Homework 9

MATH 461 (due April 12) April 9, 2024

Problem 1. Prove that if α, β are any WFF's, then

$$(\forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta))$$

is valid.

Solution. Let $\mathfrak{a}$ be an $\mathcal{L}$-structure and $s : V \to A^{\mathfrak{a}}$ be an assignment to the variables. WTP

$$\mathfrak{a} \models (\forall x(\alpha \to \beta) \to (\forall x \alpha \to \forall x \beta))[s]$$

Suppose that $\mathfrak{a} \models (\forall x(\alpha \to \beta))[s]$ WTP $\mathfrak{a} \models (\forall x \alpha \to \forall x \beta)[s]$. Suppose that $\mathfrak{a} \models (\forall x \alpha)[s]$ WTP $\mathfrak{a} \models (\forall x \beta)[s]$. Let $a \in A^{\mathfrak{a}}$, WTP $\mathfrak{a} \models \beta[s(x|a)]$. Since $\mathfrak{a} \models (\forall x \alpha)[s]$, we have that $\mathfrak{a} \models \alpha[s(x|a)]$ and since $\mathfrak{a} \models (\forall x(\alpha \to \beta))[s]$ we have $\mathfrak{a} \models (\alpha \to \beta)[s(x|a)]$ which implies that $\mathfrak{a} \models \beta[s(x|a)]$ as wanted.

Problem 2. Prove that if P is a binary predicate symbol, then

$$(x = y \to (P(x, z) \to P(y, z)))$$

is valid.

Solution. Let $\mathfrak{a}$ be an $\mathcal{L}$-structure and $s : V \to A^{\mathfrak{a}}$ be an assignment to the variables. WTP

$$\mathfrak{a} \models (x = y \to (P(x, z) \to P(y, z)))[s]$$

Assume that $\mathfrak{a} \models (x = y)[s]$ WTP $\mathfrak{a} \models (P(x, z) \to P(y, z))[s]$. Suppose that $\mathfrak{a} \models (P(x, z))[s]$ WTP $\mathfrak{a} \models (P(y, z))[s]$. By assumption $s(x) = s(y)$ and $\langle s(x), s(z) \rangle \in P^{\mathfrak{a}}$ and since $s(x) = s(y)$ we have that $\langle s(y), s(z) \rangle \in P^{\mathfrak{a}}$. By definition this means that $\mathfrak{a} \models (P(y, z))[s]$

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Problem 3. Prove that if $\Gamma \vdash \alpha_1$ and $\Gamma \vdash \alpha_1 \rightarrow \alpha_2$ then $\Gamma \vdash \alpha_2$

Solution. Let $\langle b_1, \ldots, b_n = \alpha_1 \rangle$ be a deduction to α_1 from Γ and $\langle c_1, \ldots, c_m = \alpha_1 \rightarrow \alpha_2 \rangle$ be a deduction to $\alpha_1 \rightarrow \alpha_2$ from Γ, then $\langle b_1, \ldots, b_n, c_1, \ldots, c_m, \alpha_2 \rangle$ is a deduction for α_2 from Γ since α_2 is obtained by MP from b_n and c_m.

Problem 4. Show that $\vdash \exists v_1 P(v_1) \rightarrow \exists v_2 P(v_2)$

[Small Hint: Use the generalization theorem]

(1) $\forall v_2 \neg P(v_2) \rightarrow \neg P(v_1)$. [Ax 2]

(2) $\forall v_1(\forall v_2 \neg P(v_2) \rightarrow \neg P(v_1))$ [Generalization Thm.]

(3) $\forall v_1(\forall v_2 \neg P(v_2) \rightarrow \neg P(v_1)) \rightarrow (\forall v_1 \forall v_2 \neg P(v_2) \rightarrow \forall v_1 \neg P(v_1))$ [Ax 3]

(4) $\forall v_1 \forall v_2 \neg P(v_2) \rightarrow \forall v_1 \neg P(v_1)$ [MP]

(5) $\forall v_2 \neg P(v_2) \rightarrow \forall v_1 \forall v_2 \neg P(v_2)$ [Ax 4, (2),(3)]

(6) $(\forall v_2 \neg P(v_2) \rightarrow (\forall v_1 \forall v_2 \neg P(v_2)) \rightarrow \Big((\forall v_1 \forall v_2 \neg P(v_2) \rightarrow \forall v_1 \neg P(v_1)) \rightarrow (\forall v_2 \neg P(v_2) \rightarrow \forall v_1 \neg P(v_1))\Big)$ [Ax1: $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$]

(7) $(\forall v_1 \forall v_2 \neg P(v_2) \rightarrow \forall v_1 \neg P(v_1)) \rightarrow (\forall v_2 \neg P(v_2) \rightarrow \forall v_1 \neg P(v_1))$ [MP, (5),(6)]

(8) $\forall v_2 \neg P(v_2) \rightarrow \forall v_1 \neg P(v_1)$ [MP, (4),(7)]

(9) $\neg \forall v_2 \neg P(v_2) \rightarrow \neg \forall v_1 \neg P(v_1)$ [Ax1- contrapositive + MP with (8)]