Homework 11

MATH 461 (due April 26) April 19, 2024

Problem 1. Let us prove the substitution Lemma we used to prove the Completeness Theorem: For any $\mathcal{L}$-structure $\mathfrak{a}$, any ϕ, any term t which is substitutable for x in ϕ, and any $s : V \longrightarrow A^{\mathfrak{a}}$,

$$\mathfrak{a} \models \phi^x_t[s] \text{ iff } \mathfrak{a} \models \phi[s(x|\bar{s}(t))]$$

(a) First show by induction on the complexity of a term t_0, that if x is any variable in t_0, and t_1 is any other terms, then $\bar{s}((t_0)^x_{t_1}) = \big(\bar{s}(x|\bar{s}(t_1))\big)(t_0)$.
Solution. By induction on the complexity of t_0, if $t_0 = x$ then $(t_0)^x_{t_1} = t_1$ and $\bar{s}(t_1) = (\bar{s}(x|\bar{s}(t_1))(t_0)$. If $t_0 = z$ for a variable different from x, then $(t_0)^x_{t_1} = z = t_0$ and

$$\bar{s}(t_0) = s(z) = (\bar{s}(x|\bar{s}(t_1))(t_0)$$

. For constant symbols we have again $\bar{s}((c)^x_{t_1}) = \bar{s}(c) = c^{\mathfrak{a}} = (\bar{s}(x|\bar{s}(t_1))(c))$. Finally for $f(t'_0, ..., t'_n)^x_{t_1} = f((t'_0)^x_{t_1}, ..., (t'_n)^x_{t_1})$ and by the induction hypothesis

$$\bar{s}(f((t'_0)^x_{t_1}, ..., (t'_n)^x_{t_1})) = f^{\mathfrak{a}}(\bar{s}((t'_0)^x_{t_1}), ..., \bar{s}((t'_n)^x_{t_1})) =$$

$$= f^{\mathfrak{a}}(\bar{s}(((x|\bar{s}(t_1))(t'_0), ..., \bar{s}(((x|\bar{s}(t_1))(t'_n)) = \bar{s}(x|\bar{s}(t_1))(f(t'_0, ..., t'_n)).$$

(b) Prove the substitution lemma by induction on the complexity of ϕ. [Recall that if ϕ is of the form $\forall x \psi$ and t cannot substitute for x since x is not free in ϕ, also x cannot appear in t by definition of "substitutable".]

Solution. If ϕ is $t_0 = t_1$ we have that $\mathfrak{a} \models (t_0 = t_1)^x_t[s]$ iff $\bar{s}((t_0)^x_t) = \bar{s}((t_1)^x_t)$ and by the previous section this is iff $\bar{s}(x|\bar{s}(t))(t_0) = \bar{s}(x|\bar{s}(t))(t_1)$ iff $\mathfrak{a} \models (t_0 = t_1)[s(x|\bar{s}(t))]$ as wanted. For ϕ of the form $P(t_1, ..., t_n)$ we have $\mathfrak{a} \models P(t_1, \ldots, t_n)^x_t[s]$ iff $\langle \bar{s}((t_1)^x_t), \ldots, \bar{s}((t_n)^x_t) \rangle \in P^{\mathfrak{a}}$ iff $\langle \bar{s}(x|\bar{s}(t))(t_1), \ldots, \bar{s}(x|\bar{s}(t))(t_n) \rangle \in P^{\mathfrak{a}}$ iff $\mathfrak{a} \models (P(t_1, \ldots, t_n))[\bar{s}(x|\bar{s}(t))]$.

For ϕ of the form $\neg\alpha$ and $\alpha \to \beta$ this is an easy application of the induction hypothesis. Finally for ϕ of the form $\forall y \psi$, if t is substitutable for x in ϕ, then y does not appear in t and $x \neq y$. Thus, $(\forall y \psi)^x_t = \forall y (\psi)^x_t$ and $\mathfrak{a} \models (\forall y \psi)^x_t[s]$ iff for every $a \in A^{\mathfrak{a}}$, $\mathfrak{a} \models \psi^x_t[s(y|a)]$, by the induction hypothesis this is equivalent to

$$\mathfrak{a} \models \psi[s(y|a)(x|\bar{s}(t))]$$

Note that $s(y|a)(x|\bar{s}(t)) = s(x|\bar{s}(t))(y|a)$ (as $x \neq y$) hence for all $a \in A^{\mathfrak{a}}$, $\mathfrak{a} \models (\forall y \psi)[s(x|\bar{s}(t))]$ as wanted.

Problem 2. Conclude from the substitution Lemma that the Logical axiom $\forall x \phi \mapsto \phi^x_t$ (where t is substitutable for x in ϕ) is valid.

Solution. Suppose that $\mathfrak{a} \models \forall x \phi[s]$, and let us prove $\mathfrak{a} \models (\phi)^x_t[s]$. By the substitution lemma this is equivalent to showing that $\mathfrak{a} \models \phi[s(x|\bar{s}(t))]$, but $\bar{s}(t) \in A^{\mathfrak{a}}$ and by assumption, for every $a \in A^{\mathfrak{a}}$, $\mathfrak{a} \models \phi[s(x|a)]$, so we are done.

