MATH 300: CHAPTER 1- INTRODUCTION TO PROPOSITIONAL AND PREDICATE CALCULUS

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1. Notations

- $\mathbb{N} = \{0, 1, 2, 3, 4, ...\}$ is the set of natural numbers.
- $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$ is the set of integers.
- $\mathbb{Q} = \{0, \frac{1}{2}, -\frac{38}{7}, \frac{150{,}354}{243}, ...\}$ is the set of all rational numbers (all the quotients of integers).
- $\mathbb{R}$ is the set of all real numbers.
- $\mathbb{C}$, the set of complex number, consists of numbers of the form $a + ib$, where a, b are real numbers. We have that $i^2 = -1$.
- (α, β) is the open interval of all the real numbers strictly between α and β. $[\alpha, \beta]$ is the closed interval of all the real numbers x such that $\alpha \leq x \leq \beta$.
- $\emptyset$ is the empty set, a set with no elements.
- if a is an element of a set A we denote it by $a \in A$.

2. The language of mathematics

The first goal of this course is to learn how to convey formal mathematical ideas. For this purpose we need to develop a precise language which is common to all mathematicians around the globe. There are two layers to that language: **Propositional Calculus** and **Predicate Calculus**.

The use of the word "calculus" suggest that the structure of the language will enable us certain computations. Let us start with proposition calculus.

3. Propositional Calculus

Consider the following example:

> "On even weekdays, if the sun is out and there are no clouds, I am sad. If the sun is out then there are no clouds. Since I am always Happy, and I live on a planet where all weekdays are even, it is never sunny on my planet"

The intention of this bizarre paragraph is to emphasis that we do not care if the premises are realistic or "true" (whatever that means). Instead, we are interested in the logical structure of the paragraph, and the ability to analyze/calculate its logical validity.

Date: February 5, 2024.

Let us start by removing all the non-logical content from the paragraph by replacing pieces of information with letters which are called *propositions* or *atomic formula.* For this we define a *dictionary* which permits the translation:

1. $A =$"It is an even day".
2. $B =$"the sun is out".
3. $C =$"There are no clouds".
4. $D =$"I am sad".

We can now reformulate the paragraph:

Premise 1: "If A, then if B and C, then D.
Premise 2: If B then C.
Premise 3: Not D and A.
Conclusion: Not B.

Finally let us turn the premises and conclusion to be purely symbolic, for this we need the *Logical connectives*:

1. A and B is denoted by $A \wedge B$. (Conjunction)
2. A or B is denoted by $A \vee B$. (Disjunction)
3. If A then B is denoted by $A \Rightarrow B$. (Implication)
4. Not A is denoted by $\sim A$. (Negation, other notations $\neg A/\overline{A}$)

Finally, our initial paragraph has the following *symbolic representation in propositional calculus*:

Premise 1: $A \Rightarrow ((B \wedge C) \Rightarrow D)$
Premise 2: $B \Rightarrow C$
Premise 3: $(\sim D) \wedge A$
Conclusion: $\sim B$

Note that the parenthesis are crucial to avoid ambiguities. We will come back to this example later and analyse its logical validity. But before that let us define all of this in more generality.

3.1. **Formulas and statements.**

Definition 3.1. A *formula* or *statement* in propositional calculus over the atomic formulas $A_1, A_2, A_3, ...$ is a (finite) string of symbols obtained from the atomic formulas, connected by the logical connectives $\wedge, \vee, \sim, \Rightarrow$, and parenthesis to avoid ambiguity.

Example 3.2. Here are some meaningful formulas in propositional calculus (Here the atomic formulas are $A, B, C, D...$ rather than $A_1, A_2, ...$):

$$(A \Rightarrow B) \wedge (A \Rightarrow B), \quad \sim (C \Rightarrow (C \wedge C)) \vee B, \quad B, \quad C, \quad A.$$

Here are meaningless formulas in propositional calculus:

$$B \Rightarrow, \quad \wedge B, \quad BC \wedge A, \quad A \wedge \vee B \quad A \wedge B \Rightarrow C.$$

3.2. **Truth values.** As we said before, we are only interested in the logical validity of arguments rather then the actual content of the statements. Hence, we should not make any assumption about the truth of falsity of the atomic formulas. Instead, we are going to consider all the possible assignments true/false for the atomic formulas.

