Homework 3

MATH 461 — (due February 16) — Feb 9, 2024

Problem 1. A function $f : A \longrightarrow B$ is called countable-to-one if every $b \in B$ has at most countably many preimages. Namely, if for every $b \in B$, the following set is countable:

$$\{a \in A \mid f(a) = b\}$$

1. Give an example of a function which is countable-to-one but not one-to-one.

2. Suppose that A is a set such that there exists a countable-to-one function $f : A \longrightarrow \mathbb{Q}$. Prove that A is countable. [Hint: countable union of countable sets is countable]

Problem 2. Let $\Pi \subseteq P(A) \setminus \{\emptyset\}$. Define

$$F_\Pi = \{\langle x, X\rangle \in A \times \Pi \mid x \in X\}$$

prove that $F_\Pi : A \longrightarrow P(A)$ is a function of and only if Π is a partition.

Problem 3. Describe the partitions induced from the following equivalence relations (namely, compute A/E in each of the cases):

1. $A = \mathbb{Z}$, $E = \{\langle z, z'\rangle \in \mathbb{Z}^2 \mid |z| = |z'|\}$.

2. $A = \mathbb{R} \times \mathbb{R}$, $E = \{\langle\langle x, y\rangle, \langle a, b\rangle\rangle \in (\mathbb{R} \times \mathbb{R})^2 \mid \min(x, y) = \min(a, b)\}$

3. for $A = {}^{\{0,\dots,10\}}\{0, 1\}$. define

$$E = \{\langle f, g\rangle \in A \times A \ \mid \ \Big|\{n \mid f(n) = 1\}\Big| = \Big|\{n \mid g(n) = 1\}\Big| \ \}$$

Problem 4. Let $A = \mathbb{N}^{\mathbb{N}}$, and consider the equivalence relation $R = \{\langle f, g\rangle \in (\mathbb{N}^{\mathbb{N}})^2 \mid f(0) = g(0)\}$ in A (no need to prove that). Prove that $A/R \approx \mathbb{N}$.

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Problem 5 (Optional)**.** On ${}^{\mathbb{N}}\{0,1\}$, define the equivalence relation E by $f E g$ if and only if there is N such that for every $n \geq N$, $f(n) = g(n)$.

Prove that ${}^{\mathbb{N}}\{0,1\}/E \approx {}^{\mathbb{N}}\{0,1\}$. [Guidence: In order to prove that ${}^{\mathbb{N}}\{0,1\} \preceq {}^{\mathbb{N}}\{0,1\}/E$, decompose $\mathbb{N}$ to infinitely many infinite disjoint sets $\mathbb{N} = \biguplus_{n\in\mathbb{N}} A_n$. Try to use such a decomposition to define a function $F : {}^{\mathbb{N}}\{0,1\} \to {}^{\mathbb{N}}\{0,1\}$ which duplicates each value of the in input value f (i.e. duplicates the values $f(n)$) infinitely many times]