Homework 9

MATH 300 (due April 12) April 5, 2024

Problem 1. Prove that if $A \sim B$ and $B \sim C$ then $A \sim C$.

Solution. Let $f : A \to B$ and $g : B \to C$ be bijections witnessing that $A \sim B$ and $B \sim C$. WTP there is a bijection $h : A \to C$. Define $h = g \circ f$, we have proved that the composition of bijections is a bijection,

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Problem 2. Prove the following items:

1. $\mathbb{N} \setminus \{2023, 2024\} \sim \mathbb{N}_{even}$.

   Solution. Define $f : \mathbb{N}_{even} \to \mathbb{N} \setminus \{2023, 2024\}$ by $f(n) = \begin{cases} \frac{n}{2} & n < 4046 \\ \frac{n+4}{2} & n \geq 4046 \end{cases}$

2. $P(\mathbb{N}) \setminus \{\emptyset\} \sim P(\mathbb{N})$.

   Solution Define $f : P(\mathbb{N}) \setminus \{\emptyset\} \to P(\mathbb{N})$, by $f(X) = \begin{cases} \emptyset & X = \{0\} \\ \{\min(X) - 1\} & |X| = 1 \wedge \min(X) > 0 \\ X & otherwise \end{cases}$

3. $(0, 1) \sim (0, \infty)$.

   Solution. Define $f : (0, 1) \to (0, \infty)$ by $f(x) = \frac{1}{x} - 1$

4. $\mathbb{Z} \times [0, 1) \sim \mathbb{R}$.

   Solution. Define $f : \mathbb{Z} \times [0, 1) \to \mathbb{R}$ by $f(z, r) = z + r$.

In all the above solutions you need to prove that the functions defined are bijections.

Problem 3. Prove that for every $\alpha < \beta$ real numbers $(\alpha, \beta) \approx (0, 1)$. [Hint: First stretch/shrink $(0, 1)$ to have length $\beta - \alpha$, then shift it by $+c$ as we did in class.]

Solution. Define $f : (0, 1) \to (\alpha, \beta)$ by $f(x) = (\beta - \alpha)x + \alpha$. Then f is abijection and the inverse function is $g : (\alpha, \beta) \to (0, 1)$ defined by $g(y) = \frac{y-\alpha}{\beta-\alpha}$.

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Problem 4. Show that ${}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \approx {}^{\mathbb{N}}\{0,1\}$.

[Hint: Use the interleaving function $F : ({}^{\mathbb{N}}\{0,1\})^2 \to {}^{\mathbb{N}}\{0,1\}$ defined by

$$F(\langle f,g\rangle)(n) = \begin{cases} f(\frac{n}{2}) & n \in \mathbb{N}_{even} \\ g(\frac{n-1}{2}) & n \in \mathbb{N}_{odd} \end{cases}$$

as the witnessing bijection.]

solution. Let us prove that it is one-to-one, Suppose that $F(\langle f,g\rangle) = F(\langle f',g'\rangle)$, this is a function equality and therefore for every n, $F(\langle f,g\rangle)(n) = F(\langle f',g'\rangle)(n)$. We want to prove that $f = f'$ and $g = g'$. Let $n \in \mathbb{N}$,

$$f(n) = F(\langle f,g\rangle)(2n) = F(\langle f',g'\rangle)(2n) = f'(n)$$

and therefore $f = f'$, similarily, let $n \in \mathbb{N}$ then

$$g(n) = F(\langle f,g\rangle)(2n+1) = F(\langle f',g'\rangle)(2n+1) = g(n)$$

Let us prove that F is onto. Let $h : \mathbb{N} \to \{0,1\}$ be any function, we need to find $\langle f,g\rangle$ such that $F(\langle f,g\rangle) = h$. Define $f(n) = h(2n)$ and $g(n) = h(2n+1)$, then it is not hard to show that $F(\langle f,g\rangle)(n) = h(n)$ for every n and therefore $F(\langle f,g\rangle) = h$.