Homework 2

MATH 461 (due February 9) Feb 2, 2024

Problem 1. Let $C(\mathbb{R})$ be the set of all continuous function $f : \mathbb{R} \to \mathbb{R}$. Prove that

$$C(\mathbb{R}) \preceq {}^{\mathbb{Q}}\mathbb{R}$$

[Hint: use that fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to prove that the restriction function $G : C(\mathbb{R}) \to {}^{\mathbb{Q}}\mathbb{R}$ defined by $G(f) = f \restriction \mathbb{Q}$ is one-to-one.]

Problem 2. Prove that for every $\alpha < \beta$ real numbers $(\alpha, \beta) \approx (0, 1)$. [Hint: First stretch/shrink $(0, 1)$ to have length $\beta - \alpha$, then shift it by $+c$ as we did in class.]

Problem 3. Show that ${}^{\mathbb{N}}\{0, 1\} \times {}^{\mathbb{N}}\{0, 1\} \approx {}^{\mathbb{N}}\{0, 1\}$. [Hint: see HW1 Problem 5.]

Problem 4. Prove by a diagonalization argument that $\mathbb{N} \prec {}^{\mathbb{N}}\mathbb{N}_{even}$.

Problem 5. Prove that $\{X \in P(\mathbb{N}) \mid X \approx \mathbb{N}\} \approx P(\mathbb{N})$. [Hint: Cantor-Bernstein]

Problem 6. Let A be any set. Let us define recursively $A_0 = A$ and $A_{n+1} = P(A_n)$. Define $A_\omega = \bigcup_{n<\omega} A_n$. Prove that for every set A and any $n < \omega$, $A_n \prec A_\omega$.