cap \alpha_0][Cl(C^*)]$. ∎

Work in $N := V[C_G \cap \alpha_0]$. Since $C^* \cap \alpha_0 \in V[C_G \cap \alpha_0]$, we can assume $\min(C^*) > \alpha_0$. Since $I = I(C^*, C_G \setminus \alpha_0) \subseteq \mathrm{otp}(C_G)$, it follows that $I \in N$. In $N$, consider the coherent sequence

$$\vec{W} = \vec{U}^* \restriction (\alpha_0, \kappa] = \langle U^*(\beta, \delta) \mid \delta < o^{\vec{U}}(\beta), \alpha_0 < \delta < \kappa \rangle$$

where $U^*(\beta, \delta)$ is the ultrafilter generated by $U(\beta, \delta)$ in $N$. Also denote $G^* = G \restriction (\alpha_0, \kappa)$. The following proposition is to be compared with Remark 2.8.

Proposition 4.13: *$N[G^*]$ is a $\mathbb{M}[\vec{W}]$-generic extension of $N$.*

*Proof.* Let us argue that the Mathias criteria holds. Let $X \in \cap \vec{W}(\delta)$ where $\delta \in \mathrm{Lim}(C_{G^*})$. By definition of $\vec{W}$, for every $i < o^{\vec{W}}(\delta)$, there is $X_i \in U(\delta, i)$ such that $X_i \subseteq X$. The choice of $X_i$'s is done in $N$ and the sequence $\langle X_i \mid i < o^{\vec{U}}(\delta) \rangle$ might not be in $V$. Fortunately, $\mathbb{M}[\vec{U}] \restriction \alpha_0$ is $\alpha_0^+$-c.c. and $\alpha_0^+ < \delta$, so in $V$ we can find sets

$$E_i := \{X_{i,j} \mid j \leq \alpha_0\} \subseteq U(\delta, i)$$

such that $X_i \in E_i$. By $\delta$-completness of $U(\delta, i)$, the set $X_i^* := \cap E_i \in U(\delta, i)$ and $X_i^* \subseteq X_i \subseteq X$. Note that

$$X^* := \bigcup_{i < o^{\vec{U}}(\delta)} X_i^* \in \cap \vec{U}(\delta)$$

and therefore by genericity of $G$ there is $\xi < \delta$ such that

$$C_G \cap (\xi, \delta) \subseteq X^* \subseteq X.$$

Hence $C_{G^*} \cap (\max(\alpha_0, \xi), \delta) \subseteq X$. ∎

[5] For a set of ordinals $X$, $Cl(X) = X \cup \mathrm{Lim}(X) = \{\xi \mid \xi \in X \vee \sup(X \cap \xi) = \xi\}$.

Note that $o^{\vec{W}}(\kappa) < \min\{\nu \mid o^{\vec{W}}(\nu) = 1\}$ and $I(C^*, C_G) \in N$. In [1], this is the situation dealt with, a forcing denoted by $\mathbb{M}_I[\vec{W}] \in N[C^*]$ was defined where $I = I(C^*, C_G)$ and used to conclude the theorem. We only state here the main results and definitions and refer the reader to [1] for the full definition and proofs.

PROPOSITION 4.14: *Let $G^* \subseteq \mathbb{M}[\vec{W}]$ be an $N$-generic filter and $C \subseteq C_{G^*}$ be closed. Assume that $I = I(C, C_{G^*}) \in N$. Then there is a forcing notion $\mathbb{M}_I[\vec{W}] \in N$ and a projection $\pi_I : \mathbb{M}[\vec{W}] \to \mathbb{M}_I[\vec{W}]$ such that $N[G_I] = N[C]$, where $G_I = \overline{\pi_I'' G^*} \subseteq \mathbb{M}_I[\vec{W}]$ is the $N$-generic filter obtained by projecting $G^*$.*

LEMMA 4.15: *Let $G^* \subseteq \mathbb{M}[\vec{W}]$ be an $N$-generic filter. Then the forcing $\mathbb{M}[\vec{W}]/G_I$ satisfies $\kappa^+$-c.c. in $N[G^*]$.*

The referee pointed out a simpler argument than the one given in [1] for the continuation of the proof. First we conclude the following (see for example [4, Thm. 16.4]:

COROLLARY 4.16: *The forcing $\mathbb{M}[\vec{W}]/G_I \times \mathbb{M}[\vec{W}]/G_I$ satisfies $\kappa^+$-c.c.*

The next theorem is needed in order to apply Lemma 4.11 and to conclude the case for $A \subseteq \kappa^+$.

THEOREM 4.17: $A \in N[C^*]$.

*Proof.* Let $I = I(Cl(C^*), C_{G^*})$. Then

$$I, \mathbb{M}_I[\vec{W}], \pi_I \in N.$$

Let $G_I$ be the generic induced for $\mathbb{M}_I[\vec{W}]$ from $G$. It follows that $\mathbb{M}[\vec{W}]/G_I$ is defined in $N$. Toward a contradiction, assume that $A \notin N[C^*]$. By Lemma 4.12, $N[C^*] = N[Cl(C^*)]$, hence $A \notin N[Cl(C^*)]$. Let $\underset{\sim}{A}$ be a name for $A$ in $\mathbb{M}[\vec{U}]/G_I$. Work in $N[G_I]$. By corollary 4.14, $N[G_I] = N[Cl(C^*)]$. We define a tree $T \in N[G_I]$ of height $\kappa^+$. For every $\alpha < \kappa^+$ define the $\alpha$th level of the tree by

$$\mathrm{Lev}_\alpha(T) = \{B \subseteq \alpha \mid ||\underset{\sim}{A} \cap \alpha = B|| \neq 0\},$$

where the truth value is taken in $RO(\mathbb{M}[\vec{W}]/G_I)$—the complete Boolean algebra of regular open sets for $\mathbb{M}[\vec{W}]/G_I$. The order of the tree $T$ is simply end-extension. Different $B$'s in $\mathrm{Lev}_\alpha(T)$ yield incompatible conditions of $\mathbb{M}[\vec{W}]/G_I$

and we have $\kappa^+$-c.c. by Lemma 4.15, thus

$$\forall\alpha<\kappa^+\ |\mathrm{Lev}_\alpha(T)|\leq\kappa.$$

Work in $N[G^*]$; denote $A_\alpha=A\cap\alpha$. Recall that

$$\forall\alpha<\kappa^+\ A_\alpha\in N[Cl(C^*)]=N[G_I],$$

thus $A_\alpha\in\mathrm{Lev}_\alpha(T)$ which makes $A$ a branch through $T$. At this point, the referee pointed out an argument by Unger [7] showing that a forcing $\mathbb{P}$ such that $\mathbb{P}\times\mathbb{P}$ satisfies $\kappa^+$-c.c. has the $\kappa^+$-approximation property and, in particular, cannot add new branches to $\kappa^+$ trees in the ground model (see Definition 2.2, the discussion succeeding it, and Lemma 2.4 in [7]). By Corollary 4.16, the product of $\mathbb{M}[\vec{W}]/G_I$ in $\kappa^+$-c.c. in $N[G_I]$ and therefore $\mathbb{M}[\vec{W}]/G_I$ does not add new branches to $\kappa^+$ which implies that $A\in N[G_I]$.

