THE VARIETY OF PROJECTIONS OF A TREE-PRIKRY FORCING

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ABSTRACT. We study which κ-distributive forcing notions of size κ can be embedded into tree Prikry forcing notions with κ-complete ultrafilters under various large cardinal assumptions. An alternative formulation - can the filter of dense open subsets of a κ-distributive forcing notion of size κ be extended to a κ-complete ultrafilter.

1. INTRODUCTION

In this paper we will study possibilities of embedding of κ-distributive forcing notions of size κ into Prikry forcings with non-normal ultrafilter or into tree Prikry forcing notions with κ-complete ultrafilters.

By the result of Kanovei, Koepke and the second author [11] every subforcing of the standard Prikry forcing is either trivial or equivalent to the Prikry forcing with the same normal ultrafilter. However, the situation changes drastically if non-normal ultrafilters are used.

Existence of such embedding allows one to iterate distributive forcing notions on different cardinals, see [9, Section 6.4].

A closely related problem is the possibility of extension of the filter of dense open subsets of a κ-distributive forcing notion of size κ to a κ-complete ultrafilter, the exact statement is given in theorem 13.

Clearly, if κ is a κ-compact cardinal, then this follows. Actually more is true—there is a single Prikry type forcing, such that any κ-distributive forcing notion of size κ embeds into it, see [10].

However, there are κ-distributive forcing notion of size κ which can be embedded into Prikry forcing notions under much weaker assumptions. Thus, for example, in [4] starting from a measurable cardinal, a generic extension in which there is a κ-complete ultrafilter on κ, $\mathcal{U}$, such that the tree Prikry forcing using $\mathcal{U}$ introduces a Cohen subset of κ was constructed.

This paper investigates different possibilities which are intermediate between those two extremes. More specifically, let H be a subclass of the κ-distributive of size κ forcings, we examine the following question:

Can the dense open filter, $D(\mathbb{Q})$, of any $\mathbb{Q} \in H$
be extended to a κ-complete ultrafilter?

Our notations are mostly standard. For general information about Prikry type forcing we refer the reader to [9]. For general information about large cardinals we refer the reader to [14].

Throughout the paper, $p \leq q$ means that p is weaker than q.

The structure of the paper is as follows:

- Section 2 is intended to give the reader background and basic definitions which appear in this paper.

- The main result of section 3 is theorem 13: Let $\mathbb{Q}$ be a σ-distributive forcing of size κ. Then $B(\mathbb{Q})$ is a projection of the tree Prikry forcing if and only if $D(\mathbb{Q})$ can be extended to a κ-complete ultrafilter. Moreover the ultrafilter extending $D(\mathbb{Q})$ must be Rudin-Keisler below the ultrafilters of the tree Prikry forcing.
- Section 4 deals with the class of κ-strategically closed and $<\kappa$-strategically closed forcings. Lemma 17 establishes that $\mathrm{Add}(\kappa, 1)$ projects onto every κ-strategically closed forcing of cardinality κ. Also, we present the forcing that adds a Jensen square (see definition 20) and prove that it maximal among all the $<\kappa$-strategically closed forcings, this is formulated in Lemma 22.
- Section 5 focuses on upper bounds. In theorem 24 we give an upper bound for the claim "For every $<\kappa$-strategically closed forcing of size κ $\mathbb{P}$ and every $p \in \mathbb{P}$, $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter". In the rest of the section we discuss some weaker version of Π_1^1-subcompact cardinal which is an upper bound for the claim "For every κ-distributive forcing of size κ $\mathbb{P}$ and every $p \in \mathbb{P}$, $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter.
- Section 6 is devoted to the forcing Q, of shooting a club through the singulars. This forcing is a milestone for the class of $<\kappa$-strategically closed forcing of size κ. In theorem -34, we prove that if we can extend $D(Q)$, then either $\exists \lambda o(\lambda) = \lambda^{++}$ or $o^{\mathcal{K}}(\kappa) \geq \kappa + \kappa$.
- Section 7 provides a strengthening of results of section 6 to $o^{\mathcal{K}}(\kappa) \geq \kappa^+ + 1$.
- Section 8 defines a class called *masterable forcing*. We show, starting with a measurable, that one can force that the filter of $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter for every masterable forcing $\mathbb{P}$. In this generic extension we give examples of many important forcing notions which are masterable.
- Section 9 presents forcing notions which do not fall under the examples considered in this paper and present further research directions.

