Cofinal types of ultrafilters on measurable and non-measurable cardinals

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Sequencial Continuity Vs. Continuity

Let $(X,\tau_X),(Y,\tau_Y)$ be Hausdorff topological spaces. Recall that

Definition 1

A function $f:X\to Y$ is continuous in the sequential sense if whenever $(x_n)_{n=0}^\infty \subseteq X$ is a sequence converging to $x\in X$ (namely, for every neighborhood $x\in U\in\tau_X$ there is N such that for all $n\geq N$, $x_n\in U$), the sequence $(f(x_n))_{n=0}^\infty$ converges to $f(x)$.

It is well known that first-countable spaces a function f is continuous if and only if f is continuous in the sequential sense. In general the two are not equivalent (For example $f:\omega_1+1\to\mathbb{R}$ defined by $f(x)=0$ if $x<\omega_1$ and $f(\omega_1)=1$ is not continuous but sequentially continuous.)

Definition 2

A *net* is a function $\vec{x}=(x_a)_{a\in A}$ such that $(A,\leq_A)$ is a directed set. x is a limit of $\vec{x}$ if for every $x\in U\in\tau_X$ there is a such that , $b\geq a$, $x_b\in U$ (AKA Moore-Smith convergence).

Now a function $f:X\to Y$ is continuous iff for every net $(x_a)_{a\in A}$ with limit x, $(f(x_a))_{a\in A}$ has limit $f(x)$.

Cofinal maps

Some "types" of directed sets actually give essentially the same notion of net, for example, $\mathbb{N}$ and $\mathbb{N}_{even}$ or even $fin = \{X \in P(\mathbb{N}) \mid X \text{ is finite}\}$. More generally we would like to find an equivalence relation that reduces to the "essential" ordered sets. This is given by the Tukey order which was defined by J. Tukey [7]:

Definition 3

Let $(P, \leq_P), (Q, \leq_Q)$ be two partially ordered (directed) sets. Define $(P, \leq_P) \leq_T (Q, \leq_Q)$ iff there is a cofinal map[a] $f : Q \to P$. Define $(P, \leq_P) \equiv_T (Q, \leq_Q)$ iff $(P, \leq_P) \leq_T (Q, \leq_Q)$ and $(Q, \leq_Q) \leq_T (P, \leq_P)$.

[a] if for every cofinal $B \subseteq Q$, $f[B] \subseteq P$ is cofinal.

If $B \leq_T A$, then any B-net $(x_b)_{b \in B}$ can be now replaced by $(x_{f(a)})_{a \in A}$ and if x is a limit point of (x_b) then x must be a limit of $(x_{f(a)})_{a \in A}$.
The research of what are the "essential" A's is a completely set theoretic (order theoretic) question.

Classic results of Todorcevic

Theorem 4 (Todorcevic 85[6])

Assuming $MA_{\aleph_1}$ it is consistent that there are exactly 5 Tukey classes of directed posets of cardinality at most $\aleph_1$.

Theorem 5 (Todorcevic 85[6])

for any regular $\kappa > \omega$, there are 2^κ-many distinct Tukey classes of cardinality $\kappa^{\aleph_0}$. In particular, there are at least $2^{cf(\mathfrak{c})}$ many distinct Tukey classes of cardinality $\mathfrak{c}$.

Definition 6

Given a net $\vec{x} = (x_a)_{a \in A}$, define for each $a \in A$, $x_{\geq a} = \{x_b \mid b \geq a\}$. The filter associated with $\vec{x}$, denoted by $F_{\vec{x}}$ is the filter generated by the sets $x_{\geq a}$. Namely, $T \in F_{\vec{x}}$ iff $\exists a \in A,\ x_{\geq_a} \subseteq T$.