For self-inclusion reasons and for the convenience of the reader, let us give another argument. For every $B\in\mathrm{Lev}_\alpha(T)$ define

$$b(B)=||\underset{\sim}{A}\cap\alpha=B||.$$

Assume that $B'\in\mathrm{Lev}_\beta(T)$ and $\alpha\leq\beta$; then $B=B'\cap\alpha\in\mathrm{Lev}_\alpha(T)$. Moreover, $b(B')\leq_B b(B)$ (we switch to Boolean algebra notation: $p\leq_B q$ means $p$ extends $q$). Note that for such $B,B'$, if $b(B')<_B b(B)$ then there is

$$0<p\leq_B(b(B)\setminus b(B'))\leq_B b(B).$$

Therefore

$$p\cap b(B')\leq_B(b(B)\setminus b(B'))\cap b(B')=0$$

meaning $p\perp b(B')$. As before, in $N[G^*]$ we denote $A_\alpha=A\cap\alpha\in\mathrm{Lev}_\alpha(T)$. Consider the $\leq_B$-non-increasing sequence $\langle b(A_\alpha)\mid\alpha<\kappa^+\rangle$. If there exists some $\gamma^*<\kappa^+$ on which the sequence stabilizes, define

$$A'=\bigcup\{B\subseteq\kappa^+\mid\exists\alpha\ b(A_{\gamma^*})\Vdash\underset{\sim}{A}\cap\alpha=B\}\in N[Cl(C^*)].$$

We claim that $A'=A$. Notice that if $B,B',\alpha,\alpha'$ are such,

$$b(A_{\gamma^*})\Vdash\underset{\sim}{A}\cap\alpha=B,\quad b(A_{\gamma^*})\Vdash\underset{\sim}{A}\cap\alpha'=B'.$$

Without loss of generality $\alpha\leq\alpha'$; then we must have $B'\cap\alpha=B$, otherwise the non zero condition $b(A_{\gamma^*})$ would force contradictory information. Consequently, for every $\xi<\kappa^+$ there exists $\xi<\gamma<\kappa^+$ such that

$$b(A_{\gamma^*})\Vdash\underset{\sim}{A}\cap\gamma=A\cap\gamma,$$

hence $A' \cap \gamma = A \cap \gamma$. This is a contradiction to $A \notin N[Cl(C^*)]$. We conclude that the sequence $\langle b(A_\alpha) \mid \alpha < \kappa^+\rangle$ does not stabilize. By regularity of $\kappa^+$, there exists a subsequence

$$\langle b(A_{i_\alpha}) \mid \alpha < \kappa^+\rangle$$

which is strictly decreasing. Use the observation we made to find $p_\alpha \leq_B b(A_{i_\alpha})$ such that $p_\alpha \perp b(A_{i_{\alpha+1}})$. Since $b(A_{i_\alpha})$ are decreasing, for any $\beta > \alpha$ $p_\alpha \perp b(A_{i_\beta})$ thus $p_\alpha \perp p_\beta$. This shows that $\langle p_\alpha \mid \alpha < \kappa^+\rangle \in N[G^*]$ is an antichain of size $\kappa^+$ which contradicts Lemma 4.15. ∎

Sets of ordinals above $\kappa^+$: By induction on $\sup(A) = \lambda > \kappa^+$. It suffices to assume that $\lambda$ is a cardinal.

Case 1: $cf^{V[G]}(\lambda) > \kappa$, the arguments for $\kappa^+$ works.

Case 2: $cf^{V[G]}(\lambda) \leq \kappa$ and since $\kappa$ is singular in $V[G]$ then $cf^{V[G]}(\lambda) < \kappa$. Since $\mathbb{M}[\vec{U}]$ satisfies $\kappa^+$-c.c. we must have that $\nu := cf^V(\lambda) \leq \kappa$. Fix

$$\langle \gamma_i \mid i < \nu\rangle \in V$$

cofinal in $\lambda$. Work in $V[A]$, for every $i < \nu$ find $d_i \subseteq \kappa$ such that

$$V[d_i] = V[A \cap \gamma_i].$$

By induction, there exists $C^* \subseteq C_G$ such that $V[\langle d_i \mid i < \nu\rangle] = V[C^*]$, therefore:

(1) $\forall i < \nu \; A \cap \gamma_i \in V[C^*]$.
(2) $C^* \in V[A]$.

Work in $V[C^*]$. For $i < \nu$ fix

$$\langle X_{i,\delta} \mid \delta < 2^{\gamma_i}\rangle = P(\gamma_i).$$

Then we can code $A \cap \gamma_i$ by some $\delta_i$ such that $X_{i,\delta_i} = A \cap \gamma_i$. By Corollary 4.9, we can find $C'' \subseteq C_G$ such that

$$V[C''] = V[\langle \delta_i \mid i < \nu\rangle].$$

Finally we can find $C' \subseteq C_G$ such that $V[C'] = V[C^*, C'']$; it follows that $V[A] = V[C']$. ∎ Theorem 1.3

## 5. Classification of intermediate models

Let $G \subseteq \mathbb{M}[\vec{U}]$ be a $V$-generic filter. Assume that for every $\alpha \leq \kappa$,

$$o^{\vec{U}}(\alpha) < \alpha.$$

Let $M$ be a transitive $ZFC$ model such that $V \subseteq M \subseteq V[G]$. We would like to prove it is a generic extension of a "Magidor-like" forcing which will be defined shortly.

