Homework 4-Sols

MATH 461 (due February 23) Feb 16, 2024

Problem 1. Prove that $\langle \mathbb{Q} \setminus \mathbb{Z}, < \rangle \simeq \langle \mathbb{Q}, < \rangle$

Solotion. We use Cantor's theorem for DLO's. The set $\mathbb{Q} \setminus \mathbb{Z}$ is countable (as a subset of $\mathbb{Q}$) and for every $q_1 < q_2$ in $\mathbb{Q} \setminus \mathbb{Z}$, Let $q' = \min(\lfloor q + 1 \rfloor, q_2)$, note that $q_1 < q'$, and there are no integers in the interval (q_1, q'). Let $q = \frac{q_1 + q'}{2}$, then $q_1 < q < q' \leq q_2$. Since $q \in (q_1, q')$, $q \in \mathbb{Q} \setminus \mathbb{Z}$. Similarly we prove that there are no least and last elements. Hence by Cantor's theorem, $\langle \mathbb{Q} \setminus \mathbb{Z} \rangle \simeq \langle \mathbb{Q}, < \rangle$

Problem 2. Prove that every order R over a finite set A can be extended to a linear order.

Solution. By induction on the number of elements in A. For $A = \emptyset$, this is trivial. Let R be a any order on A, and $|A| = n + 1$. If there is $a \in A$ such that for every $b \in A$, bRa, then we can take $A' = A \setminus \{a\}$, and $R' = R \cap A' \times A'$. This is an order of A' which now have n-elements. by the induction hypothesis, there is R_0 linear on A' such that $R' \subseteq R_0$. Note that since $R = R' \cup \{\langle b, a \rangle b \in A'\}$ (as a is R-above every element of A), we have that $R \subseteq R_0 \cup \{\langle b, a \rangle \mid b \in A'\} = R_1$, and R_1 is a linear ordering of A. In the general case, since A is finite, we can always find a maximal element a^* (i.e. $a \in A$ such that there is no $b \in A$, $b \neq a$ and aRb. To prove that, just assume otherwise, and produce an infinite subset of A). Now extend R to $R^* = R \cup \{\langle b, a^* \rangle \mid b \in A \setminus \{a^*\}\}$. Check that this is still an order of A, but now a^* is the greatest element, and by the previous case it can be extended to a linear order.

Problem 3. Let $\langle A, < \rangle$ be an ordered set. A is called separable if there is a countable set $B \subseteq A$ which is dense in A. Namely, for every $a, a' \in A$, if

$a < a'$ then there is $b \in B$ such that $a < b < a'$.

(a) Convince yourselves that $\mathbb{R}$ is separable (no action required for this item) **Solution.** I am convinced.

(b) Consider the set $A = {}^{\mathbb{N}}\mathbb{N}$ with the following order:

$$f \prec g \text{ iff } f(n^*) < g(n^*), \text{ where } n^* = \min\{n \mid f(n) \neq g(n)\}.$$

Prove that $\langle A, \prec \rangle$ is separable.

Solution. the set of all function $f : \mathbb{N} \to \mathbb{N}$ which are eventually constant is countable (as a countable union of countable set- we have seen similar arguments in class) and it is dense: indeed, given any two functions $f \prec g$, define

$$f'(n) = \begin{cases} f(n) & n \leq n^* \\ f(n^* + 1) + 1 & n > n^* \end{cases}$$

Then f' is eventually constant. The minimal n such that $f(n) \neq f'(n)$ is $n^* + 1$ and $f(n^* + 1) < f'(n^* + 1)$, thus $f \prec f'$. Also, since the minimal n such that $f'(n) \neq g(n)$ is n^* and $f'(n^*) = f(n^*) < g(n^*)$. Thus $f' \prec g$.

(c) Prove that if A is separable then $|A| \leq 2^{\aleph_0}$.

Solution. Let $B \subseteq A$ be countable such that B is countable. Define $f : A \to P(B)$ by $f(a) = \{b \in B \mid b < a\}$. It is not hard to prove (as we did in class for to prove that $|\mathbb{R}| \leq |P(\mathbb{Q})|$) that f is one-to-one.