Problem 3 (Optional)**.** Let us show the existence of alphabetical variants: Suppose that ϕ is a formula, x is a variable and t is a term. There is ϕ' (which is called an alphabetical variant) such that:

(1) ϕ and ϕ' only differ on quantifies variables.

(2) $\phi \vdash \phi'$ and $\phi' \vdash \phi$,

(3) t is substitutable for x in ϕ'.

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Let us define ϕ' by induction on ϕ. If ϕ is atomic, then $\phi' = \phi$. Then $(\phi \to \psi)' = \phi' \to \psi'$ and $(\neg\phi)' = \neg\phi'$. Finally, $(\forall y \phi) = \forall z (\phi')^y_z$ where $z \neq x$ does not appear in ϕ', nor in t.

(a) Prove that t is substitutable for x is ϕ' (again, by induction).

(b) Let us prove that $\phi \vdash \phi'$ and $\phi' \vdash \phi$, by induction on ϕ:

   (i) Prove that for atomic formulas, $\phi \to \psi$ and $\neg\phi$.

   (ii) For formulas of the form $\forall y \phi$, first prove that $\phi \vdash \phi'$.

   [Hint: note that the choice of z is substitutable for y in ϕ' and therefore we can use axiom 2. Then use generalization.]

   (iii) Now prove $\phi \vdash \phi'$ [Hint: Explain why $((\phi')^y_z)^z_y = \phi'$, then the induction hypothesis, and the generalization theorem.]

Problem 4. (a) Let $\mathcal{L}$ have the following nonlogical symbols:

   (i) a binary predicate symbol $<$; and

   (ii) two constant symbol a and b.

Let T be the theory in $\mathcal{L}$ with the following axioms:

(1) $\forall x \neg(x < x)$.

(2) $\forall x \forall y (x < y \vee y < x \vee x = y)$.

(3) $\forall x \forall y \forall z ([x < y \wedge y < z] \to [x < z])$.

(4) $\forall x \forall y ([x < y] \to \exists z [x < z \wedge z < y])$.

(5) $\forall x \exists y \exists z (y < x \wedge x < z)$.

(6) $a < b$.

Prove that T is consistent and complete.

(b) Prove that $\langle \mathbb{Q}, <, 2, 3\rangle \equiv \langle \mathbb{R}, <, \sqrt{2}, \sqrt{3}\rangle$.

Solution. To prove that T is consistent, by the completeness theorem, it suffices to prove it is satifiable. The model $\langle \mathbb{Q}, <, 2, 3\rangle$ satisfy the above. To prove the completeness, let us use the Los-Vaught theorem, clearly, T has no finite model (as any model of T is in particular a dense linear order with no least and last element). Suppose that $\mathfrak{a} = \langle A, <_A, a^{\mathfrak{a}}, b^{\mathfrak{a}}\rangle$ and $\mathfrak{b} = \langle B, <_B, a^{\mathfrak{b}}, b^{\mathfrak{b}}\rangle$ are two countable models of T. To see that $\mathfrak{a} \simeq \mathfrak{b}$, let us prove that $\mathfrak{a} \simeq \langle \mathbb{Q}, <, 2, 3\rangle$, and by symmetry this will also be true for $\mathfrak{b}$. Define $f : A \to \mathbb{Q}$ an isomorphism as follows: by definition of isomorphism, we have to map $f(a^{\mathfrak{a}}) = 2$ and $f(b^{\mathfrak{a}}) = 3$. It is not hard to check that $\{x \in A \mid x <_A a^{\mathfrak{a}}\}$, $\{x \in A \mid a^{\mathfrak{a}} <_A x <_A b^{\mathfrak{a}}\}, \{x \in A \mid x >_A b^{\mathfrak{a}}\}$ are three dense linear orders without least and last elements, hence by Cantor's theorem, they are isomorphic to $\mathbb{Q}$. By the same reason, $\mathbb{Q}\cap(-\infty, 2), \mathbb{Q}\cap(2, 3), \mathbb{Q}\cap(3, \infty)$ these are also isomorphic to $\mathbb{Q}$ and therefore there is an isomorphism

$$f_1 : \{x \in A \mid x <_A a^{\mathfrak{a}}\} \to \mathbb{Q} \cap (-\infty, 2)$$

$$f_2 : \{x \in A \mid a^{\mathfrak{a}} <_A x <_A b^{\mathfrak{a}}\} \to \mathbb{Q} \cap (2, 3)$$

$$f_3 : \{x \in A \mid x >_A b^{\mathfrak{a}}\} \to \mathbb{Q} \cap (3, \infty)$$

Define $f : A \to \mathbb{Q}$ by

$$f(x) = \begin{cases} f_1(x) & x <_A a^{\mathfrak{a}} \\ 2 & x = a^{\mathfrak{a}} \\ f_2(x) & a^{\mathfrak{a}} <_A x <_A b^{\mathfrak{a}} \\ 3 & x = b^{\mathfrak{a}} \\ f_3(x) & x >_A b^{\mathfrak{a}} \end{cases}$$

It is easy to check that f is an isomorphism.

(b) Both sides are models of T. Since T is complete, every two models of T are elementary equivalent, and in particular the ones in the problem.