By Example 1.4, the class of forcings $\mathbb{M}_I[\vec{U}]$ does not capture all the intermediate models of a generic extension by $\mathbb{M}[\vec{U}]$. The reason is that if

$$o^{\vec{U}}(\kappa) \geq \min\{\alpha \mid o^{\vec{U}}(\alpha) = 1\},$$

there are subsets $C \subseteq C_G$ such that $I(C, C_G)$ does not necessarily exist in the ground model, which was crucial in the definition of $\mathbb{M}_I[\vec{U}]$. Here we generalize this class to a class of forcings denoted by $\mathbb{M}_f[\vec{U}]$. We will prove that every intermediate model is a generic extension for a finite iteration of forcings of the form $\mathbb{M}_f[\vec{U}]$. The major difference between $\mathbb{M}_f[\vec{U}]$ and $\mathbb{M}_I[\vec{U}]$ is the existence of a concrete projection of $\mathbb{M}[\vec{U}]$ onto $\mathbb{M}_I[\vec{U}]$ which keeps only the ordinals which will sit at index $i \in I$ in the generic club. As for the generic set produced by $\mathbb{M}_f[\vec{U}]$, we cannot determine in advance how this set sits inside $C_G$. For example if $\mathbb{M}_I[\vec{U}]$ turns out to be the standard Prikry forcing, then the projection tells us what indices the Prikry sequence fill in $C_G$, and the forcing made sure to leave "room" for the missing elements of $C_G$. On the other hand, if $\mathbb{M}_f[\vec{U}]$ produces a Prikry sequence, there will be many ways to place this Prikry sequence inside $C_G$. One might claim that this is only a technicality, but if we aim to describe a forcing which produces a generic extension for an intermediate model of the form $V[C]$, where $C \subseteq C_G$, then Example 5.1 below describes a situation that $I(C, C_G) \notin V[C]$, and in particular there is no model $V \subseteq N \subseteq V[C]$ such that $V[C]$ is a generic extension of $N$ by $\mathbb{M}_I[\vec{U}]$. Instead of using $I(C, C_G)$, the forcing $\mathbb{M}_f[\vec{U}]$ uses the sequence $\langle o^{\vec{U}}(\alpha) \mid \alpha \in C\rangle$ which is definable in $V[C]$.

*Example 5.1:* Consider $\kappa$ such that $o^{\vec{U}}(\kappa) = \delta_0 := \min\{\alpha \mid o^{\vec{U}}(\alpha) = 1\}$. Let

$$p = \langle\langle\delta_0, A\rangle, \langle\kappa, B\rangle\rangle \in \mathbb{M}[\vec{U}];$$

then $p \Vdash C_{\underset{\sim}{G}}(\omega) = \delta_0$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be such that $p \in G$, and consider the first Prikry sequence for $C_G(\omega) = \delta_0$, namely $\{C_G(n) \mid n < \omega\}$, and let

$$C = \{C_G(C_G(n) + 1) \mid n < \omega\}.$$

Since for each $n<\omega$, $C_G(C_G(n)+1)$ is successor in $C_G$,

$$o^{\vec{U}}(C_G(C_G(n)+1))=0$$

and therefore $C$ is a Prikry sequence for $U(\kappa,0)$. Note that

$$I(C,C_G)=\{C_G(n)+1\mid n<\omega\}$$

and $I(C,C_G)\notin V[C]$. Otherwise $\{C_G(n)\mid n<\omega\}\in V[C]$, which is a contradiction since Prikry extensions do not add bounded subsets to $\kappa$.

Proposition 5.2: *Let $C,D\subseteq C_G$. There exists $E$ such that*

$$C\cup D\subseteq E\subseteq C_G\cap\sup(C\cup D)\quad\textit{and}\quad V[C,D]=V[E].$$

*Proof.* By induction on $\sup(C\cup D)$. If $\sup(C\cup D)\leq C_G(\omega)$ then $|C|,|D|\leq\aleph_0$. We can take $E=C\cup D$, clearly

$$I(C,C\cup D),I(D,C\cup D)\subseteq\omega,$$

thus these sets belong to $V$. In the general case, consider $I(C,C\cup D)$ and $I(D,C\cup D)$. Since

$$o^{\vec{U}}(\sup(C\cup D))<\sup(C\cup D)$$

it follows that

$$\mathrm{otp}(C\cup D)\leq\mathrm{otp}(C_G\cap\sup(C\cup D))<\sup(C\cup D).$$

Denote $\lambda=\mathrm{otp}(C_G\cap\sup(C\cup D))$. By Theorem 1.3, there is $F\subseteq C_G\cap\lambda$ such that

$$V[I(C,C\cup D),I(D,C\cup D)]=V[F].$$

Apply the induction hypothesis to $F,(C\cup D)\cap\lambda$ and find $E_*\subseteq\lambda$ such that

$$V[E_*]=V[F,(C\cup D)\cap\lambda].$$

Let $E=E_*\cup(D\cup C)\setminus\lambda$; then $E\in V[C,D]$ as both $E_*,D\cup C$ are in $V[C,D]$. In $V[E]$ we can find

$$E_*=E\cap\lambda\quad\text{and}\quad(D\cup C)\setminus\lambda=E\setminus\lambda.$$

Thus $F,(C\cup D)\cap\lambda\in V[E]$ and therefore also

$$D\cup C,\ I(C,C\cup D),\ I(D,C\cup D)\in V[E].$$

It follows that $C,D\in V[E]$. ∎

Corollary 5.3: *For every $C' \subseteq C_G$ there is $C^* \subseteq C_G \cap \sup(C')$ such that $C^*$ is closed and $V[C'] = V[C^*]$.*

*Proof.* Again we proceed by induction on $\sup(C')$. If $\sup(C') = C_G(\omega)$ then $C^* = C'$ is already closed. For general $C'$, consider $C' \subseteq Cl(C')$; then $I(C', Cl(C'))$ is bounded by some $\nu < \sup(C')$. So there is $D \subseteq C_G \cap \nu$ such that $V[D] = V[I(C', Cl(C'))]$. By Proposition 5.2, we can find $E$ such that

$$D \cup Cl(C') \cap \nu \subseteq E \subseteq C_G \cap \nu$$

and

$$V[E] = V[D, Cl(C')].$$

By the induction hypothesis there is a closed $E_*$ such that $E \subseteq E^* \subseteq C_G \cap \nu$ and $V[E] = V[E_*]$. Finally, let

$$C^* = E_* \cup \{\sup(E_*)\} \cup Cl(C') \setminus \nu.$$

Then $C^* \in V[C']$, and also $Cl(C')$ and $I(C', Cl(C'))$ can be constructed in $V[C^*]$ so $C' \in V[C^*]$. Obviously, $C^*$ is closed, hence $C^*$ is as desired. ∎

*Definition 5.4:* Let $\lambda < \kappa$ be ordinal. A function $f : \lambda \to \kappa$ is **suitable** if, for all $\delta \in \mathrm{Lim}(\lambda)$,

$$\limsup_{\alpha<\delta} f(\alpha) + 1 \leq f(\delta).$$

We would like to define $\mathbb{M}_f[\vec{U}]$ for a suitable $f$ to be the forcing which constructs a continuous sequence such that the order of the elements of the sequence is prescribed by $f$. However, we must require some connection to $\vec{U}$. In Example 5.5 below, we provide a suitable function which cannot describe the orders of any generic subsequence.

