Problem 1. Prove that if A, B, C are sets then

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Solution. We prove a double inclusion:

$\subseteq$: Let $x \in A \cup (B \cap C)$. The by the definition of union either $x \in A$ or $x \in B \cap C$. We split into cases:

(a) If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$ by the definition of union. By definition of intersection $x \in (A \cup B) \cap (A \cup C)$.

(b) If $x \in B \cap C$, then by definition on intersection $x \in B$ and $x \in C$. Hence $x \in A \cup B$ and $x \in A \cup C$ by the definition of union and again by definition of intersection $x \in (A \cup B) \cap (A \cap C)$.

In any case $x \in (A \cup B) \cap (A \cup C)$.

$\supseteq$: Let $x \in (A \cup B) \cap (A \cup C)$. Then by definition of intersection $x \in A \cup B$ and $x \in A \cup C$. LEt us split into cases:

(a) If $x \in A$, then by definition of union $x \in A \cup (B \cap C)$.

(b) If $x \notin A$, since $x \in A \cup B$ and $x \in A \cup C$, then by definition of union $x \in B$ and $x \in C$. By definition of intersection $x \in B \cap C$. By definition of union $x \in A \cup (B \cap C)$.

Problem 2. Let $\mathcal{B}$ be a nonempty set of sets and let A be any set. Show that

(a) $A \cap \bigcup \mathcal{B} = \bigcup \{A \cap B \mid B \in \mathcal{B}\}$.

(b) $A \setminus \bigcap \mathcal{B} = \bigcup \{A \setminus B \mid B \in \mathcal{B}\}$.

Solution. We will prove item (1) as an example: By double inclusion:

- $\subseteq$: Let $x \in A \cap \bigcup \mathcal{B}$. By definition of intersection $x \in A$ and $x \in \bigcup \mathcal{B}$. By definition of generalized union, there is $B_0 \in \mathcal{B}$ such that $x \in B_0$. It follows that $x \in A \cap B_0$. Since $A \cap B_0 \in \{A \cap B \mid B \in \mathcal{B}\}$, and by the definition of generalized union, $x \in \bigcup\{A \cap B \mid B \in \mathcal{B}\}$.

- $\supseteq$: Let $x \in \bigcup\{A \cap B \mid B \in \mathcal{B}\}$. Then by the definition of generalized union, there is $B \in \mathcal{B}$ such that $x \in A \cap B$. By definition of intersection $x \in A$ and $x \in B$. It follows the $x \in \bigcup \mathcal{B}$ and by definition on intersection $x \in A \cap \bigcup \mathcal{B}$.

Problem 3. For a function $f : A \longrightarrow B$ and $C \subseteq A$ define the *pointwise image of* C *by* f as

$$f''C = \{f(c) \mid c \in C\}$$

(a) Prove that if $f : A \longrightarrow B$ is a function and $C \subseteq A$, then

$$(f''A) \setminus (f''C) \subseteq f''[A \setminus C].$$

(b) Give an example of a function $f : A \longrightarrow B$ and a subset $C \subseteq A$ such that

$$(f''A) \setminus (f''C) \neq f''[A \setminus C].$$

(c) Prove that if $f : A \longrightarrow B$ is an injection and $C \subseteq A$, then

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

Solution. (a) Let $b \in f''A \setminus f''C$. Since $b \in f''A$, there is $a \in A$ such that $b = f(a)$. Since $b \notin f''C$, $a \notin C$. It follows that $a \in A \setminus C$. We conclude that $b = f(a) \in f''[A \setminus C]$.

(b) Let $f : \{1,2\} \to \{1,2\}$ defined by $f(1) = f(2) = 1$. Let $A = \{1,2\}$, and $C = \{1\}$. Then

$$f''\{1,2\} = \{1\},\ f''\{1\} = \{1\} \Rightarrow f''\{1,2\} \setminus f''\{1\} = \emptyset$$

Also

$$\{1,2\} \setminus \{1\} = \{2\} \Rightarrow f''[\{1,2\} \setminus \{1\}] = \{1\}$$

Hence

$$f''\{1,2\} \setminus f''\{1\} = \emptyset \neq \{1\} = f''[\{1,2\} \setminus \{1\}].$$

(c) Suppose that f is injective and we would like to prove that

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

By a double inclusion. In section (a) we proved $\subseteq$. For the other direction, let $x \in f''[A \setminus C]$. Then there is $a \in A \setminus C$ such that $f(a) = x$. By the definition of difference, we would like to prove that $x \in f''A$ and $x \notin f''C$. Since $a \in A$, it follows that $x = f(a) \in f''A$. Suppose towards a contradiction that there is $c \in C$ such that $f(c) = x$. Then $f(c) = f(a)$. Since f is injective, $c = a$. However $c \in C$ and $a \notin C$, contradiction. Hence $x \in f''C$.

