Intermediate Models of Prikry-Type Forcings

Tom Benhamou

Department of Mathematics, Statistics and Computer Science
University of Illinois at Chicago

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Outline

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   - The Forcing Notion
   - Examples Main result

3. Tree-Prikry Forcing
   - Known Results Regarding the Tree-Prikry forcing
   - Under very large cardinals
   - Cardinality greater than κ

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- In many mathematical theories, such as groups, vector spaces, topological spaces, graphs etc., the study of submodels of a given model is indispensable to the understanding of the model and in some sense measures its complexity.
- In forcing theory, subforcings of a given forcing generate intermediate models to a generic extension by the forcing. Hence, in order to understand the subforcings of a given forcing it suffices to consider the following question, which will be the central to this talk:

Question

Given a forcing notion $\mathbb{P}$, what forcing notions $\mathbb{Q}$ have (consistently have) generic extensions which are intermediate to a generic extension by $\mathbb{P}$?

There are numerous classification results in this spirit, for example:

Theorem 1

1. *(folklore [13]) Any intermediate model of a Cohen generic extension is a Cohen generic extension.*
2. *(D.Maharam [16]) Any intermediate model of a random real generic extension is a random real generic extension.*
3. *(Sacks [21]) There are no proper intermediate models to a generic extension by the Sacks forcing.*

Prikry-Type Forcing

In this talk we will focus on a class of forcing notion called *Prikry-Type* forcing, which is among the most important today tools in the realm of singular cardinals arithmetics and combinatorics. It traces back to Karel Prikry's celebrated work [19], where he defined the standard Prikry forcing, denoted by $\mathbb{P}(U)$ which was designated to be an example of a forcing which preserves cardinals and changes cofinalities:

Definition 2 (Prikry forcing)

Let U be a **normal** measure over a measurable cardinal κ. The conditions of $\mathbb{P}(U)$ are of the form $\langle \alpha_1, ..., \alpha_n, A\rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence of ordinals below κ.
2. $A \in U$, $\min(A) > \alpha_n$ is the set of candidates for the continuation.

The order is define as follows $\langle \alpha_1, ..., \alpha_n, A\rangle \leq \langle \beta_1, ..., \beta_m, B\rangle$ iff:

1. $n \leq m$ and $\alpha_i = \beta_i$ for $1 \leq i \leq n$.
2. $\beta_{n+1}, ..., \beta_m \in A$.
3. $B \subseteq A$.

Prikry sequence illustration

$$\langle A \rangle$$

Prikry sequence illustration

$\langle A \rangle$, Choose $\alpha_1 \in A, A_1 \subseteq A$

Prikry sequence illustration

$$\langle \alpha_1, A_1 \rangle$$

Prikry sequence illustration

$\langle \alpha_1, A_1 \rangle$, Choose $\alpha_2 \in A_1$, $A_2 \subseteq A_1$

Prikry sequence illustration

$$\langle \alpha_1, \alpha_2, A_2 \rangle$$

Prikry sequence illustration

$$\langle \alpha_1, \alpha_2, \alpha_3, A_3 \rangle$$

Prikry sequence illustration

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, A_4 \rangle$$

Prikry sequence illustration

$$\langle \alpha_1, \alpha_2, \alpha_3, ...\alpha_n, A_n \rangle$$

Prikry sequence illustration

$$\langle \alpha_1, \alpha_2, \alpha_3, ...\rangle$$

Prikry sequence illustration

$$\langle \alpha_1, \alpha_2, \alpha_3, ...\rangle$$

This sequence, which we denote by C_G (where G is the generic filter), produced generically by $\mathbb{P}(U)$ is an unbounded and cofinal sequence in κ called a *Prikry sequence for the measure* U. It diagonalizes U.

Prikry forcing with a normal filter

The intermediate models of the Prikry forcing are completely classified:

Theorem 3 (Gitik, Kanovei, Koepke, 2010 [12])

Let U be a normal measure over κ and $G \subseteq \mathbb{P}(U)$ be a V-generic set producing the Prikry sequence $C_G := \{\kappa_n \mid n < \omega\}$. Then for every set of ordinals $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$ [a]

[a] For $A \subseteq On$, $V[A]$ is the minimal ZFC model which includes $V \cup \{A\}$.