*Example 5.5:* Assume that $o^{\vec{U}}(\kappa) = \omega_1$ and $\forall \alpha < \kappa . o^{\vec{U}}(\alpha) < \omega_1$. Let $f : \omega+1 \to \kappa$ be defined by $f(0) = f(\omega) = \omega_1$ and $f(n+1) = 0$. There is no $C \subseteq C_G \cup \{\kappa\}$ with $\mathrm{otp}(C) = \omega + 1$ such that $o^{\vec{U}}(C(i)) = f(i)$. There are two reasons for that: The first, is that there is no $\alpha < \kappa$ that can be $C(0)$, since by assumption $o^{\vec{U}}(\alpha) < \omega_1 = f(0)$. The second reason is that $cf^{V[G]}(\kappa) = \omega_1$, hence there is no unbounded $\omega$-sequence of ordinals of order 0 below $\kappa$.

Let us restrict our attention to a more specific family of suitable functions.

*Definition 5.6:* Let $G \subseteq \mathbb{M}[\vec{U}]$ be $V$-generic and $C \subseteq C_G$ be closed, $\lambda + 1 = \text{otp}(C \cup \{\sup(C)\})$, and $\langle C(i) \mid i \leq \lambda\rangle$ be the increasing continuous enumeration of $C$. The **suitable function derived** from $C$, denoted by $f_C$, is the function $f_C : \lambda+1 \to \kappa$, defined by $f_C(i) = o^{\vec{U}}(C(i))$. A suitable function is called a **derived suitable function** if it is derived from some closed $C \subseteq C_G$.

Proposition 5.7: *If $C \subseteq C_G$ is a closed subset, then $f_C$ is suitable.*

*Proof.* Let $\delta \in \text{Lim}(\lambda + 1)$. Then $C(\delta) \in \text{Lim}(C_G \cup \{\kappa\})$ and therefore there is $\xi < C(\delta)$ such that for every $x \in C_G \cap (\xi, C(\delta))$, $o^{\vec{U}}(x) < o^{\vec{U}}(C(\delta))$. Let $\rho < \delta$ be such that for every $\rho < i < \delta$, $\xi < C(i) < C(\delta)$. Then

$$\sup_{\rho<i<\delta} o^{\vec{U}}(C(i)) + 1 \leq o^{\vec{U}}(C(\delta))$$

and also

$$\min\{(\sup_{\alpha<i<\delta} o^{\vec{U}}(C(i)) + 1) \mid \alpha < \delta\} \leq o^{\vec{U}}(C(\delta)). \qquad \blacksquare$$

*Definition 5.8:* Let $f : \lambda + 1 \to \kappa$ be a derived suitable function. Define the forcing $\mathbb{M}_f[\vec{U}]$. The conditions are functions $F$ such that:

(1) $F$ is a finite partial function, with $\text{Dom}(F)\subseteq\lambda+1$. such that $\lambda\in\text{Dom}(F)$.
(2) For every $i \in \text{Dom}(F) \cap \text{Lim}(\lambda + 1)$:
  (a) $F(i) = \langle\kappa_i^{(F)}, A_i^{(F)}\rangle$.
  (b) $o^{\vec{U}}(\kappa_i^{(F)}) = f(i)$.
  (c) $A_i^{(F)} \in \cap\vec{U}(\kappa_i^{(F)})$.
  (d) Let $j = \max(\text{Dom}(F) \cap i)$ or $j = -1$ if $i = \min(\text{Dom}(F))$. Then for every $j < k < i$, $f(k) < f(i)$.
(3) For every $i \in \text{Dom}(F) \setminus \text{Lim}(\lambda)$:
  (a) $F(i) = \kappa_i^{(F)}$.
  (b) $o^{\vec{U}}(\kappa_i^{(F)}) = f(i)$.
  (c) $i - 1 \in \text{Dom}(F)$.
(4) The map $i \mapsto \kappa_i^{(F)}$ is increasing.

*Definition 5.9:* The order of $\mathbb{M}_f[\vec{U}]$ is defined as follows; $F \leq G$ iff:

(1) $\text{Dom}(F) \subseteq \text{Dom}(G)$.
(2) For every $i \in \text{Dom}(G)$, let $j = \min(\text{Dom}(F) \setminus i)$.
  (a) If $i \in \text{Dom}(F)$, then $\kappa_i^{(F)} = \kappa_i^{(G)}$, and $A_i^{(G)} \subseteq A_i^{(F)}$.
  (b) If $i \notin \text{Dom}(F)$, then $\kappa_i^{(G)} \in A_j^{(F)}$, and $A_i^{(G)} \subseteq A_j^{(F)}$.

Proposition 5.10: *Let $f$ be a suitable derived function. Then $\mathbb{M}_f[\vec{U}]$ is a forcing notion.*

*Proof.* It is not hard to check that $\leq$ is a partial order on $\mathbb{M}_f[\vec{U}]$. To see $\mathbb{M}_f[\vec{U}] \neq \emptyset$, let $C$ be such that $f = f_C$. We define a finite sequence $\alpha_0 = \lambda$,if $\alpha_0$ is successor, $\alpha_1 = \alpha_0 - 1$. Otherwise, if there is no $\beta$ such that $f(\beta) \geq f(\alpha_0)$; then we halt the definition. If there is such $\beta$, let

$$\alpha_1 = \max\{\beta < \alpha_0 \mid f(\beta) \geq f(\alpha_0)\}.$$

By the suitability requirement, this maximum is defined and $\alpha_1 < \alpha_0$. In a similar fashion if $\alpha_1$ is successor, let $\alpha_2 = \alpha_1 - 1$, if there is no $\beta$ such that $f(\beta) \geq f(\alpha_1)$, then we halt the definition, otherwise,

$$\alpha_2 = \max\{\beta < \alpha_1 \mid f(\beta) \geq f(\alpha_1)\}$$

and $\alpha_2 < \alpha_1 < \alpha_0$. After finitely many steps we reach $\alpha_k$ such that for every $\beta < \alpha_k$, $f(\beta) < f(\alpha_k)$. The function $F$ defined by $\mathrm{Dom}(F) = \{\alpha_k, \ldots, \alpha_1\}$ and

$$F(\alpha_i) = \langle C(\alpha_i), C(\alpha_i) \setminus C(\alpha_{i+1}) + 1\rangle$$

satisfies Definition 5.8. ∎

*Example 5.11:* Assume that $f : \omega + 1 \to \kappa$, defined by $f(n) = 0$ and $f(\omega) = 1$. Then $\mathbb{M}_f[\vec{U}]$ first picks some measurable $\kappa^F_\omega$ of order 1, then adds a Prikry sequence to the measure $U(\kappa^F_\omega, 0)$.

If we only change $f$ at $\omega$, $f(\omega) = 2$, then we still force a Prikry sequence for the measure $U(\kappa^F_\omega, 0)$, but the first part chooses a measurable of order 2.

