Galvin's property at large cardinals

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The Galvin property

Galvin's Theorem

In a paper by Baumgartner, Hajnal and Maté [2], the following theorem due to F. Galvin was published:

Theorem 1 (Galvin's Theorem 1973)

Suppose that $\kappa^{<\kappa} = \kappa$. Then for every normal filter U over κ, and for any collection $\langle A_\alpha \mid \alpha < \kappa^+ \rangle \in [U]^{\kappa^+}$ consisting of κ^+-many sets, there is a subcollection $\langle A_i \mid i \in I \rangle$, of size κ (i.e. $I \in [\kappa^+]^\kappa$) such that $\cap_{i \in I} A_i \in U$.

In particular, if *GCH* holds and κ is a regular cardinal then from κ^+-many clubs, one can always extract κ-many for which the intersection is a club.
Let us put this combinatorical/saturation property into a definition:

Definition 2 (Galvin's Property)

Let $\mathcal{F}$ be a filter over κ and $\mu \leq \lambda$. Denote by $Gal(\mathcal{F}, \mu, \lambda)$ the following statement:

$$\forall \langle A_i \mid i < \lambda \rangle \in [\mathcal{F}]^\lambda . \exists I \in [\lambda]^\mu . \cap_{i \in I} A_i \in \mathcal{F}$$

Galvin's Property.

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□

Example 3

1. Galvin's Theorem $\equiv$ If $\kappa^{<\kappa} = \kappa$ the $Gal(U, \kappa, \kappa^+)$ holds for every normal U over κ.
2. If $\mu' \leq \mu \leq \lambda \leq \lambda'$ then $Gal(\mathcal{F}, \mu, \lambda) \Rightarrow Gal(\mathcal{F}, \mu', \lambda')$.
3. If (e.g.) $\mathcal{F}$ contains all the final segments and $\mu = cf(\kappa)$ then $\neg Gal(\mathcal{F}, \mu, \mu)$.

Most of the work presented here is the results of two papers: [3], and a joint paper [4] with **Alejandro Poveda**, **Shimon Garti** and **Moti Gitik**.

An applications of the Galvin property

It is known that [11] every κ-distributive forcing notion of **cardinality** κ is consistently a projection of the Tree-Prikry forcing with some non-normal measure U, denoted by $\mathbb{P}_U$.

Question

Is it consistent to have some measure U and some κ-distributive forcing notion of cardinality greater than κ is a projection of $\mathbb{P}_U$?[11]

The first forcing to consider is $Cohen(\kappa^+, 1)$: It is not hard to prove that this forcing cannot be a projection $\mathbb{P}_U$ for any U. The next forcing to consider is $Cohen(\kappa, \kappa^+) = \{f : \kappa^+ \to 2 \mid |f| < \kappa\}$.

Proposition 1 (Gitik, B.[10] 2022)

If $Gal(U, \kappa, \kappa^+)$ then the $\mathbb{P}_U$ does not project to $Cohen(\kappa, \kappa^+)$.

Sketch of proof.

- Note that a V-generic function $f_G : \kappa^+ \to 2$ for $Cohen(\kappa, \kappa^+)$ has the property that there is no subset $X \in V$, $|X| = \kappa$ of $f_G^{-1}[\{1\}]$ or $f_G^{-1}[\{0\}]$.
- In V: Suppose that $\underset{\sim}{f}_G$ is a $\mathbb{P}_U$-name for f_G and fix $p \in \mathbb{P}$.
- For every $\alpha < \kappa^+$ find a condition $p \leq p_\alpha = \langle t_\alpha, A_\alpha \rangle$ such that $p_\alpha \Vdash \underset{\sim}{f}_G(\alpha) = i_\alpha$ for $i_\alpha \in \{0, 1\}$.
- Find $X' \subseteq \kappa^+$, $|X'| = \kappa^+$, i^* and t^* such that for every $\alpha \in X'$, $\langle t_\alpha, i_\alpha \rangle = \langle t^*, i^* \rangle$ (WLOG $i^* = 1$).
- Apply the Galvin property to the sequence $\langle A_\alpha \mid \alpha \in X' \rangle$,