2. Preliminaries

Let us recall some basic concepts about forcing notions and Tree Prikry forcing. First, our forcing notions are always separative and have a minimal element. We force upward i.e. $p \leq q$ means that $q \Vdash p \in \dot{G}$. Let us start with the concept of projection:

Definition 1. Let $\mathbb{P}, \mathbb{Q}$ be forcing notions, $\pi : \mathbb{P} \to \mathbb{Q}$ is a projection if
  (1) π is order preserving.
  (2) $\forall p \in \mathbb{P} \forall \pi(p) \leq q \exists p' \geq p.\pi(p') \geq q$. projection.
  (3) $Im(\pi)$ is dense in $\mathbb{P}$.

Definition 2. Let $\pi : \mathbb{P} \to \mathbb{Q}$ be a function
  (1) If $G \subseteq \mathbb{P}$ is V-generic, define
  $$\pi_*(G) = \{q \in \mathbb{Q} \mid \exists p \in G.q \leq \pi(p)\}$$
  (2) If $H \subseteq \mathbb{Q}$ is V-generic, define the quotient forcing
  $$\mathbb{P}/H = \pi^{-1}[H] = \{p \in \mathbb{P} \mid \pi(p) \in H\}$$

  With the separative order $p \leq_{\mathbb{P}/H} q$ if an only if for every $q \leq_{\mathbb{P}} r$, r is compatible with p.

Claim 3. *Let $\mathbb{P}, \mathbb{Q}$ be any forcing notions, then:*
  (1) *Let $G \subseteq \mathbb{P}$ be V-generic and $\pi : \mathbb{P} \to \mathbb{Q}$ a projection, then $\pi_*(G) \subseteq \mathbb{Q}$ is V-generic*

(2) *Let $H \subseteq \mathbb{Q}$ be V-generic and $\pi : \mathbb{P} \to \mathbb{Q}$ a projection, then if $G \subseteq \mathbb{P}/H$ is V[H]-generic, then $G \subseteq \mathbb{P}$ is V-generic, moreover, $\pi_*(G) = H$.*
(3) *Let $G \subseteq \mathbb{P}$ be V-generic and $\pi : \mathbb{P} \to \mathbb{Q}$ a projection, then $G \subseteq \mathbb{P}/\pi_*(G)$ is V[\pi_*(G)]-generic.*

Definition 4. Let $\mathbb{P}$ be a forcing notion, denote by $B(\mathbb{P})$ the complete boolean algebra of regular open sets of $\mathbb{P}$.

If is known that $\mathbb{P}$ can be identify with a dense subset of $B(\mathbb{P})$ and that $B(\mathbb{P})$ is the unique (up to isomorphism) complete boolean algebra we a dense subset isomorphic to $\mathbb{P}$. Moreover, $\mathbb{P}$ and $B(\mathbb{P})$ yield the same generic extensions. Let $G \subseteq \mathbb{P}$ be a V-generic filter then $\bar{G} = \{b \in B(\mathbb{P}) \mid \exists p \in G.b \le p\} \subseteq B(\mathbb{P})$ is V-generic and if $\bar{G} \subseteq B(\mathbb{P})$ is V-generic then $G = \bar{G} \cap \mathbb{P} \subseteq \mathbb{P}$ is V-generic. For more information about boolean algebras see [20] or [1].

Claim 5. *Let $\mathbb{P}, \mathbb{Q}$ be forcing notions. Then:*
(1) *There is a projection $\pi \colon \mathbb{P} \to B(\mathbb{Q})$ if and only if there is a $\mathbb{P}$-name $\underset{\sim}{H}$ such that for every generic filter H for $\mathbb{Q}$ there is a generic filter G for $\mathbb{P}$ such that $(\underset{\sim}{H})_G = H$.*
(2) *There is a strong projection $\pi : \mathbb{P} \to B(\mathbb{Q})$ iff there is a $\mathbb{P}$-name $\underset{\sim}{H}$ such that for every V-generic filter H for $\mathbb{Q}$ there is a V-generic filter G for $\mathbb{P}$ such that $(\underset{\sim}{H})_G = H$.*

Definition 6. Let $\mathbb{P}$ be a forcing notion and let κ be a cardinal. $\mathbb{P}$ is κ-distributive if for every collection $\mathcal{D}$ of dense open subsets of $\mathbb{P}$, $|\mathcal{D}| < \kappa$, the intersection $\bigcap \mathcal{D}$ is also a dense open subset of $\mathbb{P}$.

Note that if $\mathbb{P}$ is κ-distributive then the filter generated by the dense open subsets of $\mathbb{P}$ is κ-complete.

Notation. Let $\mathbb{P}$ be a forcing notion. We denote by $\mathcal{D}(\mathbb{P})$ the filter for dense open subsets of $\mathbb{P}$. For $p \in \mathbb{P}$ let $D_p(\mathbb{P})$ be the filter generated by $D(\mathbb{P})$ and the set $\{q \in \mathbb{P} \mid q \ge p\}$.