*Example 5.12:* Let $f : \omega^2 + \omega + 1 \to \kappa$ defined by

$$f(\omega \cdot n + m) = n, \quad f(\omega^2) = \omega, \quad f(\omega^2 + m + 1) = 1, \quad f(\omega^2 + \omega) = 2.$$

Clearly, $f$ is suitable. Now $\mathbb{M}_f[\vec{U}]$ first picks a measurable $\kappa^{(F)}_{\omega^2+\omega}$ of order 1. By condition (2)(d) of Definition 5.8, we must also pick $\kappa^{(F)}_{\omega^2}$ of order $\omega$, since $f(\omega^2) > f(\omega^2 + \omega)$. Then in the interval $(\kappa^{(F)}_{\omega^2}, \kappa^{(F)}_{\omega^2+\omega})$ the forcing generates a Prikry sequence for $U(\kappa^{(F)}_{\omega^2+\omega}, 1)$ and below $\kappa^{(F)}_{\omega^2}$ the forcing generates a diagonal Prikry sequence $\{\kappa^{(F)}_{\omega^n} \mid n < \omega\}$ for the measures $\langle U(\kappa^{(F)}_{\omega \cdot n}, n) \mid n < \omega\rangle$. For each $n < \omega$, the forcing generates a Prikry sequence $\{\kappa^{(F)}_{\omega \cdot n+m} | \mid m < \omega\}$ for $U(\kappa^{(F)}_{\omega\cdot(n+1)}, n)$ in the interval $[\kappa^{(F)}_{\omega \cdot n}, \kappa^{(F)}_{\omega\cdot(n+1)})$. So in all $\mathbb{M}_f[\vec{U}]$ generates a sequence of order type $\omega^2 + \omega + 1$.

Let $f : \omega^{o^{\vec{U}}(\kappa)} + 1 \to \kappa$, defined by $f(\alpha) = o_L(\alpha)$ (see Definition 2.19). By Proposition 2.20, for every $V$-generic filter $G \subseteq \mathbb{M}[\vec{U}]$ with $p_0 : \langle\langle\kappa,\kappa\rangle\rangle \in G$, $f = f_{C_G}$. Hence above $p_0$, $\mathbb{M}[\vec{U}]$ is isomorphic to $\mathbb{M}_f[\vec{U}]$. Note that forcing with $\mathbb{M}[\vec{U}]$ above $p_0$ is in the framework of this section since $\forall\alpha \in C_G \cup \{\kappa\}$. $o^{\vec{U}}(\alpha) < \alpha$.

Similar to $\mathbb{M}[\vec{U}]$, we decompose sets $A_i^{(F)} = \biguplus_{\xi < o^{\vec{U}}(\kappa_i^{(F)})} A_{i,\xi}^{(F)}$. Also, if $j$ is as in condition (2)(d) of Definition 5.8 and $j < i_1 < \cdots < i_k < i$, then for every $\vec{\alpha} \in \prod_{r=1}^k A_{f(i_r)}^{(F)}$, $G := F^\frown\vec{\alpha}$ is such that $\mathrm{Dom}(G) = \mathrm{Dom}(F) \cup \{i_1, \ldots, i_k\}$ and $G(x) = F(x)$ unless $x = i_r$, in which case $G(x) = \vec{\alpha}(r)$.

Proposition 5.13: *Let $f : \lambda + 1 \to \kappa \in V$ be a derived suitable function and $H \subseteq \mathbb{M}_f[\vec{U}]$ be a $V$-generic filter. Let*

$$C_H^* := \{\kappa_i^{(F)} \mid i \in \mathrm{Dom}(F), F \in H\}.$$

*Then,*

(1) $\mathrm{otp}(C_H^*) = \lambda + 1$ *and* $C_H^*$ *is continuous.*
(2) *For every* $i \le \lambda$, $o^{\vec{U}}(C_H^*(i)) = f(i)$.
(3) $V[C_H^*] = V[H]$.
(4) *For every* $\delta \in \mathrm{Lim}(\lambda + 1)$, *and every* $A \in \cap\vec{U}(\delta)$, *there is* $\xi < \delta$ *such that* $C^* \cap (\xi, \delta) \subseteq A$.
(5) *For every successor* $\rho < \lambda$, $H \restriction \rho := \{F \restriction \rho \mid F \in H\}$ *is* $V$*-generic for* $\mathbb{M}_{f\restriction\rho}[\vec{U}]$.

*Proof.* To see (1), let us argue by induction on $i < \lambda$ that the set

$$E_i = \{F \in \mathbb{M}_f[\vec{U}] \mid i \in \mathrm{Dom}(F)\}$$

is dense. Let $F \in \mathbb{M}_f[\vec{U}]$; if $i \in \mathrm{Dom}(F)$ we are done. Otherwise, let

$$j_M := \min(\mathrm{Dom}(F) \setminus i) > i > \max(\mathrm{Dom}(F) \cap i) =: j_m.$$

By condition (3)(c) of Definition 5.8 and minimality of $j_M$, $j_M \in \mathrm{Lim}(\lambda + 1)$. Split into two cases. First, if $i$ is successor, then we can find $F \le G$ such that $i - 1 \in \mathrm{Dom}(G)$ by induction hypothesis. By conditions (2)(d) and (2)(b), $f(i) < o^{\vec{U}}(\kappa_{j_M}^{(F)})$. By condition (2)(c), we can find $\alpha \in A_{j_M}^{(F)}$ such that $\alpha > \kappa_{j_m}^i$, $o^{\vec{U}}(\alpha) = f(i)$ and $A_{j_M}^{(F)} \cap \alpha \in \cap\vec{U}(\alpha)$. Then

$$G' = G \cup \{\langle i, \langle\alpha, A_{j_M}^{(F)} \cap \alpha\rangle\rangle\}$$

is as wanted. If $i$ is limit, since $f$ is suitable, there is $i' < i$ such that for every $i' < k < i$, $f(k) < f(i)$. Again by induction, find $F \leq G$ such that $i' \in \mathrm{Dom}(G)$. Then the desired $G'$ is constructed as in the successor step. Denote by $F_H$, the function with domain $\lambda + 1$, and let $F_H(i) = \gamma$ be the unique $\gamma$ such that for some $F \in H$, $i \in \mathrm{Dom}(F)$ and $\kappa_i^{(F)} = \gamma$. Then it is clear that $F_H$ is order preserving and $1-1$ from $\lambda$ to $C^*_H$. By the same argument as for $\mathbb{M}[\vec{U}]$, we conclude also that $F_H$ is continuous.

For (2), note that $C^*_H(i) = F_H(i)$, thus there is a condition $F \in H$ such that $F(i) = C^*_H(i)$. Hence $o^{\vec{U}}(C^*_H(i)) = f(i)$ by the definition of the condition in $\mathbb{M}_f[\vec{U}]$.

For (3), as usual we note that $H$ can be defined in terms of $C^*_H$ as the filter $H_{C^*_H}$ of all the conditions $F \in \mathbb{M}_f[\vec{U}]$ such that for every $i \leq \lambda$:

(1) If $i \in \mathrm{Dom}(F)$, then $\kappa_i^{(F)} = C^*_H(i)$.
(2) If $i \notin \mathrm{Dom}(F)$, then $C^*_H(i) \in \bigcup_{i \in \mathrm{Dom}(F)} A_i^{(F)}$.

