Homework 11

MATH 300 | (due April 26) | April 19, 2024

Problem 1. Prove that for any n, $\mathbb{R}^n \sim \mathbb{R}$. [Hint: by induction on $n \geq 1$, you can use the result from class that $\mathbb{R} \times \mathbb{R} \sim \mathbb{R}$.]

Solution. For $n = 1$, $\mathbb{R}^1 = \mathbb{R} \sim \mathbb{R}$. Suppose this was true for n and let us prove for $n + 1$.

$\mathbb{R}^{n+1} \sim \mathbb{R}^n \times \mathbb{R} \sim^{(*)} \mathbb{R} \times \mathbb{R} \sim^{(**)} \mathbb{R}$

$(*)$- by the induction hypothesis

$(*)$- theorem in class.

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Problem 2. Prove that for every set A, $A \times A$ is an equivalence relation.

Solution. Reflexive, let $a \in A$, then $\langle a, a\rangle \in A \times A$ by definition.

Symmetric, let $a, b \in A$ and suppose that $\langle a, b\rangle \in A \times A$, then $\langle b, a\rangle \in A \times A$ by definition.

Transitive, let $a, b, c \in A$ and assume that $\langle a, b\rangle, \langle b, c\rangle \in A \times A$, then since $a, c \in A$, by definition of product $\langle a, c\rangle \in A \times A$.

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Problem 3. For each of the following relations check whether it is reflexive symmetric or transitive:

1. $\{\langle a, b\rangle \in \mathbb{N}^2 \mid a - b \equiv 0(mod\ 2)\}$

   Solution This is an ER which is in fact E_2 from class.

2. $\{\langle X, Y\rangle \in P(\mathbb{R}) \times P(\mathbb{R}) \mid X \cap Y \neq \emptyset\}$

   Solution. not reflexive ($\emptyset \cap \emptyset = \emptyset$), symmetric (as $X \cap Y = Y \cap X$) and not transitive ($\{1,2\} \cap \{2,3\} \neq \emptyset$ and $\{2,3\} \cap \{3,4\} \neq \emptyset$ by $\{1,2\} \cap \{3,4\} = \emptyset$)

3. $\{\langle x, y\rangle \in (\mathbb{R} \setminus \{0\})^2 \mid xy > 0\}$

   Solution. This relation is he same and $sign(x) = sign(y)$ where $sign(x) = +/-$ according to whether $x > 0$ and $x < 0$. from this it is trivial to prove that this is an ER.

4. (*optional) $\{\langle f, g\rangle \in {}^{\mathbb{N}}\mathbb{N} \mid \exists N \in \mathbb{N} \forall n \geq N,\ f(n) = g(n)\}$

Problem 4. Prove that if R is a reflexive relation on A if and only if $id_A \subseteq R$.

Solution. Suppose that R is reflexive, and let $\langle a, b \rangle \in Id_A$, then by definition, $a = b$ and since R is reflexive $\langle a, a \rangle \in R$. In the other direction, suppose that $Id_A \subseteq R$. To see that R is reflexive, let $a \in A$, then $\langle a, a \rangle \in Id_A \subseteq R$ and therefore $\langle a, a \rangle \in R$.