  $$(\text{Recall: } Gal(\mathcal{U}, \kappa, \kappa^+) \equiv \forall \langle A_i \mid i < \lambda \rangle \in [\mathcal{U}]^{\kappa^+}. \exists I \in [\kappa^+]^{\kappa}. \cap_{i \in I} A_i \in \mathcal{U})$$

  to find $X \in [X']^\kappa$ such that for $A := \bigcap_{\alpha \in X} A_\alpha \in U$.
- For every $\alpha \in X$ $p_\alpha \leq p^* = \langle t^*, A \rangle$ and therefore $p^* \Vdash \underset{\sim}{f}_G(\alpha) = 1$. This is a contradiction.

□

Other application: Density of old sets in Prikry extensions([13],[6]), analyzing quotients of Prikry-type forcings([9]), applications to partition relations ([6],[5]), connection to Kurepa trees([7]), relation to strong generating sequence of

How far can we push Galvin's Theorem?

Recall that a filter U over a regular cardinal κ is P-point if every sequence $\langle X_i \mid i < \kappa\rangle$ has a pseudo-intersection. Clearly normal$\Rightarrow$ P-point.

Theorem 4 (Gitik and B.)

Suppose that $\kappa^{<\kappa} = \kappa$. Then

1. *([8] 2021) Every filter U which is Rudin-Keisler equivalent to a finite product of P-point filters satisfies $Gal(U, \kappa, \kappa^+)$.*
2. *([3] 2023) The same for a filter U which is Rudin-Keisler equivalent to a filter of the form[a]:*

$$\sum_U(\sum_{U_{\alpha_1}} \cdots \sum_{U_{\alpha_1,..,\alpha_{n-1}}} (U_{\alpha_1,\dots,\alpha_n})\dots)$$

where U and each $U_{\alpha_1,\dots,\alpha_k}$ are a p-point ultrafilter.

[a] Suppose that W is an ultrafilter over κ and W_α is an ultrafilter over $\delta_\alpha \leq \kappa$. Then $\sum_W W_\alpha = \{X \subseteq [\kappa]^2 \mid \{\alpha < \kappa \mid \{\beta \mid \langle \alpha, \beta\rangle \in X\} \in W_\alpha\} \in W\}$.

Although item (2) seems like a slight improvment of (1), it turns out to be essential for our application in the next slide.
Trying to remove the assumption $\kappa = \kappa^{<\kappa}$ was proven impossible for successor of regulars by Abraham ans Shelah [1] and for successors of singulars by Garti, Poveda and B. [6]. The question regarding (weakly) inac. cardinals remains open.

Galvin's property in canonical inner models

It is well known that in the minimal inner model for a measurable cardinal $L[U]$, for a normal measure U, every κ-complete ultrafilter is Rudin-Keisler isomorphic to a finite power of U. Hence from (1) in the theorem above we get:

Corollary 5

In $L[U]$, every κ-complete (even σ-complete) ultrafilter W satisfy $Gal(W, \kappa, \kappa^+)$.

In particular, it is consistent that there are no non-Galvin ultrafilters.
To analyze the situation in canonical inner models for larger cardinals, let us use the recent analyses of ultrafilters by Goldberg [14] under the *Ultrapower axiom*(UA). The importance of (UA) is that it follows from weak comparison and therefore should hold in every canonical inner model. It has many consequences on the combinatorics of ultrafilter and the one which is relevant for us is the following generalization of the structure of ultrafilters in $L[U]$ we mentioned above:

Theorem 6 (Goldberg [14] (2020))

Assume UA, then every σ-complete ultrafilter W is Rudin-Keisler equivalent to an irreducible sum of sums of irreducible sums ... of irreducible ultrafilters.

Galvin's property in canonical inner models

The next step is to use results of Schluzeberg [16], which proved that in inner models which are extender models of the form $L[\mathbb{E}]$ up to a superstrong cardinal, every ultrafilter is equivalent to an extender on the sequence $\mathbb{E}$. In those canonical inner models up to a superstrong, irreducible ultrafilters are p-points:

Theorem 7

Suppose that there is no inner model with a superstrong cardinal, then in $L[\mathbb{E}]$ every irreducible ultrafilter is p-point.