Definition 3.3. Given atomic formulas $A_1, ..., A_n$, a *truth values assignment* for the atomic formulas is a function v which assign for **every** atomic formula $A_1, ..., A_n$ a truth value $v(A_i) = T/F$.

Example 3.4. Here there are three different truth values assignments for the atomic formulas A_1, A_2, A_3:

$$v_1(A_1) = v_1(A_2) = v_1(A_3) = T$$

$$v_2(A_1) = v_2(A_2) = T \quad v_2(A_3) = F$$

$$v_3(A_1) = F \quad v_3(A_2) = v_3(A_3) = T$$

Problem 1. *How many truth values assignments are there for atomic formulas $A_1, ..., A_n$?*

The rule to **compute** complex statements from simpler ones is given by the following truth tables:

A	$\sim$ A
T	F
F	T

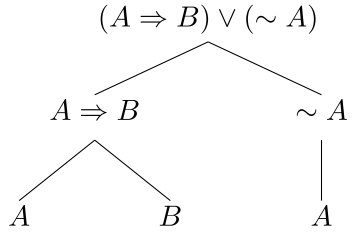
A	B	$A \wedge$B
T	T	T
T	F	F
F	T	F
F	F	F

A	B	$A \vee$B
T	T	T
T	F	T
F	T	T
F	F	F

A	B	$A \Rightarrow$B
T	T	T
T	F	F
F	T	T
F	F	T

Remark 3.5.
1. Note that each row in the truth table corresponds to a truth value assignment.
2. Even if an atomic formula does not appear in a formula we can always assume it appears and the truth value does not depend on that variable.
3. Once a truth assignment is fixed, one can calculate the truth value of any formula using the truth table.
4. in a statement $A \Rightarrow B$, A is called the *antecedent* and B the *consequent*. To see why the truth table of $A \Rightarrow B$ is defined to be true in case the antecedent A is false, thing about what it mean that A does not imply B, namely, when does $A \Rightarrow B$ should be false.

Example 3.6. Let us compute the truth table of $(A \Rightarrow B) \vee (\sim A)$. For this we need to decompose the formula:



Then we construct the truth table bottom-top:

A	B	A$\Rightarrow B$	$\sim A$	(A$\Rightarrow B) \vee (\sim A)$
T	T	T	F	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Example 3.7. Compute the truth value of the statement "Either 7 is prime and 9 is even, or else 11 is not less than 3".

Solution: This statement is of the form $(A \wedge B) \vee \sim C$ where $A =$"7 is prime", $B =$"9 is even" and $C =$"11 is less than 3". Hence $v(A) = T$, $v(B) = F$ and $V(C) = F$. and so $v(\sim C) = F$ which makes the statement true.

3.3. **Logical equivalence.** Some statements in mathematics are equivalent simply because of their logical structure and regardless of the mathematical content of the statements.

Definition 3.8. Two formulas α, β is propositional calculus are said to be *logically equivalent* if for every truth assignment v, $v(\alpha) = v(\beta)$. Equivalently, if α, β have the same truth table. We denote this by $\alpha \equiv \beta$.

