Homework 3-Sols

MATH 461 (due February 16) Feb 9, 2024

Problem 1. A function $f : A \to B$ is called countable-to-one if every $b \in B$ has at most countably many preimages. Namely, if for every $b \in B$, the following set is countable:

$$\{a \in A \mid f(a) = b\}$$

1. Give an example of a function which is countable-to-one but not one-to-one.

   Solution. $f : \mathbb{N} \to \mathbb{N}$, $f(n) = \lfloor \frac{n}{2} \rfloor$ (where $\lfloor q \rfloor$ is the greatest integer less or equal to q)

2. Suppose that A is a set such that there exists a countable-to-one function $f : A \to \mathbb{Q}$. Prove that A is countable. [Hint: countable union of countable sets is countable]

   Solution. Since f is a function $A = \bigcup_{q \in \mathbb{Q}} f^{-1}[\{q\}]$, where $f^{-1}[\{q\}] = \{a \in A \mid f(a) = q\}$ (prove this!). By the countable-to-one assumption, $f^{-1}[\{q\}]$ is countable for every q. Since $\mathbb{Q}$ is countable, we get that A is a countable union of countable sets and therefore countable.

Problem 2. Let $\Pi \subseteq P(A) \setminus \{\emptyset\}$. Define

$$F_\Pi = \{\langle x, X\rangle \in A \times \Pi \mid x \in X\}$$

prove that $F_\Pi : A \to P(A)$ is a function of and only if Π is a partition.

solution. Suppose that F_Π is a function, and let us prove that Π is a partition. By assumption, $\Pi \subseteq P(A) \setminus \{\emptyset\}$ and therefore $\emptyset \notin \Pi$. Since F_Π is total, for every $x \in A$ there is $X \in \Pi$ such that $x \in X$ and therefore $A \subseteq \bigcup \Pi$. Since $\Pi \subseteq P(A)$, $\bigcup \Pi \subseteq A$, hence $\bigcup \Pi = A$. Finally, if $X \cap Y \neq \emptyset$, then there is $x \in X \cap Y$ and therefore $\langle x, X\rangle, \langle x, Y\rangle \in F_\Pi$. Since F_Π is univalent, $X = Y$.

For the other direction, suppose that Π is a partition and let us prove that F_Π is a function. To see it is total, let $x \in A$, then since $x \in A = \bigcup \Pi$, there is $X \in \Pi$ such that $x \in X$, and therefore by definition $\langle x, X\rangle \in F_\Pi$. To see it is univalent, let $\langle x, X\rangle, \langle x, Y\rangle \in F_\Pi$. By definition, this means that $x \in X \cap Y$. Hence $X \cap Y \neq \emptyset$. Since $X, Y \in \Pi$, by definition of partition, $X = Y$. Hence F_Π is a function.

Problem 3. Describe the partitions induced from the following equivalence relations (namely, compute A/E in each of the cases):

1. $A = \mathbb{Z}$, $E = \{\langle z, z'\rangle \in \mathbb{Z}^2 \mid |z| = |z'|\}$.

   Solution. $\{\{0\}\} \cup \{\{-n, n\} \mid n \in \mathbb{N}_+\}$.

2. $A = \mathbb{R} \times \mathbb{R}$, $E = \{\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in (\mathbb{R} \times \mathbb{R})^2 \mid \min(x, y) = \min(a, b)\}$.

   Solution. $\{[r, \infty) \times \{r\} \cup \{r\} \times [r, \infty) \mid r \in \mathbb{R}\}$.

3. for $A = {}^{\{0,\dots,10\}}\{0, 1\}$. define

   $$E = \{\langle f, g\rangle \in A \times A \;\mid\; \Big|\{n \mid f(n) = 1\}\Big| = \Big|\{n \mid g(n) = 1\}\Big|\;\}$$

   Solution. $\{\{f \in A \mid |f^{-1}[\{1\}]| = k\} \mid k \in \{0, \dots, 11\}\}$.

Problem 4. Let $A = \mathbb{N}^{\mathbb{N}}$, and consider the equivalence relation $R = \{\langle f, g\rangle \in (\mathbb{N}^{\mathbb{N}})^2 \mid f(0) = g(0)\}$ in A (no need to prove that). Prove that $A/R \approx \mathbb{N}$.

Solution. Define $F : \mathbb{N} \to A/R$ defined by $F(n) = \{f \in A \mid f(0) = n\}$. Check that $F(n) \in A/R$, clearly if F is one-to-one, and check that F is surjective.

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Problem 5 (Optional)**.** On ${}^{\mathbb{N}}\{0,1\}$, define the equivalence relation E by fEg if and only if there is N such that for every $n \geq N$, $f(n) = g(n)$.

Prove that ${}^{\mathbb{N}}\{0,1\}/E \approx {}^{\mathbb{N}}\{0,1\}$. [Guidence: In order to prove that ${}^{\mathbb{N}}\{0,1\} \preceq {}^{\mathbb{N}}\{0,1\}/E$, decompose $\mathbb{N}$ to infinitely many infinite disjoint sets $\mathbb{N} = \biguplus_{n\in\mathbb{N}} A_n$. Try to use such a decomposition to define a function $F : {}^{\mathbb{N}}\{0,1\} \to {}^{\mathbb{N}}\{0,1\}$ which duplicates each value of the in input value f (i.e. duplicates the values $f(n)$) infinitely many times]