(4) is the standard density argument given for $\mathbb{M}[\vec{U}]$.

As for (5), note that the restriction function $\phi : \mathbb{M}_f[\vec{U}] \to \mathbb{M}_{f\restriction\rho}[\vec{U}]$ is a projection of forcings from the dense subset $\{F \in \mathbb{M}_f[\vec{U}] \mid \rho \in \mathrm{Dom}(F)\}$ onto $\mathbb{M}_{f\restriction\rho}[\vec{U}]$, which suffices to conclude (5). ∎

The following theorem is a Mathias criteria for $\mathbb{M}_f[\vec{U}]$.

THEOREM 5.14: *Let $f : \lambda + 1 \to \kappa \in V$ be a derived suitable function, and let $C \subseteq \kappa$ be such that:*

(1) $\mathrm{otp}(C) = \lambda + 1$ *and $C$ is continuous.*
(2) *For every $i \leq \lambda$, $o^{\vec{U}}(C(i)) = f(i)$.*
(3) *For every $\delta \in \mathrm{Lim}(\lambda + 1)$, and $A \in \cap\vec{U}(C(\delta))$, there is $\xi < \delta$ such that $C \cap (\xi, \delta) \subseteq A$.*

*Then there is a $V$-generic filter $H \subseteq \mathbb{M}_f[\vec{U}]$ such that $C^*_H = C$.*

*Proof.* Define $H_C$ to consist of all the conditions $F \in \mathbb{M}_f[\vec{U}]$ such that for every $i \in \mathrm{Dom}(F)$:

(1) $F(i) = C(i)$.
(2) $C \setminus \{\kappa_i^{(F)} \mid i \in \mathrm{Dom}(F)\} \subseteq \bigcup_{i \in \mathrm{Dom}(F)} A_i^{(F)}$.

We prove by induction on $\lambda$ that $H_C$ is $V$-generic. Assume for every $\rho < \lambda$ and any suitable function $g : \rho + 1 \to \kappa$, every $C'$ satisfying $(1) - (3)$, the definition of $H_{C'}$ is generic for $\mathbb{M}_g[\vec{U}]$. Let $f, C$ be as in the theorem. For every $\delta < \lambda$, by definition, $H_C \restriction \delta+1 = H_{C\restriction\delta+1}$. Hence by the induction hypothesis $H_C \restriction \delta+1$ is generic for $\mathbb{M}_{f\restriction\delta+1}[\vec{U}]$. Also, it is a straightforward verification that $H_C$ is a filter. Let $D$ be a dense open subset of $\mathbb{M}_f[\vec{U}]$.

Claim 1: *For every $F \in \mathbb{M}_f[\vec{U}]$, there is $F \leq G_F$ such that:*

(1) $\xi := \max(\mathrm{Dom}(F)\cap\lambda)) = \max(\mathrm{Dom}(G_F)\cap\lambda)$.
(2) *There are $\xi < i_1 < \cdots < i_k < \lambda+1$ such that every $\vec{\alpha} \in \prod_{j=1}^k A^{(F)}_{\lambda,f(i_j)}$, $G_F^\frown\vec{\alpha} \in D$.*

*Proof.* For every $i_1 < \cdots < i_k < \lambda+1$ and every $F \leq G$ such that

$$\max(\mathrm{Dom}(F)\cap\lambda) = \max(\mathrm{Dom}(G)\cap\lambda) \quad \text{and} \quad G(\lambda) = F(\lambda),$$

consider the set

$$B = \Big\{\vec{\alpha} \in \prod_{j=1}^k A^{(G)}_{\lambda,f(i_j)} \mid \exists R. G^\frown\vec{\alpha} \leq^* R \in D\Big\}.$$

Then

$$B \in \prod_{j=1}^k U(\kappa^{(F)}_\lambda, f(i_j)) \vee \prod_{j=1}^k A^{(F)}_{\lambda,f(i_j)} \setminus B \in \prod_{j=1}^k U(\kappa^{(F)}_\lambda, f(i_j)).$$

Denote the set which is in $\prod_{j=1}^k U(\kappa^{(F)}_\lambda, f(i_j))$ by $B'$. By normality, there are $B_{i_j} \in U(\kappa^{(F)}_\lambda, f(i_j))$ such that $\prod_{j=1}^k B_{i_j} \subseteq B'$. Let $A^*_{G,i_1,\ldots,i_k} \in \cap\vec{U}(\kappa^{(F)}_\lambda)$ be the set obtained by shrinking only the sets $A^{(F)}_{\lambda,f(i_j)}$ to $B_{i_j}$. Since $o^{\vec{U}}(\kappa^{(F)}_\lambda) < \kappa^{(F)}_\lambda$ the possibilities for $G$ (note that $G(\lambda)$ must be $F(\lambda)$) and $i_1,\ldots,i_k$ are at most $\lambda$. So by $\kappa^{(F)}_\lambda$-completness

$$A^* = \bigcap_{G,i_1,\ldots,i_k} A^*_{G,i_1,\ldots,i_k} \in \cap\vec{U}(\kappa^{(F)}_\lambda).$$

Let $F \leq^* F^*$ be the condition obtained by shrinking $A^{(F)}_\lambda$ to $A^*$. By density, there is $G \geq F$ such that $G \in D$. So there is $\vec{\alpha} \in [A^*]^{<\omega}$ such that

$$(G \restriction \max(\mathrm{Dom}(F)\cap\lambda)) \cup \{\langle\lambda, \langle\kappa^{(F)}_\lambda, A^*\rangle\}^\frown\vec{\alpha} \leq^* G.$$

Let $i_j \in \mathrm{Dom}(G)$ be such that $\kappa^{(G)}_{i_j} = \vec{\alpha}(j)$; then $o^{\vec{U}}(\alpha_j) = f(i_j)$ and $\vec{\alpha} \in \prod_{j=1}^k A^{(F^*)}_{\lambda,f(i_j)}$. Hence for every $\vec{\beta} \in \prod_{j=1}^k A^{(F^*)}_{\lambda,f(i_j)}$, there is $G_{\vec{\beta}}$ such that

$$(G \restriction \max(\mathrm{Dom}(F)\cap\lambda)) \cup \{\langle\lambda, \langle\kappa^{(F)}_\lambda, A^*\rangle\}^\frown\vec{\beta} \leq^* G_{\vec{\beta}} \in D.$$

Note that $\vec{\beta} \in [A^*]^{<\omega}$, hence we are in the same situation as in Proposition 4.1, so we can find a single $F \leq G_F$ as wanted. ∎