Proposition 3.9. *Propositional calculus logical identities:*

(1) *Commutativity:*
  (a) $A \wedge B \equiv B \wedge A$.
  (b) $A \vee B \equiv B \vee A$.
  (c) $A \Rightarrow B \not\equiv B \Rightarrow A$.
(2) *Associativity:*
  (a) $A \wedge (B \wedge C) \equiv (A \wedge B) \wedge C$.
  (b) $A \vee (B \vee C) \equiv (A \vee B) \vee C$.
(3) *Distributivity low:*
  (a) $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$.
  (b) $A \vee (B \wedge C) \equiv (A \vee B) \wedge (A \vee C)$.
(4) *Implication identities:*
  (a) $A \Rightarrow B \equiv (\sim A) \vee B$.
  (b) $A \Rightarrow B \equiv (\sim B) \Rightarrow (\sim A)$. *(contrapositive)*
(5) *Law of negation of logical connectives:*
  (a) $\sim (\sim (A)) \equiv (A)$.
  (b) $\sim (A \wedge B) \equiv (\sim A) \vee (\sim B)$. *(De-Morgan law I)*

(c) $\sim(A \vee B) \equiv (\sim A) \wedge (\sim B)$. *(De-Morgan law II)*
(d) $\sim(A \Rightarrow B) \equiv A \wedge \sim B$.

Proof. We provide proof only the distributivity law $A \wedge (B \vee C) \equiv (A \wedge B) \vee (A \wedge C)$ as a demonstration:

For the left hand side we the following truth table

A	B	C	B$\vee C$	A$\wedge(B \vee C)$
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	F
F	T	T	T	F
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

The truth table of the right hand side is given by:

A	B	C	A$\wedge B$	A$\wedge C$	(A$\wedge B) \vee (A \wedge C)$
T	T	T	T	T	T
T	T	F	T	F	T
T	F	T	F	T	T
T	F	F	F	F	F
F	T	T	F	F	F
F	T	F	F	F	F
F	F	T	F	F	F
F	F	F	F	F	F

When comparing[1] the truth tables we see that the statements are logically equivalent.

Next let us prove the law of negation for implication based on the other identities $4.a, 5.a, 5.c$:[2]

$$\sim(A \Rightarrow B) \underset{(4.a)}{\equiv} \sim((\sim A) \vee B) \underset{(5.c)}{\equiv} (\sim(\sim A)) \wedge \sim B \underset{(5.a)}{\equiv} A \wedge \sim B$$

□

Definition 3.10. We define an additional logical connective $A \Leftrightarrow B$ by the following truth table:

A	B	$A \Leftrightarrow$B
T	T	T
T	F	F
F	T	F
F	F	T

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[1] Note that we should make sure that the truth assignments in both tables are ordered in the same rows (the first three columns) and beside that we are only interested in the right most column.

[2] The other identities can be proven by computing the truth tables as above.

Problem 2. *Prove that* $\alpha \Leftrightarrow \beta \equiv (A \Rightarrow B) \wedge (B \Rightarrow A)$.

The previous exercise shows that the logical connective $\Leftrightarrow$ is redundant and can be expressed using the other logical connectives. Nonetheless, it turns out that $\Leftrightarrow$ is quit useful.

Problem 3. *Prove that all the logical connectives can be expressed with* $\vee, \sim$.

Definition 3.11. A statement α is called a *tautology* if for every truth assignment v, $v(\alpha) = T$. We denote it by $\alpha \equiv T$. Similarly, a *contradiction* is a statement α such that for every v, $v(\alpha) = F$, we denote it by $\alpha \equiv F$.

Example 3.12. $A \vee (\sim A)$ is a tautology and $A \wedge (\sim A)$ is a contradiction.

Problem 4. *Show that if* α *is a tautology then* $\sim \alpha$ *is a contradiction.*

Proposition 3.13. *Tautology and contradiction identities:*

1. $\alpha \wedge T \equiv \alpha$.
2. $\alpha \vee T \equiv T$.
3. $\alpha \Rightarrow T \equiv T$, $T \Rightarrow \alpha \equiv \alpha$.
4. $\alpha \wedge F \equiv F$.
5. $\alpha \vee F \equiv \alpha$.

