Homework 11

MATH 461 (due April 26) April 19, 2024

Problem 1. Let us prove the substitution Lemma we used to prove the Completeness Theorem: For any $\mathcal{L}$-structure $\mathfrak{a}$, any ϕ, any term t which is substitutable for x in ϕ, and any $s : V \to A^{\mathfrak{a}}$,

$$\mathfrak{a} \models \phi^x_t[s] \text{ iff } \mathfrak{a} \models \phi[s(x|\bar{s}(t))]$$

(a) First show by induction on the complexity of a term t_0, that if x is any variable in t_0, and t_1 is any other terms, then $\bar{s}((t_0)^x_{t_1}) = \big(\bar{s}(x|\bar{s}(t_1))\big)(t_0)$.

(b) Prove the substitution lemma by induction on the complexity of ϕ. [Recall that if ϕ is of the form $\forall x \psi$ and t cannot substitute for x since x is not free in ϕ, also x cannot appear in t by definition of "substitutable".]

Problem 2. Conclude from the substitution Lemma that the Logical axiom $\forall x \phi \mapsto \phi^x_t$ (where t is substitutable for x in ϕ) is valid.

Problem 3 (Optional)**.** Let us show the existence of alphabetical variants: Suppose that ϕ is a formula, x is a variable and t is a term. There is ϕ' (which is called an alphabetical variant) such that:

(1) ϕ and ϕ' only differ on quantifies variables.

(2) $\phi \vdash \phi'$ and $\phi' \vdash \phi$,

(3) t is substitutable for x in ϕ'.

Let us define ϕ' by induction on ϕ. If ϕ is atomic, then $\phi' = \phi$. Then $(\phi \to \psi)' = \phi' \to \psi'$ and $(\neg\phi)' = \neg\phi'$. Finally, $(\forall y \phi) = \forall z (\phi')^y_z$ where $z \neq x$ does not appear in ϕ', nor in t.

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(a) Prove that t is substitutable for x is ϕ' (again, by induction).

(b) Let us prove that $\phi \vdash \phi'$ and $\phi' \vdash \phi$, by induction on ϕ:

(i) Prove that for atomic formulas, $\phi \to \psi$ and $\neg\phi$.

(ii) For formulas of the form $\forall y \phi$, first prove that $\phi \vdash \phi'$.

[Hint: note that the choice of z is substitutable for y in ϕ' and therefore we can use axiom 2. Then use generalization.]

(iii) Now prove $\phi \vdash \phi'$ [Hint: Explain why $((\phi')^y_z)^z_y = \phi'$, then the induction hypothesis, and the generalization theorem.]

Problem 4. (a) Let $\mathcal{L}$ have the following nonlogical symbols:

(i) a binary predicate symbol $<$; and

(ii) two constant symbol a and b.

Let T be the theory in $\mathcal{L}$ with the following axioms:

(1) $\forall x \neg(x < x)$.

(2) $\forall x \forall y (x < y \vee y < x \vee x = y)$.

(3) $\forall x \forall y \forall z ([x < y \wedge y < z] \to [x < z])$.

(4) $\forall x \forall y ([x < y] \to \exists z [x < z \wedge z < y])$.

(5) $\forall x \exists y \exists z (y < x \wedge x < z)$.

(6) $a < b$.

Prove that T is consistent and complete.

(b) Prove that $\langle \mathbb{Q}, <, 2, 3 \rangle \equiv \langle \mathbb{R}, <, \sqrt{2}, \sqrt{3} \rangle$.