For every possible lower part $F_0$ below $C(\lambda)$ i.e., $F_0 = F \restriction \lambda$ for some $F \in \mathbb{M}_f[\vec{U}]$ with $\kappa_\lambda^{(F)} = C(\lambda)$, use the claim to find $F_0 \cup \{\langle \lambda, \langle C(\lambda), C(\lambda)\rangle\rangle\} \leq G_{F_0}$. Let

$$\begin{aligned} A^* &= \Delta_{F_0} A_{F_0} \\ &:= \{\alpha < C(\lambda) \mid \forall F_0. F_0(\max(\mathrm{Dom}(F_0))) < \alpha \to \alpha \in A_{F_0}\} \in \cap \vec{U}(C(\lambda)). \end{aligned}$$

There is $\xi < C(\lambda)$ such that $C \cap (\xi, C(\lambda)) \subseteq A^*$. Pick any $\kappa' \in C \cap [\xi, C(\lambda))$ and let $\delta < \lambda$ be such that $C(\delta) = \kappa'$. By the claim, the set

$$E = \left\{ F \in \mathbb{M}_{f\restriction\delta+1}[\vec{U}] \mid \exists \delta < i_1 < \cdots < i_k.\ \forall \vec{\alpha} \in \prod_{j=1}^{k} A^*_{f(i_j)}.\ G_F^\frown \vec{\alpha} \in D \right\}$$

is dense. Since $H_C \restriction \delta + 1$ is generic, there is $G^* \in (H_C \restriction \xi + 1) \cap E$. By condition (2) of the assumption of the theorem, $f(i_j) = o^{\vec{U}}(C(i_j))$ and since $\xi < i_1 < \cdots < i_k$, $\langle C(i_1), C(i_2), \ldots, C(i_k)\rangle \in \prod_{j=1}^{k} A^*_{f(i_j)}$. Thus

$$(G^* \cup \{\langle \lambda, \langle \kappa, A^*\rangle\rangle\})^\frown \langle C(i_1), C(i_2), \ldots, C(i_k)\rangle \in H_C \cap D,$$

which concludes the proof that $H_C$ is generic. Obviously condition (1) of the definition of $H_C$ ensures that $C^*_{H_C} = C$. ∎

Theorem 5.15: *Let $G \subseteq \mathbb{M}[\vec{U}]$ be $V$-generic and let $C \subseteq C_G$ be any closed subset. Let $f_C$ be the suitable function derived from $C$. If $f_C \in V$, then there is a $V$-generic $H \subseteq \mathbb{M}_{f_C}[\vec{U}]$ such that $C^*_H = C$.*

*Proof.* Let us certify that $C$ satisfies the assumptions of Theorem 5.14 with respect to $f_C$. (1), (2) are immediate from the definition of $f_C$ and by closure of $C$. To see condition (3), let $\delta \in \mathrm{Lim}(\lambda + 1)$ and $A \in \cap \vec{U}(C(\delta))$. Since $C(\delta) \in \mathrm{Lim}(C)$, and $C \subseteq C_G$, $C(\delta) \in \mathrm{Lim}(C_G)$. By Proposition 2.16(3), there is $\xi < \delta$ such that $C_G \cap (\xi, \delta) \subseteq A$ and also $C \cap (\xi, \delta) \subseteq A$. ∎

*Example 5.16:* Consider the Prikry forcing with $U(\kappa, 0)$, take $C = C_G \restriction_{\text{even}}$. Then

$$\mathrm{otp}(C \cup \{\kappa\}) = \omega + 1 \quad f_C(n) = o^{\vec{U}}(C_G(2n)) = 0, \quad f_C(\omega) = o^{\vec{U}}(\kappa) > 0.$$

The forcing $\mathbb{M}_{f_C}[\vec{U}]$ is simply the Prikry forcing with $U(\kappa, 0)$. Distinguishing from the forcing $\mathbb{M}_I[\vec{U}]$, where we must leave "room" for the missing elements of the full generic $C_G$, it is possible that $\mathbb{M}_{f_C}[\vec{U}]$ did not leave ordinals between successive points of the Prikry sequence.

Theorem 5.17: *Assume that $\forall\alpha \leq \kappa.o^{\vec{U}}(\alpha) < \alpha$. Let $G \subseteq \mathbb{M}[\vec{U}]$ be a $V$-generic filter and let $V \subseteq M \subseteq V[G]$ be an intermediate ZFC model. Then there is a closed subset $C^*_{\text{fin}} \subseteq C_G$ such that $M = V[C^*_{\text{fin}}]$ and $V[C^*_{\text{fin}}]$ is a generic extension of a finite iteration of the form*

$$\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_2}}[\vec{U}] * \cdots * \mathbb{M}_{\underset{\sim}{f_n}}[\vec{U}].$$

*Proof.* By [4, Thm. 15.43], there is $A \in V[G]$ such that $V[A] = M$. By Theorem 1.3, there is $C' \subseteq C_G$ such that $M = V[A] = V[C']$. Apply Corollary 5.3 to find a closed $C^* \subseteq C_G \cup \{\kappa\}$ such that $V[C'] = V[C^*]$. Let $\lambda_0 = \kappa$, recursively define $\lambda_{i+1} = \text{otp}(C_G \cap \lambda_i)$. By the assumption $\forall\alpha \leq \kappa.o^{\vec{U}}(\alpha) < \alpha$ and Proposition 2.18, $\text{otp}(C_G \cap \lambda_i) < \lambda_i$. Hence after finitely many steps, $\lambda_n \leq C_G(\omega)$, denote $\kappa_i = \lambda_{n-i}$. Let $C^*_n := C^*$ and consider the derived suitable function

$$f_n := f_{C^*_n \cap (\kappa_{n-1}, \kappa_n]} : \text{otp}(C^*_n \cap (\kappa_{n-1}, \kappa_n]) \to \kappa$$

Since for each $x \in C^*_n \cap (\kappa_{n-1}, \kappa_n)$,

$$o^{\vec{U}}(x) < \text{otp}(C_G \cap \kappa_n) \quad \text{and} \quad \text{otp}(C^* \cap (\kappa_{n-1}, \kappa_n)) \leq \kappa_{n-1},$$

by Proposition 2.16(6), $f_n \in V[C^*_n] \cap V[C_G \cap \kappa_{n-1}]$. By Proposition 1.3 there is $D \subseteq C_G \cap \kappa_{n-1}$ such that $V[f_n] = V[D]$; apply Proposition 5.2 to $D, C^*_n \cap \kappa_{n-1}$ to find $E \subseteq \kappa_{n-1}$ such that $V[D, C^*_n \cap \kappa_{n-1}] = V[E]$. Next, apply Corollary 5.3 to $E$ in order to find a closed subset $C^*_{n-1} \subseteq C_G \cap \kappa_{n-1} \cup \{\kappa\}$ such that $V[C^*_{n-1}] = V[E]$. Now consider the derived suitable function

$$f_{n-1} := f_{C^*_{n-1} \cap (\kappa_{n-2}, \kappa_{n-1}]} : \text{otp}(C^*_{n-1} \cap (\kappa_{n-2}, \kappa_n - 1]) \to \kappa.$$

