Homework 3

MATH 361 (due October 9) September 29, 2023

Problem 1. Prove that the collection of all ordered pairs of the form $\langle 1, x\rangle$ does not form a set. [Hint: Prove by contradiction]

Problem 2. Let R be a relation on A. Prove the following statements:

1. R is reflexive if and if $id_A \subseteq R$.
2. R is symmetric if and only if $R = R^{-1}$.
3. R is transitive if and only if $R \circ R \subseteq R$.
4. R is an equivalence relation if and only if R is reflexive and $R \circ R^{-1} \subseteq R$.

Problem 3. Let $\Pi \subseteq P(A) \setminus \{\emptyset\}$. Define

$$F_\Pi = \{\langle x, X\rangle \in A \times \Pi \mid x \in X\}$$

prove that $F_\Pi : A \longrightarrow P(A)$ is a function of and only if Π is a partition.

Problem 4. For each of the following relation check whether it is reflexive symmetric or transitive:

1. $\{\langle a, b\rangle \in \mathbb{R}^2 \mid a + b = 350\}$.
2. $\{\langle a, b\rangle \in \mathbb{R}^2 \mid |a - b| < 1\}$.
3. $\{\langle a, b\rangle \in \mathbb{N}^2 \mid a - b \equiv 0 (mod\ 2)\}$
4. $\{\langle X, Y\rangle \in P(\mathbb{R}) \times P(\mathbb{R}) \mid X \cap Y \neq \emptyset\}$

Problem 5. For each of the following equivalence relation find a system of representatives. No proof required here.

1. id_A, where A is a general set.

2. $A \times A$, where A is a general set.

3. The relation Res on $\mathbb{R}$ defined as follows: for $a, b \in \mathbb{R}$

$$\langle a, b\rangle \in Res \leftrightarrow b - a \in \mathbb{Z}$$

4. $\{\langle X, Y\rangle \in P(\mathbb{R})^2 \mid 3 \notin X \Delta Y\}$.

5. $\{\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in (\mathbb{R} \times \mathbb{R})^2 \mid \min(x, y) = \min(a, b)\}$

6. for $X = \{0, 1\}^{\{0,\dots,10\}}$. define

$$\{\langle f, g\rangle \in X \times X \ \mid \ \Big|\{n \mid f(n) = 1\}\Big| = \Big|\{n \mid g(n) = 1\}\Big| \ \}$$

7. $\{\langle a, b\rangle \in \mathbb{N}^+ \times \mathbb{N}^+ \mid \exists i \in \mathbb{Z}. \frac{a}{b} = 2^i\}$

8. $\{\langle x, y\rangle \in (\mathbb{R} \setminus \{0\})^2 \mid xy > 0\}$

Additional Problems

Problem 6. Let $f : A \to B$ be a function. Prove that if $X \subseteq A$, then $f \cap X \times B$ is a function and equals $f \upharpoonright X$.

Problem 7. Show that if $f : A \to B$, $g : B \to C$ are functions then $g \circ f$ (the composition of the relations) is a function from A to C and that for every $a \in A$, $g \circ f(a) = g(f(a))$.

Problem 8. Prove that if f is one-to-one and onto B then f^{-1} (the inverse relation) is a function and moreover that $f^{-1} \circ f = Id_A$ and $f \circ f^{-1} = Id_B$.