Homework 9-Solutions

MATH 361 (due December 8) November 25, 2023

Problem 1. In this exercise, we consider the statement α: "Suppose that $\mathcal{A}$ is a set of open pairwise disjoint intervals in $\mathbb{R}$, then $|\mathcal{A}| \leq \aleph_0$."

(1) Prove α: for every $\emptyset \neq I \in \mathcal{A}$ there is a rational $q_I \in I \cap \mathbb{Q}$ (why?), the define $f(I) = q_I$ and prove that f is injective.

Solution. $I \cap \mathbb{Q}$ is not embety since the rationals are dense in the reals. Suppose that $f(I) = f(J)$, then $q = q_I = q_J \in I \cap J$ but then $I = J$ since distinct intervals must be disjoint.

(2) Did you use the axiom of choice in the previous proof? if so, where did you use it?

Solution. We used the the axiom of choice when we chose $q_I \in I \cap \mathbb{Q}$, as there are potentially infinitely many intervals in $\mathcal{A}$.

Problem 2. Show that if A is countable then there is a choice function for $P(A) \setminus \{\emptyset\}$

Solution. Suppose that A is countable and let $f : A \to \mathbb{N}$ be injective. so f is invertible on $Im(f)$ and $f^{-1} : Im(f) \to A$ is defined. Define $F : P(A) \setminus \{\emptyset\} \to A$ by $F(A) = f^{-1}(\min f''A)$. Namely, $F(A) = a$ such that $f(a)$ is minimal among $f''A$. Then clearly, $F(A) \in A$ since $F(A) = f^{-}1(f(a)) = a$ for some $a \in A$.

Problem 3. Let SRP be the principle that for every set A and for every equivalence relation E on A, there exists a system of representatives.

Prove that the Axiom of Choice is equivalent to SRP.

Proof. The axiom of choice implies SRP since if E is an equivalence relation of A, then A/E is a set of non-empty sets. By the axiom of choice there is a choice function $F : A/E \to A$. Then $Im(F)$ is a system of representatives. In the other direction, Suppose that SRP holds and let $\mathcal{A}$ be a set of non-empty sets. Define $\mathcal{B} = \{\{A\} \times A \mid A \in \mathcal{A}\}$. Note that If $A \neq B$ then $\{A\} \times A$ and $\{B\} \times B$ are disjoint. On $X = \bigcup \mathcal{B}$, we let R be the equivalence relation induced from the partition $\mathcal{B}$ (it is not hard to check that the relation R is just $\langle x, y\rangle R \langle z, t\rangle$ iff $x = z$). Let $A' \subseteq X$ be a system of representatives, then for every $A \in \mathcal{A}$, there is a unique pair $\langle A, a\rangle \in A'$. Hence $F = A' : \mathcal{A} \to \bigcup \mathcal{A}$ is a choice function. □

1 Additional problems

Problem 4. The principle of Dependent choice (DC) is the following: For every total relation R on A there is a sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that for every n. $x_n R x_{n+1}$. Prove that DC follows from AC.

Remark: It is known that DC does not imply AC. For more information about DC see Link