Homework 7

MATH 361 (due November 11) November 4, 2023

Problem 1. Prove that if E is an equivalence relation on A and let A' be a system of representatives.

1. Prove that $A/E \approx A'$.

2. Conclude that $A/E \preceq A$. [Remark: first prove it using the previous item (one line proof). Then try to prove it directly by finding an onto function from A to A/E].

Problem 2. Let $A = \mathbb{N}^{\mathbb{N}}$, and consider the equivalence relation $R = \{\langle f, g\rangle \in (\mathbb{N}^{\mathbb{N}})^2 \mid f(0) = g(0)\}$ in A (no need to prove that). Prove that $A/R \approx \mathbb{N}$. [Hint: Use problem 1]

Problem 3. Prov by a diagonalization argument that $\mathbb{N} \prec {}^{\mathbb{N}}\mathbb{N}_{even}$.

Problem 4. Prove that the set of all matrices (of any dimension) with rational entries is countable. [Hint: a countable union of countable sets]

Problem 5. Prove that $\{X \in P(\mathbb{N}) \mid X \approx \mathbb{N}\} \approx P(\mathbb{N})$. [Hint: Cantor-Schroeder-Bernstein]

1 Additional Problems

Problem 6. A function $f : A \to B$ is called countable-to-one if every $b \in B$ has at most countably many preimages. Namely, if for every $b \in B$, the following set is countable:

$$\{a \in A \mid f(a) = b\}$$

1. Give an example of a function which is countable-to-one but not one-to-one.

2. Suppose that A is a set such that there exists a countable-to-one function $f : A \to \mathbb{Q}$. Prove that A is countable. [Hint: countable union of countable sets is countable]

Problem 7. On ${}^{\mathbb{N}}\{0,1\}$, define the equivalence relation E by fEg if and only if there is N such that for every $n \geq N$, $f(n) = g(n)$.

Prove that ${}^{\mathbb{N}}\{0,1\}/E \approx {}^{\mathbb{N}}\{0,1\}$. [Guidence: In order to prove that ${}^{\mathbb{N}}\{0,1\} \preceq {}^{\mathbb{N}}\{0,1\}/E$, decompose $\mathbb{N}$ to infinitely many infinite disjoint sets $\mathbb{N} = \biguplus_{n\in\mathbb{N}} A_n$. Try to use such a decomposition to define a function $F : {}^{\mathbb{N}}\{0,1\} \to {}^{\mathbb{N}}\{0,1\}$ which duplicates each value of the in input value f (i.e. duplicates the values $f(n)$) infinitely many times]