Corollary 8

Suppose that there is no inner model with a superstrong cardinal, then in $L[\mathbb{E}]$, every σ-complete ultrafilter is Rudin-Keisler isomorphic to an ultrafilter of the form:

$$\sum_{U}(\sum_{U_{\alpha_1}}\cdots\sum_{U_{\alpha_1,\ldots,\alpha_{n-1}}}(U_{\alpha_1,\ldots,\alpha_n})\ldots)$$

where U and each $U_{\alpha_1,\ldots,\alpha_k}$ are p-point ultrafilters. In particular, every κ-complete ultrafilter in $L[\mathbb{E}]$ satisfies the Galvin property.

Some related questions

The following two questions are open:

Question

Is there (is it consistent to have) a filter/ultrafilter U which is not of the previous form (Sum of Sums...) for which Gal(U, κ, κ^+) *holds?*

Question

Does UA *imply that every* κ*-complete ultrafilter over* κ *satisfies the Galvin property?*

The following question was asked in [8]:

Question

Is the existence of a κ*-complete ultrafilter over* κ *which fails to satisfy the Galvin property Gal*(U, κ, κ^+) *consistent?*

The second part of the talk addresses this question.

Ultrafilters violating $Gal(U, \kappa, \kappa^+)$

Non-Galvin Ultrafilters

It is possible to construct a non-Galvin κ-complete ultrafilter:

Theorem 9 (Garti, Shelah and B.[7] (2021))

Suppose that κ is a supercompact cardinal, then there is a cofinality preserving forcing extension where there is a κ-complete W over κ, such that $Cub_\kappa \cup \{cf(\omega)\} \subseteq W$ and W is non-Galvin.

- Shortly after, together with Gitik [10], we have managed to force non-Galvin ultrafiler $Cub_\kappa \cup \{Reg_\kappa\} \subseteq W$ from the optimal assumption (a measurable cardinal).
- In that paper (assuming larger cardinals) we also obtain failures of $Gal(W, \kappa, \kappa^{++})$.
- As for ultrafilters with extends the singulars, in a joint paper with Garti, Gitik and Poveda [4], we forced a κ-complete ultrafilter W such that $Cub_\kappa \cup \{sing_\kappa\} \subseteq U$ and $\neg Gal(U, \kappa, \kappa^+)$ (from $o(\kappa) = 2$).

Question

Is it true that supercompact cardinals always admit non-Galvin κ-complete ultrafilter (which extend the club filter) [[7, Question 4.5],[5, Question 2.26]]?

Theorem 10 (B. [3] (2023))

Let κ be a 2^κ-supercompact[a] cardinal, then κ carries a κ-complete ultrafilter W such that $\neg Gal(W, \kappa, \kappa^+)$.

[a] A cardinal κ is called λ-supercompact if there is an elementary embedding $j : V \to M$, such that $crit(j) = \kappa$ and $M^\lambda \subseteq M$.

For the proof we will need the definition of a κ-independent family:

Definition 11

A family of subsets of κ, $\langle A_i \mid i < \lambda\rangle$ with the property that for every $I, J \in [\lambda]^{<\kappa}$, $I \cap J = \emptyset \Rightarrow (\cap_{i\in I} A_i) \cap (\cap_{j\in J} A_j^c) \neq \emptyset$ is called a κ-independent family of size λ,

κ-independent families of size 2^κ always exist given that $\kappa^{<\kappa} = \kappa$. Moreover, without this cardinal arithmetic assumptions, λ-many mutually generic Cohen functions over a regular κ form a $\kappa-$independent family.

Proof

Proof of theorem.

