Homework 8

MATH 361 | (due November 20) | November 11, 2022

Problem 1. Prove that $\{X \in P(\mathbb{N}) \mid X \text{ is infinite}\} \simeq P(\mathbb{N})$

Problem 2. Determine the cardinality ($\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$) of the following sets (submit only 3 of the items):

(1) $A = \{f \in {}^{\mathbb{N}}\{0,1\} \mid \forall n \in \mathbb{N}_{even}, f(n) = 1\}$.

(2) $B = \{X \in P(\mathbb{N}) \mid X \text{ contains no consecutive numbers}\}$.

(3) The set of all arithmetic progressions of integers. [Recall: an arithmetic progression of integers is a sequence $(a_n)_{n=0}^{\infty}$ such that for some d, for any n, difference $a_{n+1} - a_n = d$.]

(4) The set of all circles in the plain.[Given a point $p = \langle x_0, y_0 \rangle \in \mathbb{R}^2$ ("the center") and $r \in (0, \infty)$ ("the radious"), the circle $C = C(p, r) = \{\langle x, y \rangle \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 = r^2\}$. A]

(5) The set of all circles C in $\mathbb{R}^2$ which intersect the x-axis at two points $\langle 0, q_1 \rangle, \langle 0, q_2 \rangle$, where $q_1, q_2 \in \mathbb{Q}$.

Problem 3. A straight line in the plain is a set of the following forms:

- $L_c = \{c\} \times \mathbb{R}$ for some $c \in \mathbb{R}$ (lines which are parallel to the y-axis).
- $L_{a,b} = \{\langle x, y \rangle \in \mathbb{R} \mid y = ax + b\}$ for some $a, b \in \mathbb{R}$. (lines which are not parallel to the y-axis)

Answer the following questions:

1. What is the cardinality of the set of all lines in the plain?
2. Prove that there is a line L which contains no rational point, namely $L \cap \mathbb{Q} \times \mathbb{Q} = \emptyset$.

3. Prove that every line L contains an irrational point, namely $L \cap (\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$.

Problem 4. A function $f : \mathbb{N} \to \mathbb{N}$ is increasingly monotone, if for every n, $f(n) < f(n+1)$. Prove that the set A of all increasingly monotone functions $f : \mathbb{N} \to \mathbb{N}$ has cardinality $2^{\aleph_0}$. [Hint: CSB. One direction is easy. For the other, given a function $f : \mathbb{N} \to \mathbb{N}_+$, define $F(f)(n) = \sum_{k=0}^{n} f(n)$.]

Problem 5. Prove that $\aleph_0^{(2^{\aleph_0})} = 2^{(2^{\aleph_0})}$.

Problem 6. Prove that $\kappa^{\lambda+\sigma} = \kappa^{\lambda} \cdot \kappa^{\sigma}$.

1 Additional problems- preparation for midterm II

Problem 7. Compute the cardinality of the set of all function $f : \mathbb{N} \to \{0,1\}$ with no consecutive zeros. Namely, there is no $n \in \mathbb{N}$ such that $f(n) = f(n+1) = 0$.

Problem 8. Consider the relation E om ${}^{\mathbb{N}}\mathbb{N}$ by fEg if and only if for every $n \geq 100$, $f(n) = g(n)$.

1. Prove that E is an equivalence relation.
2. Compute the cardinality of ${}^{\mathbb{N}}\mathbb{N}/E$.

Problem 9. Let $\leq_A, \leq_B$ be two weak linear orderings of A, B (resp.), where A, B are disjoint. We define $\leq_A + \leq_B$ which we abbreviate by $\leq_+$ on AB as follows:

$$x \leq_+ y \leftrightarrow (x, y \in A \wedge x \leq_A y) \vee (x, y \in B \wedge x \leq_B y) \vee (x \in A \wedge y \in B)$$

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1. Prove that $\leq_+$ is a linear ordering of $A \cup B$.

2. Let $\mathbb{N}^* = \{0\} \times \mathbb{N}$ and define $\leq^*$ on $\mathbb{N}^*$ by $\langle 0, n\rangle \leq^* \langle 0, m\rangle$ if and only if $m \leq n$. Prove that $\leq^*$ is a linear ordering of $\mathbb{N}^*$.

3. Prove that $\langle \mathbb{N}^* \cup \mathbb{N}, \leq^* + \leq\rangle \simeq \langle \mathbb{Z}, \leq\rangle$.

Problem 10. Define recursively $A_0 = \emptyset$ and $A_{n+1} = P(A_n)$. Prove by induction that for every n, $A_n \subseteq A_{n+1}$.

Problem 11. Prove that the intersection of finitely many Dedekind cuts is a Dedekind cut.

Problem 12. Prove that if $x \in \mathbb{R}$ and $y \in \mathbb{R}$ is positive (namely $0 < y$), then $x < x + y$.