Mathematical Reasoning
2nd Midterm Perparation-Solutions

March 22, 2024

Instruction

The midterm consists of 3 problems, each worth 34 points (The maximal grade is 100). For this you will have 1 hour during class. The solutions to the problems should be written in the designated areas.

Examples for problems

Problem 1. Prove that $(A \setminus B) \setminus C = (A \setminus C) \setminus (B \setminus C)$.

Proof. We need to prove a set equality. We will prove that by a double inclusion:

- $\underline{(A \setminus B) \setminus C \subseteq (A \setminus C) \setminus (B \setminus C)}$: Let $a \in (A \setminus B) \setminus C$, we want to prove that $a \in (A \setminus C) \setminus (B \setminus C)$. By definition of difference we know that $a \in A \setminus B$ and $a \notin C$ and therefore $a \in A$ and $a \notin B$. Since $a \in A$ and $a \notin C$, it follows that $a \in A \setminus C$ and since $a \notin B$ it follows that $a \notin B \setminus C$. Thus $a \in (A \setminus C) \setminus (B \setminus C)$.

- $\underline{(A \setminus C) \setminus (B \setminus C) \subseteq (A \setminus B) \setminus C}$: Let $a \in (A \setminus C) \setminus (B \setminus C)$ we want to prove that $a \in (A \setminus B) \setminus C$. By definition $a \in A \setminus C$ and $a \notin B \setminus C$. By definition, $a \in A$ and $a \notin C$. Also, since $a \notin B \setminus C$, then $a \notin B$ **or** $a \in C$. Since $a \notin C$, it follows that $a \notin B$. Since $a \in A$ and $a \notin B$, $a \in A \setminus B$. Since $a \notin C$, it follows that $a \in (A \setminus B) \setminus C$.

□

Problem 2. Prove by induction that for every $n \in \mathbb{N}_+$, $1+3+...+(2n-1) = n^2$.

Proof.
- Base: For $n = 1$, we want to prove that $1 = 1^2$ which is clear.
- Induction hypothesis: Suppose that

  $$1 + 3 + 5 + ... + (2n - 1) = n^2$$

  for some general n.
- Induction step: We want to prove that

  $$1 + 3 + 5... + (2n - 1) + (2(n + 1) - 1) = (n + 1)^2$$

  Indeed, by the induction hypothesis

  $$[1 + 3 + .. + (2n - 1)] + (2n + 1) = [n^2] + (2n + 1) = (n + 1)^2$$

□

Problem 3. Prove that for every integer $n > 0$, $n, n + 1$ are coprime.

Proof. Let $n > 0$ be any integer. We want to prove that $n, n+1$ are coprime. Note that

$$1 \cdot (n + 1) + (-1) \cdot n = n + 1 - n = 1$$

Thus 1 is a linear combination of $n + 1$ and n. By the Beźout Identity, this implies that $n, n + 1$ are coprime. □

Problem 4. Prove that for all $n \in \mathbb{N}$, $9^n - 5^n$ is divisible by 4.

Proof. By induction on n,

- Base: For $n = 0$ we have that $9^0 - 5^0 = 1 - 1 = 0$ which is divisible by 4 (since $0 = 4 \cdot 0$).
- Induction hypothesis: Suppose that for some general n, $9^n - 5^n$ is divisible by 4.

- Induction step: We want to prove that $9^{n+1} - 5^{n+1}$ is divisible by 4. Indeed,

  $$9^{n+1} - 5^{n+1} = 9 \cdot 9^n - 5 \cdot 5^n = 9 \cdot 9^n - 9 \cdot 5^n + 4 \cdot 5^n = 9(9^n - 5^n) + 4 \cdot 5^n$$

  By the induction hypothesis $9^n - 5^n$ is divisible by 4 and therefore the term $9(9^n - 5^n)$ divisible by 4. Clearly, the $4 \cdot 5^n$ is divisible by 4 and therefore $9^{n+1} - 5^{n+1}$ is divisible by 4.

$\square$

Problem 5. Express the following sets using the list principle. No proof required.

1. $(-5, 5) \cap \mathbb{Z} = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$.
2. $\{\emptyset, 1\} \times \{n \in \mathbb{N} \mid |P(\{0, \dots, n\})| < 4\} = \{\emptyset, 1\} \times \{0\} = \{\langle \emptyset, 0\rangle, \langle 1, 0\rangle\}$.
   Explanation: Note that $\{n \in \mathbb{N} \mid |P(\{0, \dots, n\})| < 4 = \{n \in \mathbb{N} \mid 2^{n+1} < 4\} = \{0\}$
3. $\{\langle x, y\rangle \in \mathbb{R}^2 \mid x^2 + y^2 = 1\} \cap \{\langle x, x\rangle \mid x \in \mathbb{R}\} = \{\langle -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\rangle, \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle\}$.
   Explanation: a pair in the intersection is a pair of the form $\langle x, x\rangle$ such that $x^2 + x^2 = 1$, namely $2x^2 = 1$ and therefore $x^2 = \frac{1}{2}$. It follows that there exactly two such x's $x = \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}$. Note that this is the intersection of the unit circle or radius 1 around the origin with the line $y = x$.

Problem 6. Compute the following

1. Compute A_3, where A_n is defined recursively by $A_0 = \emptyset$ and $A_{n+1} = A_n \cup \{A_n\}$.
   Solution: $A_0 = \emptyset$
   $A_1 = A_0 \cup \{A_0\} = \emptyset \cup \{\emptyset\} = \{\emptyset\}$
   $A_2 = A_1 \cup \{A_1\} = \{\emptyset\} \cup \{\{\emptyset\}\} = \{\emptyset, \{\emptyset\}\}$
   $A_3 = A_2 \cup \{A_2\} = \{\emptyset, \{\emptyset\}\} \cup \{\{\emptyset, \{\emptyset\}\}\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
2. a_4 where a_n is defined recursively by $a_0 = 0$ and $a_{n+1} = 2^{a_n}$
   Solution: $a_0 = 0$
   $a_1 = 2^{a_0} = 2^2 = 1$
   $a_2 = 2^{a_1} = 2^1 = 2$
   $a_3 = 2^{a_2} = 2^2 = 4$
   $a_4 = 2^{a_3} = 2^4 = 16$