Let $\langle A_i \mid i < \kappa^+\rangle$ be a κ-independent family and $j : V \to M$ be a 2^κ-supercompact embedding. Let $i_U : V \to M_U$ be the normal embedding derived from j and $k : M_U \to M$ be the factor map satisfying $k \circ i_U = j$. Denote by $\langle A'_\alpha \mid \alpha < j(\kappa)^+\rangle = j(\langle A_i \mid i < \kappa^+\rangle)$. Since $j''\kappa^+, k''i_U(\kappa)^+ \in M$, in M we have the sequence $\langle A'_r \mid r \in j''\kappa^+\rangle$ and $\langle A'_s \mid s \in k''i_U(\kappa)^+ \setminus j''\kappa^+\rangle$. Since $M \models |j''\kappa^+| = \kappa^+ < j(\kappa)$ and $M \models |k''i_U(\kappa)^+| = 2^\kappa < j(\kappa)$, the κ-independence of the family implies that there is $\kappa \leq \delta \in (\bigcap_{r \in j''\kappa^+} A'_r) \cap (\bigcap_{s \in k''i(\kappa)^+ \setminus j''\kappa^+} (A'_s)^c)$. Let $W = \{X \subseteq \kappa \mid \delta \in j(X)\}$. Then $\{A_i \mid i < \kappa^+\} \subseteq W$ and for every $I \in [\kappa^+]^\kappa$, consider $i_U(I)$, there is $r \in i_U(I) \setminus i''_U\kappa^+$ e.g the κ-th elements in $i_U(I)$ in the increasing enumeration. then $k(r) \in j(I)$ and not in $j''\kappa^+$, thus $\delta \notin A'_{k(r)}$. It follows that $\delta \notin \cap_{s \in j(I)} A'_s = j(\cap_{i \in I} A_i)$. This implies that $\cap_{i \in I} A_i \notin W$. □

Corollary 12

If UA *implies that every* κ*-complete ultrafilter satisfies the Galvin property then there is no inner model with a supercompact cardinal.*

Note that W above does not nesseccerily extends Cub_κ.

Extending the Club Filter

In order to modify the construction to obtain an ultrafilter which extends the club filter, we will need an independent family with a special property

Definition 13

A sequence $\langle A_i \mid i < \kappa^+ \rangle$ of subsets of κ is called a *normal*-independent family if for every disjoint subseqeuces $\{A_{i_\alpha} \mid \alpha < \kappa\}, \{A_{j_\alpha} \mid \alpha < \kappa\}$, $\Delta_{\alpha<\kappa} A_{i_\alpha} \setminus A_{j_\alpha}$ is stationary.

Theorem 14 (Hayut [15])

$\diamond(\kappa)$ *implies the existence of a normal-independent family.*

It is known that if κ is supercompact (or even measurable or subtle cardinals) then $\diamond(\kappa)$ holds, hence:

Corollary 15

If κ is supercompact then there is a κ-complete ultrafilter W such that $Cub_\kappa \subseteq W$ and W fails to satisfy the Galvin property.

Non-Galvin filters

Finding non-Galvin filters is relatively easy.

Proposition 2

Let $\mathcal{F}$ be the κ-complete filter generated by a κ-independent family of size λ, then $\neg Gal(\mathcal{F}, \kappa, \lambda)$.

Using our observation about normal independent families we can make sure that $\mathcal{F}$ extends the club filter:

Proof.

Suppose otherwise that there is $I \in [\lambda]^\kappa$ such that $\cap_{i\in I} A_i \in \mathcal{F}$. Find $J \in [\lambda]^{<\kappa}$ such that $\cap_{j\in J} A_j \subseteq \cap_{i\in I} A_i$. Fix any $\alpha \in I \setminus J$, by κ-independence there is $\nu \in \cap_{j\in J} A_j$ such that $\nu \notin A_\alpha$, and in particular not in $\cap_{i\in I} A_i$, contradiction. □

Corollary 16

If $\kappa^{<\kappa} = \kappa$ and $\diamond(\kappa)$ holds, then there is a non-Galvin κ-complete filter $Cub_\kappa \subseteq \mathcal{F}$. In particular, $V = L$ implies every regular cardinals κ carries a non-Galvin κ-complete filter which extends the club filter.

Thank you for your attention!

