Math 300 Intro Math Reasoning
Worksheet 10: Equivalence relations

(1) For each of the following relation check whether it is reflexive symmetric or transitive:

(1) $E = \{\langle a, b\rangle \in \mathbb{R}^2 \mid a + b = 350\}$.
Solution
Not reflexive (counterexample $0 + 0 \neq 350$), symmetric: suppose that aEb, then $a + b = 350$, since $a + b = b + a$ then bEa. Not transitive (counterexample $0 + 350 = 350$ and $350 + 0 = 350$ but $0 + 0 \neq 350$)

(2) $E = \{\langle a, b\rangle \in \mathbb{R}^2 \mid |a - b| < 1\}$.
Solution. Reflexive let $a \in \mathbb{R}$, then $|a - a| = 0 < 1$, and therefore aEa. Symmetric let $a, b \in \mathbb{R}$, suppose aEb, then $|a - b| < 1$. Since $|a - b| = |b - a|$, we have $|b - a| < 1$ and therefore bEa. Not transitive counterexample $|1.5 - 1| < 1$ and $|1 - 0.5| < 1$ but $|1.5 - 0.5| = 1$.

(3) $E = \{\langle X, Y\rangle \in P(\mathbb{R})^2 \mid 3 \notin X\Delta Y\}$.
Solution. This is an equivalence relation.
Reflexive. Let $X \in P(\mathbb{R})$, then $3 \notin \emptyset X\Delta X$. Thus, XEX.
Symmetric, let $X, Y \in P(\mathbb{R})$, suppose that $3 \notin X\Delta Y$. Since $X\Delta Y = Y\Delta X$, we have that $3 \notin T\Delta X$ and therefore YEX.
Transitive, let $X, Y, Z \in P(\mathbb{R})$, suppose that XEY and YEZ WTP XEZ. By definition $3 \notin X\Delta Y$ and $3 \notin Y\Delta Z$. Note that $X\Delta Z \subseteq X\Delta Y \cup Y\Delta Z$ and therefore $3 \notin X\Delta Z$. It follows that XEZ.

(2) Let $E_n = \{\langle z_1, z_2\rangle \in \mathbb{Z}^2 \mid n \text{ divides } z_1 - z_2\}$. For $n > 1$, Show that $z_1 E_n z_2$ if and only if $z_1 \mod n = z_2 \mod n$

[Hint: write $z_i = q_i n + r_i$ where $0 \leq r_i < n$ for $i = 1, 2$, and recall that if $n|m$ where $0 < n, m$ are natural numbers then $n \leq m$.]

solution. write $z_i = q_i n + r_i$ where $0 \leq r_i < n$ for $i = 1, 2$. Suppose that $z_1 E_n z_2$, then $n|z_1 - z_2$. WLOG suppose that $r_2 \leq r_1$. We have that $z_1 - z_2 = (q_1 - q_2)n + (r_1 - r_2)$ and therefore $n|r_1 - r_2$. But if $r_1 - r_2 > 0$ this would imply that $n \leq r_1 - r_2 \leq r_1 < n$, contradiction. therefore $r_1 - r_2 = 0$ which implies that $r_1 = r_2$. In the other direction, suppose that $r_1 = r_2$, then $z_1 - z_2 = (q_1 - q_2)n$ which is divisible by n as wanted.