Let us define the tree Prikry forcing. Let κ be a cardinal, and let $\vec{\mathcal{U}} = \langle U_\eta \mid \eta \in [\kappa]^{<\omega} \rangle$ be a sequence of ultrafilters on κ, indexed by $[\kappa]^{<\omega}$ which is the set of all finite sequences below κ. Such that U_η concentrate on the set $\kappa \setminus \max(\eta) + 1$.

Let us define the forcing $\mathbb{T}_{\vec{\mathcal{U}}}$. An element in $\mathbb{T}_{\vec{\mathcal{U}}}$ is a pair $\langle s, T \rangle$ where:
(1) $s \in [\kappa]^{<\omega}$.
(2) $T \subseteq [\kappa]^{<\omega}$, and for all $t \in T$, $s \trianglelefteq t$.
(3) T is $\vec{\mathcal{U}}$-splitting: for all $t \in T$, $\{\nu < \kappa \mid t^\frown \nu \in T\} \in U_t$.

For $T \subseteq [\kappa]^{<\omega}$ and $\eta \in T$ we denote $T_\eta = \{s \in [\kappa]^{<\omega} \mid \eta^\frown s \in T\}$.

For $p = \langle s, T \rangle$, $p' = \langle s', T' \rangle \in \mathbb{T}_{\vec{\mathcal{U}}}$, $p' \le p$ and say that p extends p' if $s \in T'$ and $T \subseteq T'_s$. We denote $p' \le^* p$ and say that p is a direct extension of p' if $p' \le p$ and $s = s'$.

We will assume always that each U_η is κ-complete. In this case, the relation $\le^*$ is κ-complete.

The following claim is well known [3, Lemma 3.16]:

Lemma 7 (Strong Prikry Lemma)**.** *Let $D \subseteq \mathbb{T}_{\vec{\mathcal{U}}}$ be dense open and let $p = \langle s, T \rangle \in \mathbb{T}_{\vec{\mathcal{U}}}$ be a condition. There is a direct extension of $p \le^* p^* = \langle s, T^* \rangle$, and a natural number n such that for all $\eta \in T^*$, with $\operatorname{len} \eta = n$, $\langle s^\frown \eta, T^*_\eta \rangle \in D$ and for all η such that $\operatorname{len} \eta < n$, $\langle s^\frown \eta, T^*_\eta \rangle \notin D$.*

When analyzing a tree of measures there is a natural iteration of ultrapowers to consider.

Definition 8. Let $\vec{\mathcal{U}}$ be a tree of κ-complete ultrafilters and $\eta \in [\kappa]^{<\omega}$. For $\vec{\mathcal{U}}$ and $0 < n < \omega$, define recursively the nth ultrafilter above η derived from $\vec{\mathcal{U}}$, denoted $(\mathcal{U}_\eta)_n$, to be the following ultrafilter over $[\kappa]^n$:

$$(\mathcal{U}_\eta)_1 = \mathcal{U}_\eta$$

For $A \subseteq [\kappa]^{n+1}$ define

$$A \in (\mathcal{U}_\eta)_{n+1} \iff \{\gamma \in [\kappa]^n \mid A_\gamma \in \mathcal{U}_{\eta^\frown \gamma}\} \in (\mathcal{U}_\eta)_n$$

where

$$A_\gamma = \{\alpha < \kappa \mid \gamma^\frown \alpha \in A\}$$

Definition 9. Let $\vec{\mathcal{U}}$ be a tree of κ-complete ultrafilters, define recursively the *iteration corresponding to* $\vec{\mathcal{U}}$ above $\eta \in [\kappa]^{<\omega}$.

$$j_0 = j_{\mathcal{U}_\eta} : V \to M_0 \simeq Ult(V, \mathcal{U}_\eta), \quad \delta_0 = [id]_{\mathcal{U}_\eta}$$

$$j_{n,n+1} : M_n \to \mathrm{Ult}(M_n, j_n(\vec{\mathcal{U}})_{\vec{\eta}^\frown \langle \delta_0, \ldots, \delta_n \rangle}) \simeq M_{n+1}$$

$\delta_{n+1} = [id]_{j_n(\vec{\mathcal{U}})_{\vec{\eta}^\frown \langle \delta_0, \ldots, \delta_n \rangle}}, j_{n+1} = j_{n,n+1} \circ j_n$ and $j_{m,n+1} = j_{n,n+1} \circ j_{m,n}$

The following theorem can also be found in [3]:

Theorem 10. *Let M_ω be the ω-th iteration of the iteration corresponding to $\vec{\mathcal{U}}$ above $\vec{\eta}$ i.e. M_ω is the transitive collapse of the direct limit of the system $\langle M_n, j_{n,m} \mid n, m < \omega \rangle$ defined in 9, denote the direct limit embeddings by $j_{n,\omega} : M_n \to M_\omega$. Then the sequence $\vec{\eta}^\frown \langle \delta_n \mid n < \omega \rangle$ is M_ω-generic for the forcing $j_\omega(\mathbb{T}_{\vec{\mathcal{U}}})$.*