Definition 3.14. Given statements $\alpha_1, .., \alpha_n, \alpha$, we say that the premises $\alpha_1, ..., \alpha_n$ logically implies the conclusion α, if for every truth assignment v such that $v(\alpha_1) = ... = v(\alpha_n) = T$, must also satisfy $v(\alpha) = T$. Equivalently, if $(\alpha_1 \wedge ... \wedge \alpha_n) \Rightarrow \alpha$ is a tautology.

Example 3.15. premises which logically imply and do not imply the conclusion:

(1) The premises

Premise 1: $A \Rightarrow (B \wedge C)$

Premise 2:$A \vee C$

do not logically imply the conclusion

Conclusion: B.

Indeed, the truth assignment v, defined by $v(A) = F$, $V(B) = F$ and $v(C) = T$ is an example of a truth assignment for which the premises are true, namely $v(A \Rightarrow (B \wedge C)) = v(A \vee C) = T$ and the conclusion is false $v(B) = F$.

(2) Back to our first example, let us prove that the premises:

Premise 1: $A \Rightarrow ((B \wedge C) \Rightarrow D)$

Premise 2: $B \Rightarrow C$

Premise 3: $(\sim D) \wedge A$

logically imply the conclusion:

Conclusion: $\sim B$

Suppose that v is any truth assignment that satisfy

$$v(A \Rightarrow ((B \wedge C) \Rightarrow D)) = v(B \Rightarrow C) = v((\sim D) \wedge A) = T.$$

Our goal is to infer that $v(\sim B)) = T$.

- (a) Since $v((\sim D) \wedge A) = T$, we conclude that $V(A) = T$, $V(D) = F$.
- (b) Since $v(A \Rightarrow ((B \wedge C) \Rightarrow D)) = T$ and $V(A) = T$ we conclude that $v((B \wedge C) \Rightarrow D) = T$.
- (c) Since $V(D) = F$ we conclude that $V(B \wedge C) = F$.
- (d) By De Morgan law's $v((\sim B) \vee (\sim C)) = T$.
- (e) Since $v(B \Rightarrow C) = T$, by the logical identities, $v((\sim B) \vee C) = T$.
- (f) For $(d), (e)$ and the definition of $\wedge$, we conclude that $v([(\sim B) \vee C] \wedge [(\sim B) \vee (\sim C)]) = T$.
- (g) By the distributivity law, we conclude that $v((\sim B) \wedge (C \vee (\sim C))) = T$.
- (h) ince $C \vee (\sim C)$ is a tautology, by the tautology identity (1), we conclude that $v(\sim B) = T$.

Remark 3.16. Most of the times, it is simpler to use a proof by contradiction in order to prove that premises logically imply a conclusion.

4. Predicate Calculus

The second layer of the mathematical language is the *predicate calculus* or *first order logic*. We will only describe very shortly what is the structure of statements in the predicate calculus and how to intuitively grasp their meaning. For a full and comprehensive account of first order logic, students are advised to participate in the Logic class.

Definition 4.1. A *predicate* is a (mathematical) sentence with an undefined variable (free variable). A *statement* is a sentence with no free variables.

Example 4.2.

1. Predicates:
    - (a) $x + 6 > x^2$.
    - (b) The units digit of n is 5.
    - (c) $x < 0 \vee x \geq 0$.
2. Statements:
    - (a) $2 + 6 > 5$.
    - (b) $0 = 1$.
    - (c) Every even number is the sum of two primes.
    - (d) for all x, $x + 6 > x^2$.
    - (e) For every x, $x < 0 \vee x \geq 0$.
3. Meaningless statements:
    - (a) How are you?
    - (b) Bring me Gauda cheese.

The major difference between predicates and statements is that statements always have a truth value (T or F), even if we do not know what it is (as in example 2c), while predicates do not have truth values. However, *substituting* all the free variables in a predicate renders the predicate into a

statement.
Notation: In general, a predicate with a free variable x is denoted by

$$p(x), q(x), r(x), ...$$

Similarly, if there is more then one free variable we will denote it

$$p(x, y), q(x, y, z, w), r(x_1, x_2, ..., x_n), ...$$

Definition 4.3.

