# Homework 10-Solution

MATH 361 (due Dec 9) Dec 2, 2022

**Problem 1.** Suppose that $\langle A, <_A\rangle$ is a well-ordered set. Prove that if $f : A \to A$ is order-preserving then $f = id_A$.

**Solution.** Otherwise, let $D = \{x \in A \mid f(x) \neq x\} \neq \emptyset$. Let $x^* = \min(D)$. Then for every $x <_A x^*$, $f(x) = x$ by minimality of $x^*$. Hence $f(x^*) >_A x^*$ (otherwise, $f$ is not one-to-one). Since $f$ is an isomorphism, there is $y \in A$ such that $f(y) = x^*$. Since $f(y) <_A f(x^*)$, then $y <_A x^*$, but then $x^* = f(y) = y$, contradiciton.

**Problem 2.** Prove that if $A$ is countable the $A$ can be well-ordered.

Instruction: Split into two cases- first prove that every linear strong order on a finite set is a well order. If $A$ is infinitely countable, then by taking any bijection $f : \mathbb{N} \to A$, we can define $<_A$ on $A$ by $a <_A b$ if and only if $f^{-1}(a) < f^{-1}(b)$. Prove that $\langle A, <_A\rangle \simeq \langle \mathbb{N}, <\rangle$ and deduce that $\langle A, <_A\rangle$ is a well ordered set.

**Solution.** Take any injection $f : A \to \mathbb{N}$, we can define $<_A$ on $A$ by $a <_A b$ if and only if $f(a) < f(b)$. Then by definition, $f$ is order-preserving and injective. Note that $a <_A b <_A c$ then $f(a) < f(b) < f(c)$ and therefore $f(a) < f(c)$ so $a <_A c$ (namely $<_A$ is transitive). If $a <_A b$ then $f(a) < f(b)$ so $f(b) \not< f(a)$ and therefore $b \not< Aa$. It is linear since for every $a, b \in A$, $f(a), f(b)$ are comparable hence $a, b$ are $<_A$-comparable. So far we proved that $<_A$ is a strong linear order on $A$. Let us prove that it is a well order. Let $X \subseteq A$, be a non-empty set. Then $f''X \subseteq \mathbb{N}$ is non-empty and therefore there is $n^* = \min(f''X)$. Let $x \in X$ be such that $f(x) = n^*$, then for every $y \in X$, $f(y) \geq n^* = f(x)$ and therefore $y \geq_A x$. It follows that $x = \min_{<A}(X)$.

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**Problem 3.** Prove that if $\langle A, <_A \rangle$ is a well-ordered set and $X \subseteq A$ is an initial segment (i.e. $\forall x \in X \forall a \in A,\ a <_A x \Rightarrow a \in X$) then either $X = A$ or $\exists a \in A$ such that $X = A_{<A}[a]$.

Hint: If $X \neq A$ let $a = \min_{<_A}(A \setminus X)$ (why does it exists?), prove that $X = A_{<A}[a]$.

**Solution.** If $X = A$ we are done. Otherwise, $X \subsetneq A$, so $A \setminus X \neq \emptyset$. Let $a = \min_{<A}(A \setminus X)$ and let us prove that $A_{<A}[a] = X$ by double inclusion. If $b <_A a$, then $b \in A$ and $b \notin A \setminus X$ (otherwise this would contradict the minimality of $a$) and therefore $b \in X$. If $x \in X$, then $x, a$ are $<_A$-comperable. If $a = x$, then $a \in X$, contradiction. If $a <_A x$, then $a \in X$ since $X$ is an initial segment. Hence $x <_A a$, namely $x \in A_{<A}[a]$.

**Problem 4.** Prove that the axiom of foundation implies that there is no $x$ such that $x \in x$.

**Solution.** Suppose otherwise, and let $x \in x$. Define $\{x\}$. By the axiom of foundation there is $y \in \{x\}$ such that $y \cap \{x\} = \emptyset$. But then $y = x$ and $x \in x \cap \{x\}$, contradiction.