By the same arguments as before, $f_{n-1} \in V[C^*_{n-1}] \cap V[C_G \cap \kappa_{n-2}]$ and there is a closed subset $C^*_{n-2} \subseteq C_G \cap \kappa_{n-2} \cup \{\kappa_{n-2}\}$ such that $C^*_{n-2} \in V[C^*_{n-1}]$ and $V[C^*_{n-2}] = V[C^*_{n-1} \cap \kappa_{n-2}, f_{n-1}]$. In a similar fashion we define $C^*_0, C^*_1, \ldots, C^*_n$ such that:

(1) For every $0 \leq i \leq n$, $C^*_i \subseteq C_G \cap \kappa_i \cup \{\kappa_i\}$ is closed.
(2) $V[C^*_0] \subseteq V[C^*_1] \subseteq V[C^*_2] \subseteq \cdots \subseteq V[C^*_n] = M$.
(3) For every $0 \leq i \leq n$, $V[C^*_i] = V[C^*_{i+1} \cap \kappa_i, f_{i+1}]$, where $f_{i+1} = f_{C^*_{i+1} \cap (\kappa_i, \kappa_{i+1}]}$.
(4) $f_0 \in V$.

Item (4) follows from $C^*_0 \subseteq \{C_G(n) \mid n < \omega\}$,

$$C^*_{\text{fin}} = C^*_0 \uplus (C^*_1 \setminus \kappa_0) \uplus (C^*_2 \setminus \kappa_1) \uplus \cdots \uplus (C^*_n \setminus \kappa_{n-1}).$$

Claim 2: (1) *$C^*_{\text{fin}}$ is closed.*

(2) *For every $0 \le i \le n$, $V[C^*_{\text{fin}} \cap \kappa_i] = V[C^*_i]$ and, in particular,*

$$V[C^*_{\text{fin}}] = V[C^*] = M.$$

(3) *For every $0 < i \le n$, $f_i = f_{C^*_{\text{fin}} \cap (\kappa_{i-1}, \kappa_i]} \in V[C^*_{\text{fin}} \cap \kappa_{i-1}]$.*

*Proof.* $C^*_{\text{fin}}$ is closed as the union of finitely many closed sets. We prove (2) by induction, for $i = 0$, $C^*_{\text{fin}} \cap \kappa_0 = C^*_0$. Assume that $V[C^*_{\text{fin}} \cap \kappa_i] = V[C^*_i]$. Then

$$V[C^*_{\text{fin}} \cap \kappa_{i+1}] = V[C^*_{\text{fin}} \cap \kappa_i, C^*_{\text{fin}} \cap (\kappa_i, \kappa_{i+1})] = V[C^*_i, C^*_{i+1} \setminus \kappa_i].$$

To see that $V[C^*_i, C^*_{i+1} \setminus \kappa_i] = V[C^*_{i+1}]$, we use the third property of the sequence $C^*_j$, namely that $V[C^*_i] = V[C^*_{i+1} \cap \kappa_i, f_{i+1}]$ to see that $C^*_{i+1} \in V[C^*_i, C^*_{i+1} \setminus \kappa_i]$ and therefore $C^*_{i+1} \in V[C^*_i, C^*_{i+1} \setminus \kappa_i]$. As for the other direction, by the second property, $C^*_i \in V[C^*_{i+1}]$ and also $C^*_{i+1} \setminus \kappa_i \in V[C^*_{i+1}]$, so we conclude that $V[C^*_{\text{fin}} \cap \kappa_{i+1}] = V[C^*_{i+1}]$.

As for (3), note that $C^*_{\text{fin}} \cap (\kappa_{i-1}, \kappa_i] = C^*_i \cap (\kappa_{i-1}, \kappa_i]$, and by property (3) of the sequence $C^*_j$, $f_i \in V[C^*_{i-1}]$. By (2) of the claim it follows that

$$f_{C^*_{\text{fin}} \cap (\kappa_{i-1}, \kappa_i]} = f_{C^*_i \cap (\kappa_{i-1}, \kappa_i]} = f_i \in V[C^*_{i-1}] = V[C^*_{\text{fin}} \cap \kappa_{i-1}]. \quad \blacksquare$$

Therefore for every $i \le n$, $\mathbb{M}_{f_i}[\vec{U}]$ is defined in $V[C^*_{\text{fin}} \cap \kappa_{i-1}]$; denote this model by $N_i$. Recall Remark 2.8: the club $C_G \cap (\kappa_{i-1}, \kappa_i)$ is $V[C_G \cap \kappa_{i-1}]$-generic for the forcing $\mathbb{M}[\vec{U}] \restriction (\kappa_{i-1}, \kappa_i)$[6] and therefore it is $N_i$-generic as

$$N_i \subseteq V[C_G \cap \kappa_{i-1}].$$

Hence we can apply Theorem 5.15 to $C^*_{\text{fin}} \cap (\kappa_{i-1}, \kappa_i] \subseteq C_G \cap (\kappa_i + 1)$ and find a $N_i$-generic filter $H \subseteq \mathbb{M}_{f_i}[\vec{U}]$ such that

$$N_i[H] = N_i[C^*_{\text{fin}} \cap (\kappa_{i-1}, \kappa_i] = V[C^*_{\text{fin}} \cap \kappa_{i-1}][C^*_{\text{fin}} \cap (\kappa_{i-1}, \kappa_i]] = V[C^*_{\text{fin}} \cap \kappa_i].$$

In particular, $V[C^*_{\text{fin}} \cap \kappa_0]$ is a generic extension of $V$ by $\mathbb{M}_{f_0}[\vec{U}]$.

Let $\underset{\sim}{f_i}$ be a $(\mathbb{M}_{f_0}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_1}}[\vec{U}] * \cdots * \mathbb{M}_{\underset{\sim}{f_{i-1}}}[\vec{U}])$-name for $f_i$. Then there is a $V$-generic filter $H^*$ for the iteration $\mathbb{M}_{f_1}[\vec{U}] * \mathbb{M}_{\underset{\sim}{f_2}}[\vec{U}] * \cdots * \mathbb{M}_{\underset{\sim}{f_n}}[\vec{U}]$ such that $V[H^*] = V[C^*_{\text{fin}}] = M$ (see, for example, [4, Thm. 16.2]). $\blacksquare$

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[6] Alternatively, it is $V[C_G \cap \kappa_{i-1}]$-generic for $\mathbb{M}[\vec{W}] \restriction (\kappa_{i-1}, \kappa_i)$, where $\vec{W}$ is the coherent sequence generated by $\vec{U}$ in $V[C_G \cap \kappa_{i-1}]$.

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