Homework 2

MATH 361 (due September 29) September 22, 2023

Problem 1. Prove or disprove the following items:

1. If $f : A \to B$ is injective, then for every $X \subseteq A$, $f \restriction X$ is injective.

2. If $f : A \to B$ is surjective, then for every $X \subseteq A$, $f \restriction X$ is surjective.

Problem 2. Prove that if $f : A \to B$ is a function such that for some $X \subsetneq A$, $f \restriction X : X \to B$ is onto B, then f is not injective.

Problem 3. For each of the following functions, determine if it is injective/ surjective and prove your answer for two of the items which are not the first ones.

1. $f_1 : \mathbb{R} \to \mathbb{R}$, defined by $f_1(x) = 5x - x^2$.

2. $f_2 : \mathbb{R} \to P(\mathbb{R})$, defined by $f_2(x) = \{x^2\}$.

3. $f_3 : P(\mathbb{R}) \to P(\mathbb{N})$, defined by $f_3(x) = x \cap \mathbb{N}$.

4. $f_4 : P(\mathbb{N}) \to \mathbb{N}$, defined by $f_4(x) = \begin{cases} \min(x) & 4 \in x \\ 0 & else \end{cases}$.

5. $f_5 : P(\mathbb{R}) \to P(\mathbb{N}) \times P(\mathbb{Z}) \times P(\mathbb{Q})$, defined by

$$f_5(X) = \langle X \cap \mathbb{N}, X \cap \mathbb{Z}, X \cap \mathbb{Q} \rangle$$

6. $f_6 : P(\mathbb{N}) \to P(\mathbb{N}_{even}) \times P(\mathbb{N}_{odd})$ defined by $f_6(X) = \langle \{2n \mid n \in X\}, \{2n+1 \mid n \in X\} \rangle$.

7. $f_7 : {}^{\mathbb{R}}R \to \mathbb{R}$ defined by $f_7(f) = f(7)$.

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Problem 4. For a function $f : A \to B$ and $C \subseteq A$ define the *pointwise image of C by f* as

$$f''C = \{f(c) \mid c \in C\}$$

(a) Prove that if $f : A \to B$ is a function and $C \subseteq A$, then

$$(f''A) \setminus (f''C) \subseteq f''[A \setminus C].$$

(b) Give an example of a function $f : A \to B$ and a subset $C \subseteq A$ such that

$$(f''A) \setminus (f''C) \neq f''[A \setminus C].$$

(c) Prove that if $f : A \to B$ is an injection and $C \subseteq A$, then

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

Problem 5. Prove that the interleaving function $F : ({}^{\mathbb{N}}\{0,1\})^2 \to {}^{\mathbb{N}}\{0,1\}$ defined by

$$F(\langle f, g \rangle)(n) = \begin{cases} f(\frac{n}{2}) & n \in \mathbb{N}_{even} \\ g(\frac{n-1}{2}) & n \in \mathbb{N}_{odd} \end{cases}$$

is one-to-one and onto. Prove that it is invertable and find F^{-1}.

Additional Problems

Problem 6. Let $f : A \to B$ and $g : B \to C$ be function. Prove the following items:

1. If f, g are injective then $g \circ f$ is injective.

2. If f, g are surjective, then $g \circ f$.

Problem 7. Prove that the following functions are invertible and find their inverse:

1. $h : (0, \infty) \to (0, 1)$ defined by $h(x) = \frac{1}{1+x^2}$
2. $f : \mathbb{N} \to \mathbb{N}$ defined by $f(n) = \begin{cases} n+1 & n \in \mathbb{N}_{even} \\ n-1 & n \in \mathbb{N}_{odd} \end{cases}$.
3. $g : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z} \times \mathbb{Z}$ defined by $g(\langle n, m \rangle) = \langle n, n+m \rangle$

Problem 8. Define

$$f_1 : \mathbb{N} \to \mathbb{N} \times \mathbb{N}, \ \ f_1(n) = \langle n+1, n+2 \rangle$$

$$f_2 : \mathbb{N} \to \mathbb{N}, \ \ f_2(n) = n^2$$

$$f_3 : \mathbb{N} \times \mathbb{N} \to \mathbb{Z}, \ \ f_3(\langle n, m \rangle) = n - m$$

$$f_4 : \mathbb{N} \to \mathbb{N}, \ \ f_4(n) = n + 1$$

Determine if the following compositions are defined and compute them:

1. $f_1 \circ f_2$ and $f_2 \circ f_1$.
2. $f_2 \circ f_2$. and $f_3 \circ f_3$
3. $f_4 \circ f_2$ and $f_2 \circ f_4$.
4. $f_3 \circ f_1 \circ f_2$ and $f_4 \circ f_3 \circ f_2$.

Problem 9. Let $A, B \neq \emptyset$ be any set and let $f : A \to B$ be a function. Define a new function using f, as follows, $F : P(A) \to P(B)$ defined by $F(X) = \{f(x) \mid x \in X\}$. Prove that f is invertible if and only if F is invertible.