**Problem 5.** Prove that if $A$ is a set of ordinals then $\bigcup A$ is an ordinal

**Solution.** First let us note that $\bigcup A$ is a transitive set. If $x \in y \in \bigcup A$, then there is $\alpha \in A$ such that $y \in \alpha$. Since $\alpha$ is transitive it follows that $x \in \alpha$ and therefore $x \in \bigcup A$. To see that $\in$ well-orders $\bigcup A$, let $x \in y \in z$ all in $\bigcup A$, there there are $\alpha, \beta, \gamma \in A$ such that $a \in \alpha, y \in \beta, z \in \gamma$. Since every two ordinals are comparable, WLOG $\alpha, \beta \subseteq \gamma$ and therefore $x, y, z \in \gamma$. Since $\in$ well orders $\gamma$, $x \in z$. So $\in$ is transitive on $\bigcup A$. It is then strongly anti-symmetric, since if $x \in y$ and $y \in x$, there are $\alpha, \beta \in A$ such that $a \in \alpha$ and $y \in \beta$. WLOG $\alpha \leq \beta$, $x, y \in \beta$ but $\in$ well orders $\beta$, contradiction. A similar argument shows that $\in$ is linear on $\bigcup A$. Let $X \subseteq \bigcup A$ such that $X \neq \emptyset$. Take any $\alpha \in A$ such that $X \cap \alpha \neq \emptyset$ (there exists such $\alpha$ since $X \subseteq \bigcup A$ is nonempty) The $X \cap \alpha$ is a non-empty subset of $\alpha$ and since $\in$ well-orders $\alpha$, there is $x = \min_{\in}(X \cap \alpha)$. We claim that $x = \min_{\in}(X)$. Let $y \in X$, then $y \in \cup A$, then there is $\beta$ such that $y \in \beta$. If $\beta \leq \alpha$, then $y \in \alpha$ and therefore $y \in X \cap \alpha$ in which case $x \leq y$. Otherwise, $\alpha < \beta$ and then $x, \alpha, y \in \beta$ so $x, y$ are $\in$-comparable if $y \in x$ then $y \in \alpha$ which then imply that $y \in X \cap \alpha$ contradicting the minimality of $x$. Otherwise, $x \leq y$ as wanted.

and moreover $\bigcup A = \sup(A)$ i.e.:

1. $\bigcup A$ is an upper bound for $A$, namely, for every $\alpha \in A$, $\alpha \leq \bigcup(A)$.

   **Solution.** If $\alpha \in A$, then $\alpha \subseteq \bigcup A$ and therefore by the lemma we saw in class $\alpha \leq \bigcup A$.

2. If $\beta \in On$ is an upper bound for $A$ then $\beta \geq \bigcup A$.

   **Solution.** If $\beta$ is an upper bound for $A$, then for every $\alpha \in A$, $\alpha \leq \beta$, namely $\alpha \subseteq \beta$. It follows that $\bigcup A$ is a union of subseteq pf $\beta$ and therefore itself a subset of $\beta$. we conclude that $\bigcup A \leq \beta$.

## Additional problems

**Problem 6.** Suppose that $\langle A, <_A \rangle, \langle B, <_B \rangle$ are well ordered sets such that $A \cap B = \emptyset$. Define $<_+$ on $A \uplus B$ by $x <_+ y$ if:

- $x, y \in A$ and $x <_A y$. or
- $x, y \in B$ and $x <_B y$. or
- $x \in A$ and $y \in B$.

Prove that $<_+$ is a well ordering of $A \uplus B$.

**Problem 7.** Suppose that $\langle A, <_A \rangle, \langle B, <_B \rangle$ are well orders. Define the lexicographic order on $A \times B$ as follows:

$$\langle a, b \rangle <_{Lex} \langle a', b' \rangle \text{ iff } a <_A a' \vee (a = a' \wedge b <_B b')$$

Prove that $\langle A \times B, <_{Lex} \rangle$ is a well ordering.

**Problem 8.** Prove that if $\alpha$ is an ordinal then $\alpha \cup \{\alpha\}$ is an prdinal.

**Problem 9.** Prove that if $C \neq \emptyset$ is a set of ordinals then $\bigcap C$ is an ordinal and $\bigcap C = \min_{\in}(C)$.

**Problem 10.** Prove that if $X$ is transitive than $P(X)$ is transitive.