Corollary 4

In the settings of the last theorem, let $V \subsetneq M \subseteq V[G]$ be an intermediate ZFC model definable in $V[G]$, then $M = V[G']$ where $G' \subseteq \mathbb{P}(U)$ is another V-generic filter.

Proof.

Every such model is of the form $M = V[A]$ for some set $A \in V[G]$. By theorem 3, $M = V[C]$ for some subsequence C of the Prikry sequence. By the Mathias criteria[17], C is itself a Prikry sequence. □

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Magidor-Radin Forcing

Menachem Magidor introduced [15] his forcing as an example of a forcing which preserves cardinals and changes the cofinality of some measurable cardinal κ of high Mitchell order to be uncountable by adding a club of low order type to κ. A closely related forcing is the Radin forcing[20], which also adds a club with similar to the Magidor club, but can also keep κ regular or even measurable. Nowadays, there are several variations of Magidor and Magidor-Radin forcings in use.
The following maximality result for Magidor's original variation of Magidor forcing is due to Fuchs[8]:

Theorem 5 (Fuchs, G. 2014)

Let c, d be two Magidor generic clubs over V. If $d \in V[c]$ then $d \setminus c$ is finite.

In other words, the only situation when two Magidor generic extensions are intermediate to one another, is if the generic clubs associated are almost included.

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Magidor Forcing I

Let $\vec{U} = \langle U(\alpha,\beta) \mid \alpha \leq \kappa \ , \beta < o^{\vec{U}}(\alpha)\rangle$ be a coherent sequence. We follow the variation of Magidor forcing described in [9] due to Mitchell[18]:

Definition 6

The conditions of $\mathbb{M}[\vec{U}]$ are of the form $\langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle$ where:

1. $\alpha_1 < ... < \alpha_n$ is an increasing sequence below κ.
2. $A_i = \emptyset$ unless $o^{\vec{U}}(\alpha_i) > 0$ in which case, $A_i \in \cap_{\beta < o^{\vec{U}}(\alpha_i)} U(\alpha_i, \beta)$ is a measure one set with respect to **all** the measures given on α_i.
   The order is define as follows,
   $p := \langle\langle\alpha_1, A_1\rangle, ..., \langle\alpha_n, A_n\rangle, \langle\kappa, A\rangle\rangle \leq \langle q := \langle\beta_1, B_1\rangle, ..., \langle\beta_m, B_m\rangle, \langle\kappa, B\rangle\rangle$ iff:
   $\exists 1 \leq i_1 < ... < i_n \leq m$ such that for every $1 \leq j \leq m$:
   1. If $\exists 1 \leq r \leq n$ such that $i_r = j$ then $\beta_{i_r} = \alpha_r$ and $B_{i_r} \subseteq A_r$.
   2. Otherwise let $1 \leq r \leq n+1$ such that $i_{r-1} < j < i_r$ then: $\beta_j \in A_r, \ B_j \subseteq A_r \cap \alpha_r$

Magidor Sequence illustration

Assume for simplicity that $o(\kappa) = 2$, then we have two measures on κ, $U(\kappa, 0)$ which concentrate on $\{\alpha \mid o^{\vec{U}}(\alpha) = 1\}$ and $U(\kappa, 1)$ which concentrate on $\{\alpha \mid o^{\vec{U}}(\alpha) = 0\}$. Start with a condition with no ordinals.

$$\langle\langle\kappa, A\rangle\rangle$$

Magidor Sequence illustration

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$$\langle \langle \kappa, A \rangle \rangle$$