Problem 4. Recall that the indicator function $\chi_A : P(A) \to {}^A\{0,1\}$ is defined by $\big(\chi_A(B)\big)(a) = \begin{cases} 1 & a \in B \\ 0 & a \notin B \end{cases}$. Prove that χ_A is injective.

Problem 5. Prove that the interleaving function $F : ({}^{\mathbb{N}}\{0,1\})^2 \to {}^{\mathbb{N}}\{0,1\}$ defined by

$$F(\langle f, g\rangle)(n) = \begin{cases} f(\frac{n}{2}) & n \in \mathbb{N}_{even} \\ g(\frac{n-1}{2}) & n \in \mathbb{N}_{odd} \end{cases}$$

is one-to-one and onto. Prove that it is invertable and find F^{-1}.

Solution. To see that F is one-to-one suppose that $F_1 := F(\langle f_1, g_1\rangle) = F(\langle f_2, g_2\rangle) =: F_2$. let us prove for example that $f_1 = f_2$ (as the proof the $g_1 = g_2$ is the same). Let $n \in \mathbb{N}$. Since $F_1 = F_2$, $F_1(2n) = F_2(2n)$. Hence, by definition of F, $F_1(2n) = f_1(\frac{2n}{2}) = f_1(n)$ and $F_2(2n) = f_2(n)$, hence $f_1(n) = f_2(n)$. It follows by equality of functions that $f_1 = f_2$. To that F is onto, given a function $h : \mathbb{N} \to \{0,1\}$, define $f(n) = h(2n)$ and $g(n) = h(2n+1)$, then it is not hard to check that $F(\langle f, g\rangle) = h$. This also gives the definition of F^{-1}. Indeed $F^{-1}(h) = \langle f_h, g_h\rangle$, where $f_h(n) = h(2n)$ and $g_h(n) = h(2n+1)$.

Problem 6. Prove the ollowing statements:

(a) $\left\{f \in {}^{\mathbb{R}}\mathbb{R} \mid \exists i \in \{0,1\},\ \forall x \in \mathbb{R} \setminus \mathbb{Q},\ f(x) = i\right\} \approx \{0,1\} \times {}^{\mathbb{Q}}\mathbb{R}$.

(b) If $A \approx B$ then $P(A) \approx P(B)$

Solution. (a) Define the set of the left side by A and let us just provide the bijection: $G : A \to \{0,1\} \times {}^{\mathbb{Q}}\mathbb{R}$ defined by

$$G(f) = \langle f(\sqrt{2}), f \restriction \mathbb{Q}\rangle$$

(b) Let $f : A \to B$ be a bijection and define a new function using f, as follows, $F : P(A) \to P(B)$ defined by $F(X) = f[X]$. Let us prove that F is invertible.

1. We will first show that F is injective. Let $X_1, X_2 \in P(A)$ such that $F(X_1) = F(X_2)$. Equivalently, $\{f(x)|x \in X_1\} = \{f(x)|x \in X_2\}$. We want to show that $X_1 = X_2$. So let $x_1 \in X_1$. We want to show that $x_1 \in X_2$. Denote by $y = f(X_1)$, then by the replacement principle, there exists $y \in F(X_1)$. Since $F(X_1) = F(X_2)$, $y \in F(X_2)$ and therefore, by the replacement principle, there is $x_2 \in X_2$ such that $y = f(x_2)$. We conclude that $f(x_2) = y = f(x_1)$. Since f is injective, $x_1 = x_2$. So, $x_1 \in X_2$ and thus $X_1 \subseteq X_2$. The inclusion $X_2 \subseteq X_1$ is symmetric. We conclude that $X_1 = X_2$ and therefore, f is injective.

2. We will now show that F is surjective. Let $Y \in P(B)$. Then $Y \subseteq B$. We want to show that $Y = F(X)$ for some $X \subseteq A$. Let $X = \{x \in A|f(x) \in Y\}$ and let us prove set equality $F(X) = Y$. Let $y \in Y$, since f is surjective, there exists $x \in A$ such that $f(x) = y$. Since $y \in Y$, $x \in X$ and therefore $y = f(x) \in f''X = f(X)$. For the other direction, let $y \in F(X)$. Then there is $x \in X$ such that $f(x) = y$. By definition of x, $y = f(x) \in Y$. Hence $F(X) = Y$ and therefore F is surjective. So F is a bijection, and therefore F is invertible.