Definition 4.4. Given a predicate $p(x)$, a *universe of discourse* is some set of elements U such that x can be substituted by every element $a \in U$. The *truth set*, denoted by $Tr^U(p(x))$, is the collection (set) of all those elements a in U such that $p(a)$ is a true sentence.

Usually, U would be a system of numbers for example $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, (-3, 5)$ and so on.

Example 4.5. The truth set depends on the universe. for example if $p(x)$ is $-1 \leq x < 1$ then $Tr^{\mathbb{N}}(p(x))$ would only have 0 in it, while $Tr^{\mathbb{Z}}(p(x))$ would have both 0 and -1.

Similarly, the truth set of a predicate of the form $p(x, y)$ in U is the set of tuples (a, b) such that both $a, b \in U$ and $p(a, b)$ holds true.

Example 4.6. The set $Tr^{\mathbb{R}}(y = 2x)$ is a collection of pairs which draws a straight line.

Definition 4.7. Given a universe of discourse U, two predicates $p(x), q(x)$ are equivalent (over U) if they have the same truth set.

As the truth set depends on the universe, equivalence may also depend on the universe. The two *quantifiers* of the predicate calculus are the *existential quantifier*, dented by $\exists$ (There "Exists"...) and the *universal quantifier* denoted by $\forall$ (For "All"...)

Definition 4.8. The general structure of a *statement in the predicate calculus* is $\forall x.p(x)$ or $\exists x.p(x)$ where $p(x)$ is a predicate. Given U be a universe of discourse.

- The statement $\forall x.p(x)$ is true in U (namely, has truth value T) if every possible substitution x_0 of x, where x_0 is an element of U, $p(x_0)$ holds true. Equivalently if $Tr^U(p(x)) = U$.
- The statement $\exists x.p(x)$ is true in U, if there is a specific example/substitution x_0 for x, from the set U, such that $p(x_0)$ holds true. We call the specific example a *witness* for the existential statement. Equivalently, if the set $Tr^U(p(x))$ is non-empty.

Example 4.9. (1) Example for legitimate sentence in the predicate calculus:

$$\forall x.p(x) \wedge Q(x),\ \ \exists x.\forall y.p(x, y),\ \ \forall x.\forall y.(p(x) \wedge Q(x)) \to p(x), (\exists x.p(x)) \to (\forall y.q(y))$$

(2) Examples of non legitimate statements in the predicate calculus:

$$p(x) \wedge \forall y.p(y), \quad \forall x.p(x)\forall y, \quad \exists\forall xp(x), \quad \exists xp(x) \to Q(x), \quad \forall x.p(x).Q(x)$$

(3) Determine if the following statements hold true (the universe of discourse should be understood from the context)[3]:
   (a) $\forall x.x > 0$.
   Solution. In this contex, x is a number, and there are non positive number such as -1 so the statement is **false**. ☐
   (b) $\exists x.x > 0 \wedge x^2 < 9$.
   Solution. There exists such an x, for example, $x = 1$. ☐
   (c) $\forall x.x > 0 \Rightarrow (\exists y.y > 0 \wedge x > y)$
   Proof. This is true, since for any x, if $x > 0$ the there will always be a number $0 < y < x$, for example $y = \frac{x}{2}$ ☐

It will be convenient to use the notion of quantifiers which are bounded in a given set A:

- $\forall x \in A(p(x))$ denotes the sentence $\forall x(x \in A \Rightarrow p(x))$ and should be understood as "for every x in the set A, $p(x)$ holds true".
- $\exists x \in A(p(x))$ denoted the sentence $\exists x(x \in A \wedge p(x))$ and should be understood as "there is an element x of the set A, such that $p(x)$ holds true".

We think of these quantifiers as quatifiters which range over a given set.

