Homework 4

MATH 461 (due February 23) Feb 16, 2024

Problem 1. Prove that $\langle \mathbb{Q} \setminus \mathbb{Z}, < \rangle \simeq \langle \mathbb{Q}, < \rangle$

Problem 2. Prove that every order R over a finite set A can be extended to a linear order.

Problem 3. Let $\langle A, < \rangle$ be an ordered set. A is called separable if there is a countable set $B \subseteq A$ which is dense in A. Namely, for every $a, a' \in A$, if $a < a'$ then there is $b \in B$ such that $a < b < a'$.

(a) Convince yourselves that $\mathbb{R}$ is separable (no action required for this item)

(b) Consider the set $A = {}^{\mathbb{N}}\mathbb{N}$ with the following order:

$$f \prec g \text{ iff } f(n^*) < g(n^*), \text{ where } n^* = \min\{n \mid f(n) \neq g(n)\}.$$

Prove that $\langle A, \prec \rangle$ is separable.

(c) Prove that if A is separable then $|A| \leq 2^{\aleph_0}$.

1 Preparation for midterm(Optional)

Problem 4. Compute the cardinality of the set of all function $f : \mathbb{N} \to \{0, 1\}$ with no consecutive zeros. Namely, there is no $n \in \mathbb{N}$ such that $f(n) = f(n+1) = 0$.

Problem 5. Consider the relation E om ${}^{\mathbb{N}}\mathbb{N}$ by fEg if and only if for every $n \geq 100$, $f(n) = g(n)$.

1. Prove that E is an equivalence relation.

2. Compute the cardinality of ${}^{\mathbb{N}}\mathbb{N}/E$.

Problem 6. Let $\leq_A, \leq_B$ be two weak linear orderings of A, B (resp.), where A, B are disjoint. We define $\leq_A + \leq_B$ which we abbreviate by $\leq_+$ on $A \cup B$ as follows:

$$x \leq_+ y \leftrightarrow (x, y \in A \wedge x \leq_A y) \vee (x, y \in B \wedge x \leq_B y) \vee (x \in A \wedge y \in B)$$

1. Prove that $\leq_+$ is a linear ordering of $A \cup B$.
2. Let $\mathbb{N}^* = \{0\} \times \mathbb{N}$ and define $\leq^*$ on $\mathbb{N}^*$ by $\langle 0, n \rangle \leq^* \langle 0, m \rangle$ if and only if $m \leq n$. Prove that $\leq^*$ is a linear ordering of $\mathbb{N}^*$.
3. Prove that $\langle \mathbb{N}^* \cup \mathbb{N}, \leq^* + \leq \rangle \simeq \langle \mathbb{Z}, \leq \rangle$.

Problem 7. Define recursively $A_0 = \emptyset$ and $A_{n+1} = P(A_n)$. Prove by induction that for every n, $A_n \subseteq A_{n+1}$.

Problem 8. Prove that the set of surjections $f : \mathbb{N} \to \mathbb{N}$ is uncountable.