Magidor Sequence illustration

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Option 1:

$$\langle\langle\kappa, A\rangle\rangle$$

Magidor Sequence illustration

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Option 1:

$$\langle \langle \alpha_\omega, A_\omega \rangle, \langle \kappa, A' \rangle \rangle$$

$$o^{\vec{U}}(\alpha_\omega) = 1, \ A_\omega \in U(\alpha_\omega, 0)$$

Magidor Sequence illustration

Assume for simplicity that $o(\kappa) = 2$, then we have two measures on κ, $U(\kappa, 0)$ which concentrate on $\{\alpha \mid o^{\vec{U}}(\alpha) = 1\}$ and $U(\kappa, 1)$ which concentrate on $\{\alpha \mid o^{\vec{U}}(\alpha) = 0\}$. Start with a condition with no ordinals. we now have two options.
Option 2:

$$\langle\langle\kappa, A\rangle\rangle$$

Magidor Sequence illustration

Assume for simplicity that $o(\kappa) = 2$, then we have two measures on κ, $U(\kappa, 0)$ which concentrate on $\{\alpha \mid o^{\vec{U}}(\alpha) = 1\}$ and $U(\kappa, 1)$ which concentrate on $\{\alpha \mid o^{\vec{U}}(\alpha) = 0\}$. Start with a condition with no ordinals. we now have two options.
Option 2:

$$\langle \alpha_0, \langle \kappa, A' \rangle \rangle$$

$$o^{\vec{U}}(\alpha_0) = 0$$

Magidor Sequence illustration

Assume we have chosen option 1:

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

At each stage we can do one of the following.

Magidor Sequence illustration

Assume we have chosen option 1:

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

At each stage we can do one of the following.
Option 1:(start producing a Prikry sequence for α_ω for $U(\alpha_\omega, 0)$)

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

Magidor Sequence illustration

Assume we have chosen option 1:

$$\langle\langle \alpha_\omega, A_\omega\rangle, \langle \kappa, A'\rangle\rangle$$

At each stage we can do one of the following.
Option 1:(start producing a Prikry sequence for α_ω for $U(\alpha_\omega, 0)$)

$$\langle \alpha_1, \langle \alpha_\omega, A_\omega\rangle, \langle \kappa, A'\rangle\rangle$$

$$o^{\vec{U}}(\alpha_1) = 0,$$

Magidor Sequence illustration

Assume we have chosen option 1:

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

At each stage we can do one of the following.
Option 2:(dropping another limit cardinal of the eventual sequence)

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

Magidor Sequence illustration

Assume we have chosen option 1:

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

At each stage we can do one of the following.
Option 2:(dropping another limit cardinal of the eventual sequence)

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\alpha_{\omega\cdot 2}, A_{\omega\cdot 2}\rangle, \langle\kappa, A'\rangle\rangle$$

$$o^{\vec{U}}(\alpha_{\omega\cdot 2}) = 1$$

Magidor Sequence illustration

Assume we have chosen option 1:

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

At each stage we can do one of the following.
option 3:(producing a Prikry sequence for the unknown $\alpha_{\omega\cdot 2}$)

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

Magidor Sequence illustration

Assume we have chosen option 1:

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\kappa, A'\rangle\rangle$$

At each stage we can do one of the following.
option 3:(producing a Prikry sequence for the unknown $\alpha_{\omega\cdot 2}$)

$$\langle\langle\alpha_\omega, A_\omega\rangle, \alpha_{\omega+1}, \langle\kappa, A'\rangle\rangle$$

$$o^{\vec{U}}(\alpha_{\omega+1}) = 0$$

Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle\langle\alpha_\omega, A_\omega\rangle, \langle\alpha_{\omega\cdot 2}, A_{\omega\cdot 2}\rangle, \langle\kappa, A'\rangle\rangle$$

Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle \alpha_1, \langle \alpha_\omega, A_\omega \rangle, \langle \alpha_{\omega\cdot 2}, A_{\omega\cdot 2} \rangle, \langle \kappa, A' \rangle \rangle$$

Magidor Sequence illustration

In this fashion we continue to produce the sequence

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Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle \alpha_1, \langle \alpha_\omega, A_\omega \rangle, \alpha_{\omega+1}, \langle \alpha_{\omega\cdot 2}, A_{\omega\cdot 2} \rangle, \langle \alpha_{\omega\cdot 3}, A_{\omega\cdot 3} \rangle, \langle \kappa, A' \rangle \rangle$$

Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle \alpha_1, \alpha_2, \langle \alpha_\omega, A_\omega \rangle, \alpha_{\omega+1}, \langle \alpha_{\omega\cdot 2}, A_{\omega\cdot 2} \rangle, \langle \alpha_{\omega\cdot 3}, A_{\omega\cdot 3} \rangle, \langle \kappa, A' \rangle \rangle$$

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Magidor Sequence illustration

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Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle \alpha_1, \alpha_2, \langle, \alpha_\omega, A_\omega \rangle, \alpha_{\omega+1}, \alpha_{\omega+2}, \langle \alpha_{\omega\cdot 2}, A_{\omega\cdot 2} \rangle, \alpha_{\omega\cdot 2+1}, \langle \alpha_{\omega\cdot 3}, A_{\omega\cdot 3} \rangle, \alpha_{\omega\cdot 3+1}, \langle \kappa, A' \rangle \rangle$$

Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle \alpha_1, \alpha_2, \langle, \alpha_\omega, A_\omega \rangle, \alpha_{\omega+1}, \alpha_{\omega+2}, \langle \alpha_{\omega\cdot 2}, A_{\omega\cdot 2} \rangle, \alpha_{\omega\cdot 2+1}, \langle \alpha_{\omega\cdot 3}, A_{\omega\cdot 3} \rangle, \alpha_{\omega\cdot 3+1}, \alpha_{\omega\cdot 3+2}, \langle \alpha_{\omega\cdot 4}, A_{\omega\cdot 4} \rangle, \langle \kappa, A' \rangle \rangle$$

Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle \alpha_1, \alpha_2, \alpha_3, \langle \alpha_\omega, A_\omega \rangle, \alpha_{\omega+1}, \alpha_{\omega+2}, \langle \alpha_{\omega\cdot 2}, A_{\omega\cdot 2} \rangle, \alpha_{\omega\cdot 2+1}, \langle \alpha_{\omega\cdot 3}, A_{\omega\cdot 3} \rangle, \alpha_{\omega\cdot 3+1}, \alpha_{\omega\cdot 3+2}, \langle \alpha_{\omega\cdot 4}, A_{\omega\cdot 4} \rangle, \langle \kappa, A' \rangle \rangle$$

Magidor Sequence illustration

In this fashion we continue to produce the sequence

$$\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4, \langle \alpha_\omega, A_\omega \rangle, \alpha_{\omega+1}, \alpha_{\omega+2}, \langle \alpha_{\omega\cdot 2}, A_{\omega\cdot 2} \rangle, \alpha_{\omega\cdot 2+1}, \langle \alpha_{\omega\cdot 3}, A_{\omega\cdot 3} \rangle, \alpha_{\omega\cdot 3+1}, \alpha_{\omega\cdot 3+2}, \langle \alpha_{\omega\cdot 4}, A_{\omega\cdot 4} \rangle, \langle \kappa, A' \rangle \rangle$$

Magidor Sequence illustration

Generically, this forcing produces an ω^2-sequence cofinal at κ.

$$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_\omega, \alpha_{\omega+1}, \alpha_{\omega+2}, \ldots, \alpha_{\omega\cdot 2}, \alpha_{\omega\cdot 2+1}, \ldots . \alpha_{\omega\cdot 3}, \ldots \alpha_{\omega\cdot 4}, \ldots \kappa$$

Magidor Sequence illustration

Generically, this forcing produces an ω^2-sequence cofinal at κ.

$$\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_\omega, \alpha_{\omega+1}, \alpha_{\omega+2}, \ldots, \alpha_{\omega\cdot 2}, \alpha_{\omega\cdot 2+1}, \ldots. \alpha_{\omega\cdot 3}, \ldots \alpha_{\omega\cdot 4}, \ldots \kappa$$

If $G \subseteq \mathbb{M}[\vec{U}]$ is a generic filter, we denote by C_G the Magidor generic sequence generated by G.

