Math 300 Intro Math Reasoning
Worksheet 06: Set Theory

(1) Prove by induction for $n \geq 1$ that

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n = \frac{n(n+1)(4n-1)}{3}$$

Solution: By induction.

Base $n = 1$ $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$.
I.H. Assume that

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n = \frac{n(n+1)(4n-1)}{3}$$

Step Let us prove that

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n + (2n+1)(2n+2) = \frac{(n+1)(n+2)(4n+3)}{3}$$

Indeed,

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n + (2n+1)(2n+2) = \frac{n(n+1)(4n-1)}{3} + 2(2n+1)(n+1) =$$

$$\frac{(n+1)[n(4n-1) + 6(2n+1)]}{3} = \frac{(n+1)(4n^2+11n+6)}{3} = \frac{(n+1)(n+2)(4n+3)}{3}$$

as wanted.

(2) Prove that $A \subseteq B$ if and only if $P(A) \subseteq P(B)$.

Solution Let us prove the double implication. Suppose that $A \subseteq B$. WTP $P(A) \subseteq P(B)$. Let $X \in P(A)$. WTP $X \in P(B)$. By definition of powerset, $X \in P(A)$, $X \subseteq A$ and since $A \subseteq B$, $X \subseteq B$. Hence $X \in P(B)$. In the other direction, suppose that $P(A) \subseteq P(B)$. WTP $A \subseteq B$. Note that since $A \subseteq A$, then $A \in P(A)$ and since $P(A) \subseteq P(B)$, $A \in P(B)$. By definition of powerset again, $A \subseteq B$.

(3) Define

$$t \cdot \langle \alpha_1, ..., \alpha_n \rangle = \langle t \cdot \alpha_1, ..., t \cdot \alpha_n \rangle$$

and denote by $\vec{0} = \langle 0, 0, ..., 0 \rangle$. Prove that for every $t \in \mathbb{R}$ and $\vec{\alpha} \in \mathbb{R}^n$, if $t \cdot \vec{\alpha} = \vec{0}$, then either $t = 0$ or $\vec{\alpha} = \vec{0}$.