Homework 5

MATH 461 (due March 1) Feb 26, 2024

Problem 1. Which of the following wffs are tautologies?

(a) $(Q \vee (\neg(P \Rightarrow Q)))$.

(b) $((P \Rightarrow (Q \Rightarrow R)) \Leftrightarrow ((P \wedge Q) \Rightarrow R))$.

Problem 2. In each of the following items, determine if Σ tautologically implies ϕ.

1. $\Sigma = \{(A \wedge (B \Rightarrow (\neg C))),\ (B \vee C)\},\ \phi = (\neg C)$
2. $\Sigma = \{(A \Rightarrow (B \vee (\neg C))),\ (A \wedge C),\ (D \Rightarrow (\neg B))\},\ \phi = (\neg D)$.

Problem 3. Let $\{S_n \mid n \in \mathbb{N}\}$ be a collection of finite subsets of $\mathbb{N}$ such that for each finite subset $F \subseteq \mathbb{N}$, there exists a subset $A_F \subseteq \mathbb{N}$ with $|A_F \cap S_n| = 1$ for all $n \in F$. Use the Compactness Theorem to prove that there exists a subset $A \subseteq \mathbb{N}$ such that $|A \cap S_n| = 1$ for all $n \in \mathbb{N}$.

Problem 4 (Optional)**.** In a fictional village, every inhabitant is either a truth-teller (everything they say is true) or a liar (everything they say is false). Arnie and Barnie live in the village. Suppose that Arnie says, "If I am a truth-teller, then so is Bernie." Determine if Arnie and Barnie truth-tellers or liars? Motivate your answer.
[Hint: Let A be the statement "Arnie is a truth-teller" and let B be the statement "Barnie is a truth-teller." Arnie's statement can then be expressed as $A \Rightarrow B$. Create a truth table for Arnie's statement...]

Problem 5 (Optional)**.** (a) Prove that if ϕ, ϕ' and ψ, ψ' are tautologically equivalent wff's, then so are the pairs:

(i) $(\phi \wedge \psi),\ (\phi' \wedge \psi')$.

(ii) $(\phi \vee \psi)$, $(\phi' \vee \psi')$.

(iii) $(\phi \Rightarrow \psi)$, $(\phi' \Rightarrow \psi')$.

(iv) $(\neg\phi)$, $(\neg\phi')$.

(b) Prove that every wff is tautologically equivalent to a statement which only has the logical connectives $\vee, \neg$.

Prove that every wff is tautologically equivalent to a wff where the only logical connectives used are $\vee$ and $\neg$. [Hint: by induction on the complexity of the formula].