Outline

Magidor Forcing- Examples of Intermediate Models

Intermediate Models of a generic extension by $\mathbb{M}[\vec{U}]$ are not necessarily generic extensions of $\mathbb{M}[\vec{U}]$:

Example 7

Assume that $o^{\vec{U}}(\kappa) = 2$. Then κ carries two measures: $U(\kappa, 0), U(\kappa, 1)$. This means that typically $\mathrm{otp}(C_G) = \omega^2$, denote it by $C_G = \{C_G(i) \mid i < \omega^2\}$. For example the intermediate model $V[\{C_G(n) \mid n < \omega\}]$, is a Prikry generic extension.

Example 8

Assume that $o^{\vec{U}}(\kappa) = \omega$, thus $\mathrm{otp}(C_G) = \omega^\omega$. Consider the intermediate extension $V[\{C_G(\omega^n) \mid n < \omega\}]$ it is a diagonal Prikry generic extension for the sequence of measures $\langle U(\kappa, n) \mid n < \omega \rangle$.

Example 9

Let $o^{\vec{U}}(o^{\vec{U}}(\kappa)) = 1$. There is $G \subseteq \mathbb{M}[\vec{U}]$ which produces a Magidor sequence $\{C_G(\alpha) \mid \alpha < \delta_0\}$ such that $C_G(\omega) = \delta_0$. The first Prikry sequence $\{C_G(n) \mid n < \omega\} \in V[G]$ is a cofinal sequence in $C_G(\omega) = \delta_0$. Consider the sequence $C = \{C_G(C_G(n)) \mid n < \omega\}$. It is unbounded in κ and witnesses that κ changes cofinality. This example is quite different from the previous two in the sense that the indices of C inside C_G are $I := \{C_G(n) \mid n < \omega\} \notin V$.

Example 10

Assume $o^{\vec{U}}(\kappa) = \kappa$. Let Again $C_G = \{C_G(\alpha) \mid \alpha < \kappa\}$. In $V[G]$, define $\alpha_0 = C_G(0)$, and $\alpha_{n+1} = C_G(\alpha_n)$. Then $\{\alpha_n \mid n < \omega\}$ is a cofinal ω-sequence in κ.

Theorem 11 (B. (2019)[2])

$\langle \alpha_n \mid n < \omega \rangle$ is Tree-Prikry generic sequence for the measures $\langle U(\kappa, \alpha) \mid \alpha < \kappa \rangle$.

Actually the theorem is a Mathias-like criterion for the Tree-Prikry forcing. Clearly all these example are Prikry-Type extensions.

The Main Result I

We obtained the first step toward a classification of the intermediate models of Magidor-Radin forcing:

Theorem 12 (Gitik, B.[5])

Let $G \subseteq \mathbb{M}[\vec{U}]$ be a V-generic set producing the Magidor sequence C_G. Assume that $\forall\alpha \in C_G \cup \{\kappa\}.o^{\vec{U}}(\alpha) < \alpha^+$. Then for every set of ordinals $A \in V[G]$ there is $C \subseteq C_G$, such that $V[A] = V[C]$. Where C_G is the Magidor club added by G.

As we have seen from the examples, it is not clear which are the forcings that the models $V[C]$ are generic extensions for. In [4], we restrict the order of κ to be below κ and define a class of "Magidor-Type" forcing notions, denoted by $\mathbb{M}_f[\vec{U}]$. This class is basically a Magidor forcing adding elements from measures prescribed by the function f. We then prove that the intermediate model must be finite iterations of such forcings.

Outline

The Tree-Prikry forcing

Let $\vec{U} = \langle U_a \mid a \in [\kappa]^{<\omega} \rangle$ be a tree of κ-complete ultrafilters over κ.

Definition 13 (Tree Prikry Forcing-$P_T(\vec{U})$)

Conditions of $P_T(\vec{U})$ are pairs $\langle t, T \rangle$, where T is a subtree of $[\kappa]^{<\omega}$ with stem t, which is $\vec{U}$-splitting:

$$\forall s \in T. s \geq t \to \mathrm{Succ}_T(s) := \{\alpha < \kappa \mid s^\frown\alpha \in T\} \in U_s$$

The order is defined (Israel convention: $q \leq p$ then $p \vdash q \in \dot{G}$) $\langle t, T \rangle \leq \langle s, S \rangle$ iff $S \subseteq T$ (hence $s \in T$)

There is an equivalent forcing to $P_T(W)$, where W is a non-normal κ-complete ultrafilter (We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = W$). The conditions are of the form $\langle t, A \rangle$ where $A \in W$ and the sequence t is strongly increasing. It turns out (not surprisingly) that the structure of the intermediate models of the tree Prikry forcing depends on the combinatorical properties of the measures in $\vec{U}$.