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Trying to relax the assumption $\kappa^{<\kappa} = \kappa$ in Gavin's theorem, we have the following consistency result by Abraham and Shelah.

Theorem 17 (Abraham-Shelah forcing)

Assume GCH, let κ be a regular cardinal, and $\kappa^+ < cf(\lambda) \leq \lambda$. Then there is a forcing extension by a κ-directed, cofinality preserving forcing notion such that $2^{\kappa^+} = \lambda$ and there is a sequence $\langle C_i \mid i < \lambda \rangle$ such that:

1. *C_i is a club at κ^+.*
2. *for every $I \in [\lambda]^{\kappa^+}$, $|\bigcap_{i \in I} C_i| < \kappa$.*

In particular, $\neg Gal(Cub_{\kappa^+}, \kappa^+, 2^{\kappa^+})$.

A natural question is what happens on inaccessible cardinals? of course, by Galvin's theorem, we should be interested in weakly inaccessible Cardinals.

Question

Is it consistent to have a weakly inaccessible cardinal κ such that $\neg Gal(Cub_\kappa, \kappa, \kappa^+)$?

There are some limiting results due to Garti (see [12])

At successors of singular cardinals

Our focus is on the second case which does not fall under Abraham-Shelah's- the case of successors of singulars. Is it consistent to have $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ for a singular κ? Again, by Galvin's theorem, this would require violating SCH.

Theorem 18 (Garti, Poveda and B.)

Assume GCH and that there is a (κ, κ^{++})-extender[a]. Then there is a forcing extension where $cf(\kappa) = \omega$ and $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

[a] This situation can be forced just from the assumption $o(\kappa) = \kappa^{++}$

The idea is to Easton-support iterate the Abraham-Shelah's forcing on inaccessibles $\leq \kappa$. This produces a model of $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$. Using a sophistication of Woodin's argument due to Ben-Shalom [17], we can argue that κ remains measurable after this iteration. Finally, singularize κ using Prikry/Magidor forcing. The key lemma is the following:

Lemma 19

A κ^+-cc forcing preserves a witness for $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

The strong negation at successor of singulars

The sequence of clubs $\langle C_i \mid i < \kappa^+\rangle$ produced by the Abraham-Shelah forcing, witnesses a stronger failure of $Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$, indeed for any $I \in [\kappa^{++}]^{\kappa^+}$, $\cap_{i\in I} C_i$ is actually of **size less than** κ. Let us denote this by $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.
Interestingly, the previous argument does work for the strong negation:

Proposition 3

In general κ^+-cc forcings do not preserve $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

Indeed, any forcing which adds a set of size κ which diagonalize $(Cub_\kappa)^V$ (e.g. diagonalizing the club filter, Magidor forcing with $o(\kappa) \geq \kappa$) kills $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ (namely satisfy $\neg(\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})))$.

Question

Is it a ZFC-theorem that $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ cannot hold at a successor of a singular cardinal? Explicitly, is it true that from any sequence of κ^{++}-many clubs at κ^+ one can always extract a subfamily of size κ^+ for which the intersection is of size at least κ?

Two opposite results for Prikry forcing

On one hand Prikry forcing does not add a set of cardinality κ which diagonalize $(Cub_\kappa)^V$:

Theorem 20

Let U be a normal ultrafilter over κ. Let $\langle c_n \mid n < \omega\rangle$ be V-generic Prikry sequence for U, and suppose that $A \in V[\langle c_n \mid n < \omega\rangle]$ diagonalize $(Cub_\kappa)^V$. Then, there exists $\xi < \kappa$ such that $A \setminus \xi \subseteq \{c_n \mid n < \omega\}$. In particular, $|A \setminus \xi| \leq \aleph_0$.

On the other hand, just forcing a Prikry sequence is not enough:

Theorem 21

Let $\mathcal{C}$ be a witness for the strong negation. Then there exists $\mathcal{D}$, such that:

1. *$\mathcal{D}$ is also a witness for the strong negation;*
2. *For every normal ultrafilter U over κ, forcing with $Prikry(U)$ yields a generic extension where $\mathcal{D}$ cease to be a witness.*