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1 Preparation for midterm(Optional)

Problem 4. Compute the cardinality of the set of all function $f : \mathbb{N} \to \{0,1\}$ with no consecutive zeros. Namely, there is no $n \in \mathbb{N}$ such that $f(n) = f(n+1) = 0$.

Solution. Let A be the set in the proposition. The cardinality is $|A| = 2^{\aleph_0}$. Prove it using Cantor-Bernstein, clearly, A a subset of all functions from $\mathbb{N}$ to $\{0,1\}$ and therefore $|A| \le 2^{\aleph_0}$. Let us define an injection $F : {}^{\mathbb{N}}\{0,1\} \to A$ by $F(f)(n) = \begin{cases} 1 & n \in \mathbb{N}_{odd} \\ f(\frac{n}{2}) & n \in \mathbb{N}_{even} \end{cases}$. Clearly, $F(f) \in A$ as it has no consecutive zeros (since at the odd inputs it returns 1). To see that F is injective, let $f_1 \neq f_2$, then there is n such that $f_1(n) \neq f_2(n)$, then $F(f_1)(2n) = f_1(n) \neq f_2(n) = F(f_2)(2n)$. Hence $F(f_1) \neq F(f_2)$.

Problem 5. Consider the relation E om ${}^{\mathbb{N}}\mathbb{N}$ by fEg if and only if for every $n \ge 100$, $f(n) = g(n)$.

1. Prove that E is an equivalence relation.

   Soltuion. Easy.

2. Compute the cardinality of ${}^{\mathbb{N}}\mathbb{N}/E$.

   Solution. $2^{\aleph_0}$ (Thanks to Max Romano for spotting the previous mistake)

Problem 6. Let $\le_A, \le_B$ be two weak linear orderings of A, B (resp.), where A, B are disjoint. We define $\le_A + \le_B$ which we abbreviate by $\le_+$ on $A \cup B$ as follows:

$$x \le_+ y \leftrightarrow (x, y \in A \wedge x \le_A y) \vee (x, y \in B \wedge x \le_B y) \vee (x \in A \wedge y \in B)$$

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1. Prove that $\leq_+$ is a linear ordering of $A \cup B$.

2. Let $\mathbb{N}^* = \{0\} \times \mathbb{N}$ and define $\leq^*$ on $\mathbb{N}^*$ by $\langle 0, n \rangle \leq^* \langle 0, m \rangle$ if and only if $m \leq n$. Prove that $\leq^*$ is a linear ordering of $\mathbb{N}^*$.

3. Prove that $\langle \mathbb{N}^* \cup \mathbb{N}, \leq^* + \leq \rangle \simeq \langle \mathbb{Z}, \leq \rangle$.

Problem 7. Define recursively $A_0 = \emptyset$ and $A_{n+1} = P(A_n)$. Prove by induction that for every n, $A_n \subseteq A_{n+1}$.

Solution By induction. For $n = 0$ $A_0 = \emptyset$ is a subset of every set and therefore $A_0 \subseteq A_1$. Suppose this is true for $n - 1$ and let us prove that $A_n \subseteq A_{n+1}$. Let $X \in A_n = P(A_{n-1})$. Then $X \subseteq A_{n-1}$. By the induction hypothesis, $A_{n-1} \subseteq A_n$ and therefore $X \subseteq A_n$. By definition $X \in P(A_n) = A_{n+1}$. It follows that $A_n \subseteq A_{n+1}$.

Problem 8. Prove that the set of surjections $f : \mathbb{N} \longrightarrow \mathbb{N}$ is uncountable.

Solution. Denote by A the set of all surjections. Assume otherwise there is a bijection $F : \mathbb{N} \longrightarrow A$. Define

$$g(n) = \begin{cases} \frac{n-1}{2} & n \in \ltimes_{odd} \\ F(\frac{n}{2})(n) + 2 & otherwise \end{cases}$$

Then g is subjective and $g \in A$. By assumption there is $n \in \mathbb{N}$ such that $F(n) = g$. In particular, $F(n)(2n) = g(2n) = F(n)(2n) + 1$. Hence $0 = 1$, contradiction.