Intermediate Models of Tree Prikry forcing

Theorem 14 (Koepke, Räsch, Schlicht (2013)[14])

Assume that $\vec{U} = \langle U_\alpha \mid \alpha < \kappa \rangle$ is a sequence of distinct normal measures. Then for every V-generic filter $G \subseteq P_T(\vec{U})$[a], there is no proper intermediate model $V \subsetneq M \subsetneq V[G]$.

[a] We view $\vec{U}$ as a tree by defining for every $a \in [\kappa]^{<\omega}$, $U_a = U_{\max(a)}$.

On the other hand:

Theorem 15 (Gitik, B. (2021)[5])

Assume GCH and let κ be a measurable cardinal. There is a cofinality preserving forcing extension $V \subseteq N$ and an ultrefilter $W \in N$ such that forcing with $P_T(W)$ over N adds a κ-Cohen real.

Prikry introduce Cohen- Proof

Sketch of the Proof.

The model N is obtained by forcing the Easton support iteration $\langle P_\alpha, \underset{\sim}{Q}_\beta \mid \alpha \leq \kappa, \beta < \kappa\rangle$: Each $\underset{\sim}{Q}_\beta$ is trivial, unless β is inaccessible. For inaccessible β, Q_β is the lottery sum of the trivial forcing $\{0\}$ and the β-Cohen real forcing $\widetilde{Add}(\beta, 1)$. Let $G_\kappa \subseteq P_\kappa$ be V-generic and $N := V[G_\kappa]$. The idea is to take $U \in V$ be a normal measure over κ extend it to a (non-normal) κ-complete ultrafilter W which concentrate on the set

$$L_0 = \{\alpha < \kappa \mid G_\alpha \text{ is generic for } Add(\alpha, 1)\}$$

This measure W is obtained by looking at the second iteration of U. For each $\alpha \in L_0$, let f_α be the Cohen function added by G_κ. Force $P_T(W)$ over N, and denote by $C_G := \{\kappa_n \mid n < \omega\}$ the Prikry sequence. There is $n_0 < \omega$ such that for every $n \geq n_0$, $\kappa_n \in L_0$ and therefore f_{κ_n} is defined. It remains to see that

$$f = \cup_{n_0 \leq n < \omega} f_{\kappa_n} \restriction [\kappa_{n-1}, \kappa_n) \in N[C_G]$$

is N-generic for $Add(\kappa, 1)$. □

Outline

Assuming κ is κ-compact

Definition 16 (κ-compact Cardinal)

κ is called a κ-compact cardinal if every κ-complete filter over κ can be extended to a κ-complete ultrafilter over κ

The ability to extend κ-complete filters is deeply connected to our problem:

Theorem 17 (Gitik, Hayut, B. 2021[7])

Let $\mathbb{P}$ be a σ-distributive forcing of size κ. The following are equivalent:

- *There is a tree $\vec{\mathcal{U}}$ of κ-complete ultrafilters and a projection $\pi : \mathbb{P}_T(\vec{\mathcal{U}}) \to B(\mathbb{P})$.*
- *For every $p \in \mathbb{P}$, $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter U_p. Where $D_p(\mathbb{P})$ is the filter of open subsets of $\mathbb{P}$ which are dense above p.*

Corollary 18

If κ is κ-compact, every κ-distributive forcing of cardinality κ is a projection of a Tree-Prikry forcing.

Lower bound for all the κ-distributive

The assumption that κ is κ-compact is quit strong:

Theorem 19 (Gitik [11])

If κ is κ-compact then there is an inner model with a Woodin cardinal.

Question

Can the assumption that κ is κ-compact be relaxed?