Indeed, $F_{\vec{x}} \subseteq P(X)$ is a filter over X:

1. $\emptyset \notin F_{\vec{x}}$, $X \in F_{\vec{x}}$.
2. Upward closed with respect to inclusion.
3. Closed under finite intersections

Filters catches the abstract notion of "large sets". The filter $F_{\vec{x}}$ determines the convergence properties of the net $\vec{x}$. Yet, we can restrict our attention to maximal filters, namely, ultrafilers:

Definition 7

A filter U over X is an *ultrafilter* if for every $B \subseteq X$, either $B \in U$ or $X \setminus B \in U$.

It is well known that under the axiom of choice (which we assume), every filter can be extended to an ultrafilter. If $F_{\vec{x}} \subseteq U$ and U is an ultrafilter, then whenever x is a limit of $\vec{x}$, x is a limit of U. Therefore, for most purposes, is suffices to consider only ultrafilters, or *ultranets*.

The Tukey order on ultrafilters

As we have seen earlier, it suffices to study the cofinal types of ultrafilters. This motivates the study of the directed order $(U, \supseteq)$ where U is an ultrafilter.

Proposition 1

Suppose that $U \leq_T V$ where U, V are ultrafilters, then there is a (weakly) monotone map $f : V \to U$ which is cofinal.

The Tukey order has been studied extensively on ultrafilters on ω by Blass, Dobrinen, Milovic, Raghavan, Shelah, Solecki, Todorcevic, Verner and many others. It still entails quite challenging open problems. However, the investigation regarding ultrafilters on a set of uncountable cardinality, and in particular on measurable cardinals, is limited.

The Tukey class of a Fubini product of ultrafilters.

Fact 8

Let P, Q be directed orders. Then $P \times Q$ is the least upper bound of p, Q in the Tukey order. Hence $P =_T P \times P$.

Definition 9 (Fubini product)

Suppose that U is a filter over X and for each $x \in X$, U_x is a filter over Y_x. We denote by $\sum_U U_x$ the filter over $\bigcup_{x \in X} \{x\} \times Y_x$, defined by

$$A \in \sum_U U_x \text{ if and only if } \{x \in X \mid (A)_x \in U_x\} \in U$$

where $(A)_x = \{y \in Y_x \mid \langle x, y \rangle \in A\}$. If for every x, $U_x = V$ for some fixed V over a set Y, then $U \cdot V$ is defined as $\sum_U V$, which is a filter over $X \times Y$.

It is not hard to show that $U, V \leq_T U \cdot V$ and therefore $U \times V \leq_T U \cdot V$.

Theorem 10 (Dobrinen-Todorcevic[3], Milovich[5])

For any U, V, $U \cdot V =_T U \times \prod_{n<\omega} V$.

Definition 11

Let U be an ultrafilter over $\mathbb{N}$.

- U is a p-point if every sequence $\langle X_n \mid n < \omega\rangle \subseteq U$ has a U-measure one pseudo intersection.
- U is rapid if for every function $f : \mathbb{N} \to \mathbb{N}$ there is $X \in U$ such that for every $n < \omega$, $X(n) \geq f(n)$.

These definitions are obviously generalized to any cardinal $\kappa > \omega$.

Theorem 12 (Dobrinen-Todorcevic[3])

Suppose that V, U are ultrafilters on ω, V is a rapid p-point. Then $U \cdot V \equiv_T U \times V$. In particular, if U, V are rapid p-points then $U \cdot V =_T V \cdot U$.

In particular if U is a rapid p-point then $U \cdot U \equiv_T U$. Moreover, Dobrinen and Todorcevic constructed an example of a non-rapid p-point ultrafilter U such that $U <_T U^2$.

Theorem 13 (Milovich[5])

If U is a p-point ultrafilter then on ω and V is any ultrafilter, then $V \cdot U = V \times U \times \omega^\omega$ and therefore if U, V are both p-points then $U \cdot V =_T V \cdot U$.

Theorem 14 (Dobrinen-B.[1])

Let U, V be any κ-complete ultrafilters over $\kappa > \omega$, then $U \cdot V \equiv_T U \times V$. In particular $U \cdot V =_T V \cdot U$ and $U \cdot U \equiv_T U$.

