Homework 7

MATH 461 (due March 29) March 22, 2024

Problem 1. Let $\mathcal{L}$ be the language where the non-logical 2-places function symbols are $+, \times$ and $\bar{1}$ is a non-logical constant symbol and there are no predicate symbols.

(a) Let $\mathfrak{a} = \langle \mathbb{R}, +^{\mathfrak{a}}, \times^{\mathfrak{a}}, \bar{1}^{\mathfrak{a}} \rangle$ be the $\mathcal{L}$-structure where $+^{\mathfrak{a}}, \times^{\mathfrak{a}}, \bar{1}^{\mathfrak{a}}$ are the usual $+, \cdot, 1$ on reals. Denote by $\bar{n}$ the term $\underbrace{\bar{1} + \bar{1} + \ldots + \bar{1}}_{n\text{-times}}$. Find a term for $2x^2 + x + 1$. Describe (without proof) all the terms.

(b) Find a WFF which expresses that $\sqrt{2}$ exists.

Problem 2. Let $\mathfrak{a} = \langle A^{\mathfrak{a}}, \ldots \rangle$ be a $\mathcal{L}$-structure and let t be a term. If $s_1, s_2 \colon V \longrightarrow A^{\mathfrak{a}}$ agree on all variables (if any) in t, then $\bar{s}_1(t) = \bar{s}_2(t)$. (Hint: argue by induction on the length of t.)

Definition. Suppose that $\mathfrak{a}, \mathfrak{b}$ are structures for the first order language $\mathcal{L}$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be elementarily equivalent, written $\mathfrak{a} \equiv \mathfrak{b}$, if for every sentence σ,

$$A \models \sigma \Leftrightarrow B \models \sigma.$$

Problem 3. Let $\mathcal{L}$ be the first order language such that the only nonlogical symbol is the 2-place predicate symbol $<$. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be the following $\mathcal{L}$-structures:

- $\mathfrak{a} = \langle \mathbb{N}, <^{\mathfrak{a}} \rangle$.
- $\mathfrak{b} = \langle \mathbb{Z}, <^{\mathfrak{b}} \rangle$.
- $\mathfrak{c} = \langle \mathbb{Q}, <^{\mathfrak{c}} \rangle$.

where $<^{\mathfrak{a}}, <^{\mathfrak{b}}, <^{\mathfrak{c}}$ are the usual linear orderings of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ respectively. Prove that:

(i) $\mathfrak{a} \not\equiv \mathfrak{b}$

(ii) $\mathfrak{b} \not\equiv \mathfrak{c}$.