Definition 4.10. Two sentences α, β in the predicate calculus are equivalent if they have the same truth value in every universe.

Observation: In the definition of equivalence of proposition calculus sentences we ranged over truth value assignments, while in the above definition, we ranged over universes.

The negation of the quantifiers can be computed using the following rules:

Theorem 4.11. *(1)* $\sim (\forall x(p(x))) \equiv \exists x(\sim p(x))$.
(2) $\sim (\exists x(p(x))) \equiv \forall x(\sim p(x))$.

Problem 5. *Prove that*

- $\sim (\exists x \in Ap(x)) \equiv \forall x \in A \sim p(x)$.
- $\sim (\forall x \in A, p(x)) \equiv \exists x \in A, \sim p(x)$.

Example 4.12. Negate the following statements without the negation symbol $\sim$:

(1) $(\forall x(2x \neq x)) \vee (2 = 1)$. Solution:

$$\sim [(\forall x(2x \neq x)) \vee (2 = 1)] \equiv [\sim (\forall x(2x \neq x))] \wedge [\sim (2 = 1)] \equiv (\exists x(2x = x)) \wedge 2 \neq 1$$

(2) $\forall x(\exists y(100^x = y + 1))$. Solution:

$$\sim (\forall x(\exists y(100^x = y+1))) \equiv \exists x(\sim (\exists y(100^x = y+1))) \equiv \exists x(\forall y(100^x \neq y+1))$$

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[3] In the future we will require to prove such statement but we do not require it now.

(3) $\forall x(\exists y(x < y))$. Solution:

$$\sim (\forall x(\exists y(x < y))) \equiv \exists x(\sim (\exists y(x < y))) \equiv \exists x(\forall y(\sim (x < y))) \equiv \exists x(\forall y(x \geq y))$$

(4) $\forall \epsilon(\epsilon > 0 \to (\exists \delta(\delta > 0 \wedge (\forall x((0 < x \wedge x < \delta) \to x^2 < \epsilon)))))$. Solution:

$$\begin{gathered}\sim (\forall \epsilon(\epsilon > 0 \to (\exists \delta(\delta > 0 \wedge (\forall x((0 < x \wedge x < \delta) \to x^2 < \epsilon))))) \equiv \\ \exists \epsilon(\sim (\epsilon > 0 \to (\exists \delta(\delta > 0 \wedge (\forall x((0 < x \wedge x < \delta) \to x^2 < \epsilon)))))) \equiv \\ \exists \epsilon((\epsilon > 0) \wedge \sim (\exists \delta(\delta > 0 \wedge (\forall x((0 < x \wedge x < \delta) \to x^2 < \epsilon)))))) \equiv \\ \exists \epsilon([\epsilon > 0] \wedge [\forall \delta. \sim (\delta > 0 \wedge (\forall x((0 < x \wedge x < \delta) \to x^2 < \epsilon)))]) \equiv \\ \exists \epsilon([\epsilon > 0] \wedge [\forall \delta.(\delta \leq 0) \vee \sim (\forall x((0 < x \wedge x < \delta) \to x^2 < \epsilon))]) \equiv \\ \exists \epsilon([\epsilon > 0] \wedge [\forall \delta((\delta \leq 0) \vee \exists x(\sim ((0 < x \wedge x < \delta) \to x^2 < \epsilon)]))) \equiv \\ \exists \epsilon([\epsilon > 0] \wedge [\forall \delta((\delta \leq 0) \vee (\exists x(0 < x \wedge x < \delta \wedge x^2 \geq \epsilon)])))\end{gathered}$$

Definition 4.13. $\exists! x(\ p(x))$ says that there is a unique element x such that $p(x)$ holds true

Problem 6. $\exists! x(p(x))$ *is equivalent to* $\exists x(p(x)) \wedge \forall z \forall w(p(z) \wedge p(w) \Rightarrow z = w)$.

This is equivalent to say that $Tr^U(p(x))$ has a single element.