Corollary 15 (A corollary for set theorist)

In $L[U]$ there is a single Tukey class.

On the class of $U \cdot U =_T U$

Theorem 16 (Isbell-juhazs[4])

There is an ultrafilter $\mathcal{U}_{top}$ over ω such that for every ultrafilter U over ω, $U \leq_T \mathcal{U}_{top}$

Clearly $\mathcal{U}_{top} \cdot \mathcal{U}_{top} =_T \mathcal{U}_{top}$

Question (Dobrinen-Todorcevic[3])

Suppose to U is an ultrafilter over a countable set, does $U \cdot U =_T U <_T \mathcal{U}_{top}$ implies that U is basically generated?

Recently, together with Dobrinen, we answered this question negatively:

Theorem 17 (Dobrinen-B. 2023)

It is consistent that there is an ultrafilter U over ω such that $U \cdot U =_T U < \mathcal{U}_{top}$ which is not basically generated.

Theorem 18 (B. 2024)

For any two ultrafilters U, V (on any cardinal), $U \cdot V =_T V \cdot U$.

The dual notion of a filter is an *ideal* which is just a set of the form $F^* = \{A^c \mid A \in F\}$ where F is a filter. Ideals catches the notion of smallness. For example $fin = \{A \subseteq \omega \mid A \text{ finite}\}$ is an ideal.

Definition 19 (Dobrinen-B.)

Suppose that U is an ultrafilter and $I \subseteq U^*$ is an ideal. We say that U has the I-p.i.p if for any sequence $\langle X_n \mid n < \omega \rangle$, there is $X \in U$ such that for every $n < \omega$, $X \setminus X_n \in I$.

For example, U is a p-point if and only if U has the fin-p.i.p.

Proposition 2

Suppose that U has the I-p.i.p, then $U \cdot U =_T \prod_{n<\omega} U \leq_T U \times \prod_{n<\omega} I$.

For example, if U is a p-point then $U \cdot U \leq_T U \times \omega^\omega$.

Theorem 20

Let I be a σ-ideal over a countable set X, and $G \subseteq P(X)/I$ be a V-generic ultrafilter. Then G has the I-p.i.p

We used that to prove that if $I = fin \cdot fin$ then $G \cdot G =_T G$ and by results of Blass Dobrinen and Raghavan [2], $G <_T \mathcal{U}_{top}$ and not basically generated.

Theorem 21

If U and V are ultrafilters, then U (and V of course) have the $U \cap V$-p.i.p.

Corollary 22

$U \cdot V \leq_T U \times \prod_{n<\omega} U \cap V$.

Theorem 23

For every filter $I \subseteq U^*$, $U \cdot V \geq_T \prod_{n<\omega} I$. *In particular,* $U \cdot V = U \times V \times \prod_{n<\omega} U \cap V$.

Corollary 24

For every ultrafilters U, V, $U \cdot V =_T V \cdot U$.

Corollary 25

If $U \cdot U =_T U$ *then for every* $U \leq_T V$, $V \cdot V =_T V$.

Thank you for your attention!

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In an attempt to approximate the non-Tukey-top class of ultrafilters, we have the following definition:

Definition 26

A κ-complete ultrafilter U over κ is basically generated if there is a cofinal set $B \subseteq U$ such that for every sequence $\langle A_\alpha \mid \alpha < \kappa \rangle \subseteq B$, which converges to an element of B, there is $f : \kappa \to \kappa$ such that for every $f \leq^* g$, $\bigcap_{\alpha<\kappa} A_{g(\alpha)} \in U$.

The results below are due to Dobrinen-Todorcevic for $\kappa = \omega$ and B.-Dobrinen for $\kappa > \omega$.

Theorem 27

A p-point ultrafilter over a measurable cardinal is basically generated which in turn implies non-Tukey-top

Theorem 28

Given $U, (V_\alpha)_{\alpha<\kappa}$, basically generated if $\sum_U V_\alpha$ is basically generated. In particular, the product and powers of basically generated ultrafilters are basically generated.