Homework 3-Solutions

MATH 361 (due October 9) September 29, 2023

Problem 1. Prove that the collection of all ordered pairs of the form $\langle 1, x\rangle$ does not form a set. [Hint: Prove by contradiction]

Solution. Suppose otherwise that there is a set D which is the collection of all pairs $\langle 1, x\rangle$. Then by the union axiom $E = \cup(\cup D)$ is also a set. Let us show that E is the set of all sets, let x be any set, then $\langle 1, x\rangle \in D$, and since $\{1, x\} \in \{\{1\}, \{1, x\}\} = \langle 1, x\rangle$, $\{1, x\} \in \cup D$ by definition of union. Again by definition of union it follows that $x \in \cup \cup D$. This is a contradiction to the theorem we proved in class tha the set of all sets do not exists.

Problem 2. Let R be a relation on A. Prove the following statements:

1. R is reflexive if and if $id_A \subseteq R$.

   Solution. Suppose that R is reflexive, and let $\langle a, b\rangle \in Id_A$, then by definition, $a = b$ and since R is reflexive $\langle a, a\rangle \in R$. In the other direction, suppose that $Id_A \subseteq R$. To see that R is reflexive, let $a \in A$, then $\langle a, a\rangle \in Id_A \subseteq R$ and therefore $\langle a, a\rangle \in R$.

2. R is symmetric if and only if $R = R^{-1}$.

3. R is transitive if and only if $R \circ R \subseteq R$.

   Solution. Suppose that R is transitive, and let $\langle a, c\rangle \in R \circ R$, then by definition of composition there is $b \in A$ such that $\langle a, b\rangle, \langle b, c\rangle \in R$. Since R is transitive it follows that $\langle a, c\rangle \in R$. Hence $R \circ R \subseteq R$. In the other direction, Suppose that $R \circ R \subseteq R$ and let $\langle a, b\rangle, \langle b, c\rangle \in R$. Then by definition $\langle a, c\rangle \in R \circ R$ and by inclusion $\langle a, c\rangle \in R$.

4. R is an equivalence relation if and only if R is reflexive and $R \circ R^{-1} \subseteq R$.

Solution. If R is an ER then it is reflexive. also by (2) $R = R^{-1}$ and by (3)

$$R^{-1} \circ R = R \circ R \subseteq R$$

In the other direction, suppose that R is reflexive and $R^{-1} \circ R \subseteq R$. To see that R is an ER it remains to see that R is symmetric and transitive, suppose that $\langle a, b\rangle \in R$ then $\langle b, a\rangle \in R^{-1}$ and since R is reflexive $\langle a, a\rangle \in R$. By definition of composition $\langle b, a\rangle \in R^{-1} \circ R$ an by assumption $\langle b, a\rangle \in R$. Hence R is symmetric. By (2) it follows that $R = R^{-1}$ and therefore our assumption translates to $R \circ R \subseteq R$ which by (3) implies that R is transitive.

Problem 3. Let $\Pi \subseteq P(A) \setminus \{\emptyset\}$. Define

$$F_\Pi = \{\langle x, X\rangle \in A \times \Pi \mid x \in X\}$$

prove that $F_\Pi : A \to P(A)$ is a function of and only if Π is a partition.

Solution. Suppose that F_Π is a function (namely total on A and univalent) and let us prove that Π is a partition. Clearly $\emptyset \notin \Pi$ as by assumption $\Pi \subseteq P(A) \setminus \{\emptyset\}$. Also if $X, Y \in \Pi$ are distinct then $X \cap Y = \emptyset$, just otherwise, there is $x \in X \cap Y$ and then $\langle x, X\rangle, x, Y\rangle \in F_\Pi$, contradicting F_Π being univalent. Finally, $\cup\Pi \subseteq A$ as $\Pi \subseteq P(A)$. For the other direction, let $a \in A$, since F_Π is total, there is $X \in \Pi$ such that $\langle a, X\rangle \in F_\Pi$ and therefore $a \in X$. By definition of union it follows that $a \in \cup\Pi$.

