Homework 5

MATH 461 (due March 1) Feb 26, 2024

Problem 1. Which of the following wffs are tautologies?

(a) $(Q \vee (\neg(P \Rightarrow Q)))$.

(b) $((P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R))$.

Solutions

(1) No, for example the TVA $V(Q) = V(P) = F$ makes the formula false.

(2) This is a tautology. (The proof uses truth tables for example)

Problem 2. In each of the following items, determine if Σ tautologically implies ϕ.

1. $\Sigma = \{(A \wedge (B \Rightarrow (\neg C))), (B \vee C)\}$, $\phi = (\neg C)$
2. $\Sigma = \{(A \Rightarrow (B \vee (\neg C))), (A \wedge C), (D \Rightarrow (\neg B))\}$, $\phi = (\neg D)$.

Solution.

(1) Σ does not tautologically implies ϕ, for example the TVA $V(A) = V(C) = T, V(B) = F$, makes $(B \Rightarrow (\neg C))$ true and therefore $\bar{V}(A \cap (B \Rightarrow (\neg(C)))) = T$. Also $\bar{V}(B \vee C) = T$ since $V(C) = T$, and therefore V satisfies Σ. However, $\bar{V}(\neg C) = F$.

(2) Suppose towards a contradiction that V witnesses that Σ does not tautologically implies ϕ, then $\bar{V}(\neg D) = F$ and therefore $V(D) = T$. Also $\bar{V}(D \Rightarrow (\neg B)) = T$ and therefore $V(B) = F$ and also $\bar{V}(A \wedge C) = T$ and therefore $V(A) = V(C) = T$. Bu then $\bar{V}(B \vee (\neg C)) = F$ and $\bar{V}(A \Rightarrow (B \vee (\neg C))) = F$, contracidtion.

Problem 3. Let $\{S_n \mid n \in \mathbb{N}\}$ be a collection of finite subsets of $\mathbb{N}$ such that for each finite subset $F \subseteq \mathbb{N}$, there exists a subset $A_F \subseteq \mathbb{N}$ with $|A_F \cap S_n| = 1$ for all $n \in F$. Use the Compactness Theorem to prove that there exists a subset $A \subseteq \mathbb{N}$ such that $|A \cap S_n| = 1$ for all $n \in \mathbb{N}$.

solution Let us define the sentence symbols $C_{n,m}$ for each $n \in \mathbb{N}$ and $m \in S_n$. Since each S_n is finite, there are only countably many such $C_{n,m}$'s. Consider the following wff's:

(1) $\alpha_n = (C_{n,m_1} \vee ... \vee C_{n,m_k})$ for every $n < \omega$ where $S_n = \{m_1, ..., m_k\}$.

(2) $\beta_{n,m,k} = \neg(C_{n,m} \wedge C_{n,k})$ for each $n < \omega$ and distinct $m, k \in S_n$.

Let $\Sigma = \{\alpha_n \mid n \in \mathbb{N}\} \cup \{\beta_{n,m,k} \mid n \in \mathbb{N},\ m \neq k \in S_n\}$. Then Σ is countable. Our first claim is that Σ is finitely satisfiable:

Let $\Sigma_0 \subseteq \Sigma$ be finite. Let F be the (finite) set of all n's which are mentioned in a formula in Σ_0. By the assumption, there is $A_F \subseteq \mathbb{N}$ such that $|A_F \cap S_n| = 1$ for all $n \in F$. Define $V(C_{n,m}) = T$ iff $C_{n,m} \in A_F$. Let us prove that V satisfies Σ_0. Indeed any formula $\alpha_n \in \Sigma_0$ $n \in F$ and therefore since there is $m \in A_F \cap S_n$ we will have for some $m \in S_n$ that $V(C_{n,m}) = T$ and therefore $\bar{V}(\alpha_n) = T$. If $\beta_{n,m,k} \in \Sigma_n$ for some $m \neq k \in S_n$, then $n \in F$ and since $|A_F \cap S_n| = 1$ not both $k, m \in A_F$. Hence not both $C_{n,k}$ and $C_{n,m}$ are true under V. It follows that $\bar{V}(\beta_{n,m,k}) = T$.

We conclude that Σ is finitely satisfiable and therefore by the conmpactness theorem it is satisfiable. Let V be a TVA witnessing that Σ is satisfiable. Define

$$A = \{m \in \mathbb{N} \mid m \in S_n, V(C_{n,m}) = T\}$$

and let us prove that $|A \cap S_n| = 1$ for all $n \in \mathbb{N}$. Indeed since $\bar{V}(\alpha_n) = T$, there is $m \in S_n$ such that $V(C_{n,m}) = T$ and therefore $|A \cap S_n| \geq 1$. Suppose towards a contradiction that $|A \cap S_n| > 1$ and find $m \neq k \in S_n \cap A$ and therefore $V(C_{n,k}) = V(C_{n,m}) = T$ but then $V(\beta_{n,m,k}) = F$, contradicting the fact that V satisfies Σ.

Problem 4 (Optional)**.** In a fictional village, every inhabitant is either a truth-teller (everything they say is true) or a liar (everything they say is false). Arnie and Barnie live in the village. Suppose that Arnie says, "If I am a truth-teller, then so is Bernie." Determine if Arnie and Barnie truth-tellers or liars? Motivate your answer.
[Hint: Let A be the statement "Arnie is a truth-teller" and let B be the statement "Barnie is a truth-teller." Arnie's statement can then be expressed as $A \Rightarrow B$. Create a truth table for Arnie's statement...]

Problem 5 (Optional)**.** (a) Prove that if ϕ, ϕ' and ψ, ψ' are tautologically equivalent wff's, then so are the pairs:

(i) $(\phi \wedge \psi)$, $(\phi' \wedge \psi')$.

(ii) $(\phi \vee \psi)$, $(\phi' \vee \psi')$.

(iii) $(\phi \Rightarrow \psi)$, $(\phi' \Rightarrow \psi')$.

(iv) $(\neg\phi)$, $(\neg\phi')$.

(b) Prove that every wff is tautologically equivalent to a statement which only has the logical connectives $\vee, \neg$.

Prove that every wff is tautologically equivalent to a wff where the only logical connectives used are $\vee$ and $\neg$. [Hint: by induction on the complexity of the formula].