MidTerm I- Mathematical Logic

MATH 461 (Instructor: Tom Benhamou) Feb 26

Instructions

The midterm duration is 1 hour, and the highest score possible is 106 (The maximal grade is 100). Upload your solutions to the Canvas page in the designated area. No external material is allowed.

Problems

Problem 1. Prove that $\langle \mathbb{N}\times\mathbb{N}, <_{LEX}\rangle \not\simeq \langle \mathbb{Z}, <\rangle$. (28 pt.)

[Recall: $<_{LEX}$ denotes the lexicographic order on $\mathbb{N}\times\mathbb{N}$ and $<$ is the regular order on the integers.]

Problem 2. On $P(\mathbb{N})$, consider the relation

$$E = \left\{\langle A, B\rangle \in P(\mathbb{N})^2 \mid (A\Delta B)\cap\{2023, 2024\} = \emptyset\right\}.$$

[Recall: the symmetric difference is defined by $A\Delta B = (A\setminus B)\cup(B\setminus A)$.]

(a) Prove that E is an equivalence relation on $P(\mathbb{N})$. (15 pt.)

(b) How many elements are there in the set $P(\mathbb{N})/E$? Provide a system of representatives for E. No proof required. (20 pt.)

[Instructions: your answer should look like "$P(\mathbb{N})/E$ has ... elements, and a system, of representatives is given by ...."]

Problem 3. (a) Formulate Cantor-Berstein Theorem. No proof required. (3 pt.)

(b) A function $f : \mathbb{N} \to \mathbb{N}$ is called *double-valued* if for every $n \in \mathbb{N}$, $|\{m \in \mathbb{N} \mid f(m) = n\}| = 2$. Give an example of a double-valued function. No proof required. (10 pt.)

[Instructions: your solution should look like "Here is my example: define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = \ldots$".]

(c) Compute the cardinality of the set of all double-valued functions $f : \mathbb{N} \to \mathbb{N}$. Prove your answer. (30 pt.)