In the other direction, suppose that Π is a partition. To see that F_Π is total, let $a \in A$, then $a \in \cup\Pi$ and therefore there is $X \in \Pi$ such that $a \in X$. by definition this means that $\langle a, X\rangle \in F_\Pi$. To see that F_Π is univalent, suppose that $a, X\rangle, \langle a, Y\rangle \in F_\Pi$, then $a \in X \cap Y$ and since Π is a partition, $X = Y$.

Problem 4. For each of the following relation check whether it is reflexive symmetric or transitive:

1. $\{\langle a, b\rangle \in \mathbb{R}^2 \mid a + b = 350\}$.

   Solution. We only give the short ideas of the proofs Not reflexive ($0+0 \neq 350$), symmetric ($a+b = b+a$) and not transitive ($0+350 = 350$ and $350 + 0 = 350$ but $0 + 0 \neq 350$)

2. $\{\langle a, b\rangle \in \mathbb{R}^2 \mid |a - b| < 1\}$.

   Solution. Reflexive ($|a - a| = 0 < 1$), symmetric ($|a - b| = |b - a|$) and not transitive ($|1.5 - 1| < 1$ and $|1 - 0.5| < 1$ but $|1.5 - 0.5| = 1$)

3. $\{\langle a, b\rangle \in \mathbb{N}^2 \mid a - b \equiv 0(mod\ 2)\}$

   Solution. This is an ER which is in fact E_2 from class.

4. $\{\langle X, Y\rangle \in P(\mathbb{R}) \times P(\mathbb{R}) \mid X \cap Y \neq \emptyset\}$

   Solution. not reflexive ($\emptyset \cap \emptyset = \emptyset$), symmetric (as $X \cap Y = Y \cap X$) and not transitive ($\{1, 2\} \cap \{2, 3\} \neq \emptyset$ and $\{2, 3\} \cap \{3, 4\} \neq \emptyset$ by $\{1, 2\} \cap \{3, 4\} = \emptyset$)

Problem 5. For each of the following equivalence relation find a system of representatives. No proof required here.

1. id_A, where A is a general set.

Solution. A is a system for Id_A.

2. $A \times A$, where A is a general set.

   Solution. If $a \in A$ is any element then $\{a\}$ is a system for $A \times A$

3. The relation Res on $\mathbb{R}$ defined as follows: for $a, b \in \mathbb{R}$

$$\langle a, b\rangle \in Res \leftrightarrow b - a \in \mathbb{Z}$$

   Solution. $[0, 1)$.

4. $\{\langle X, Y\rangle \in P(\mathbb{R})^2 \mid 3 \notin X\Delta Y\}$.

   Solution. $\{\{3\}, \emptyset\}$

5. $\{\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in (\mathbb{R} \times \mathbb{R})^2 \mid \min(x, y) = \min(a, b)\}$

   Solution. $\{\langle r, r + 1\rangle r \in \mathbb{R}\}$.

6. for $X = \{0, 1\}^{\{0,\dots,10\}}$. define

$$\{\langle f, g\rangle \in X \times X \ \mid \ \Big|\{n \mid f(n) = 1\}\Big| = \Big|\{n \mid g(n) = 1\}\Big|\ \}$$

   Solution. $\{f_i \mid i = 0, \dots, 11\}$ where $f_i : \{0, \dots, 10\} \to \{0, 1\}$ defined by

$$f_i(n) = \begin{cases} 1 & n < i \\ 0 & n \geq i \end{cases}$$

7. $\{\langle a, b\rangle \in \mathbb{N}^+ \times \mathbb{N}^+ \mid \exists i \in \mathbb{Z}. \frac{a}{b} = 2^i\}$

   Solution. $\mathbb{N}_{odd} \cup \{2\}$

8. $\{\langle x, y\rangle \in (\mathbb{R} \setminus \{0\})^2 \mid xy > 0\}$

   Solution. $\{-1, 1\}$

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Additional Problems

Problem 6. Let $f : A \to B$ be a function. Prove that if $X \subseteq A$, then $f \cap X \times B$ is a function and equals $f \upharpoonright X$.

Problem 7. Show that if $f : A \to B$, $g : B \to C$ are functions then $g \circ f$ (the composition of the relations) is a function from A to C and that for every $a \in A$, $g \circ f(a) = g(f(a))$.

Problem 8. Prove that if f is one-to-one and onto B then f^{-1} (the inverse relation) is a function and moreover that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.