Since we only wish to extend a relatively easily definable filter $D_p(\mathbb{P})$, it suffices to assume that κ is 1-extendable. However, we cannot hope to improve this bound much further. In [7], we found that there is a non trivial lower bound:

Theorem 20 (Gitik, Hayut, B.)

Let Q be the forcing shooting a club through the singulars below κ[a]. Assume that there is a κ-complete ultrafilter extending the filter $D(Q)$ of dense open subset of Q. Then either there is an inner model for $\exists\lambda, o(\lambda) = \lambda^{++}$, or $o^{\mathcal{K}}(\kappa) \geq \kappa^{+}$.

[a] Thus Making κ not Mahlo. It is $< \kappa$-strategically closed.

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Adding more than one Cohen and non-Galvin Ultrafilters

What limitations do we have on projections of the Tree-Prikry forcing. In terms of cardinality it should be at most 2^κ. Also, κ-centered is essential:
If $\mathbb{P} = \cup_{i<\kappa} A_i$ such that A_i is a directed set, and $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection, then $\mathbb{Q} = \cup \pi'' A_i$ and each $\pi'' A_i$ is a directed set.

Corollary 21

$Add(\kappa^+, 1)$ (Nor $B(Add(\kappa^+, 1))$) is not a projection of the Tree-Prikry forcing.

The forcing $Add(\kappa, \kappa^+)$ on the other hand is κ-centered.
In a very recent joint result with Gitik we have proved that we can actually get the consistency of κ^+-many Cohen functions of κ as an subfrorcing of the Tree-Prikry forcing is also consistent (starting from a measurable). This is done using a non-Galvin ultrafilter.

Definition 22

A κ-complete ultrafilter U is called a *Galvin*-ultrafilter, if for every $\langle X_i \mid i < \kappa^+ \rangle \in [U]^{\kappa^+}$ there is $I \in [\kappa^+]^\kappa$ such that $\cap_{i \in I} X_i \in U$.

Galvin proved that normal ultrafilters are Galvin [1].

For adding κ^+-many Cohens to κ, it is necessary to force with a non-Galvin ultrafilter:

Proposition 1

Let U is a Galvin ultrafilter and $G \subseteq \mathbb{P}_T(U)$ be V-generic. Then for any subset $A \in V[G]$, $A \subseteq V$, $|A| = \kappa^+$, there is $A' \in V$ such that $|A'| = \kappa$ and $A' \subseteq A$.

Proof.

Suppose otherwise, and let $f : \kappa^+ \to \kappa^+$ enumerating A. On one hand, translating the assumption on A, there is no $g \in V$ such that $|g| = \kappa$ and $g \subseteq f$. On the other hand, for every $\alpha < \kappa^+$ find a condition $p_\alpha = \langle t_\alpha, A_\alpha \rangle \in \mathbb{P}(U)$ such that p_α decides the value $\underset{\sim}{f}(\alpha)$. Then there is $X \subseteq \kappa^+$ and t^* such that $|X| = \kappa^+$ and for every $\alpha \in X$, $t_\alpha = t^*$. Consider $\langle A_\alpha \mid \alpha \in X \rangle$ and apply the Galvin property to find $Y \subseteq X$ such that $|Y| = \kappa$ and $A^* := \cap_{y \in Y} A_y \in U$. Then $\langle t^*, A^* \rangle$ decides κ-many values of $\underset{\sim}{f}$, contradiction.□

Actually the other direction is also true, that is there is no such subset in $V[G]$ then U must be Galvin[10],[3].

Corollary 23

If U is Galvin then U does not add a κ^+-many Cohen function.

Proof.

Indeed if $f : \kappa^+ \to 2$ is a $Add(\kappa, \kappa^+)$-generic, then by density argument the set $A = \{\alpha < \kappa^+ \mid f(\alpha) = 1\}$ has no V-subset of cardinality κ.

Theorem 24 (Gitik, B. (2022)[6])

Starting from a measurable cardinal, it is concictent that there is a non-Galvin ultrafilter U such that forcing $\mathbb{P}_T(U)$ adds a generic for $Add(\kappa, \kappa^+)$.

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Finish line

Thank you for your attention!