Claim 11. *For every $A \subseteq [\kappa]^n$*

$$\langle \delta_0, \ldots, \delta_{n-1} \rangle \in j_{n-1}(A) \iff A \in (\mathcal{U}_\eta)_n$$

Proof. For $n = 1$ it is just Łoś theorem $[id]_{\mathcal{U}_\eta} \in j_1(A) \iff A \in \mathcal{U}_\eta = (\mathcal{U}_\eta)_1$. Assume that the claim holds for n, and let $A \subseteq [\kappa]^{n+1}$. Denote by $\vec{\delta}_m = \langle \delta_0, \ldots, \delta_m \rangle$, then

$$\vec{\delta}_n \in j_n(A) \iff \delta_n \in j_n(A)_{\vec{\delta}_{n-1}} \iff j_{n-1}(A)_{\vec{\delta}_{n-1}} \in j_{n-1}(\vec{\mathcal{U}})_{\eta^\frown \vec{\delta}_{n-1}}$$

By the definition of $j_{n-1}(A)_{\vec{\delta}_{n-1}}$ and the induction hypothesis we can continue the chain of equivalences

$$\iff \{\gamma \mid A_\gamma \in \mathcal{U}_{\eta^\frown \gamma}\} \in (\mathcal{U}_\eta)_n \iff A \in (\mathcal{U}_\eta)_{n+1}$$

$\square$

3. Subforcing of the tree Prikry forcing

In this section we characterize the σ-distributive complete subforcings of a tree Prikry forcing. Since no bounded subsets of κ are introduced, such a forcing is either trivial or (κ, κ)-centered i.e. it is the union of κ many sets A_i for $i < \kappa$ such that each A_i is κ-directed. Standard arguments show that those forcing notions have to be κ-distributive. By a theorem of Gitik (see [10]), if κ is κ-compact, then there is a Prikry type forcing which absorbs every κ-distributive forcing $\mathbb{P}$ of cardinality κ. A simpler version of this theorem is stated in the following claim:

Claim 12. *Assume that for every $p \in \mathbb{P}$, we can extend $D_p(\mathbb{P})$ to a κ-complete ultrafilter U_p. Then there is a tree of κ complete ultrafilter*

$$\vec{\mathcal{W}} = \langle W_\eta \mid \eta \in [\kappa]^{<\omega} \rangle$$

and a projection $\pi : \mathbb{T}_{\vec{\mathcal{W}}} \to B(\mathbb{P})$

Proof. We would like to turn the ultrafilters U_p to ultrafilters on κ. For this, we first need to identify $\mathbb{P}$ with $[\kappa]^{<\omega}$ somehow. We define inductively for every $\eta \in [\kappa]^{<\omega}$ a condition $p_\eta \in \mathbb{P}$. First $p_{\langle\rangle} = 0_\mathbb{P}$. Assume that p_η is defined, and let $\mathbb{P}/p_\eta := \{q \in \mathbb{P} \mid q \geq p_\eta\}$. By assumption $|\mathbb{P}/p_\eta| \leq \kappa$, fix any surjection $f_\eta : (\max\{\eta\}, \kappa) \to \mathbb{P}/p_\eta$. Define for every $\alpha \in (\max\{\eta\}, \kappa)$, $p_{\eta^\frown \alpha} = f_\eta(\alpha)$.

Next we define the ultrafilters W_η for every $\eta \in [\kappa]^{<\omega}$. Let $g_\eta : \mathbb{P}/p_\eta \to (\max(\eta), \kappa)$ be a right inverse of f_η such that $f_\eta \circ g_\eta = id_{\mathbb{P}/p_\eta}$. Define $W_\eta = g_{\eta*}(U_{p_\eta})$ to be the Rudin-Keisler projection of U_{p_η} to κ i.e. for $A \subseteq \kappa$:

$$A \in W_\eta \iff g_\eta^{-1}[A \setminus \max(\eta) + 1] \in U_{p_\eta}$$

In particular $\vec{\mathcal{W}} := \langle W_\eta \mid \eta \in [\kappa]^{<\omega}\rangle$ is defined.

Let us define the following name

$$\underset{\sim}{H} = \{\langle \dot{q}, \langle t, T\rangle\rangle \mid q \in \mathbb{P},\ q \leq p_t,\ \langle t, T\rangle \in \mathbb{T}_{\vec{\mathcal{W}}}\}$$

Then $\Vdash_{\mathbb{T}_{\vec{\mathcal{W}}}} \underset{\sim}{H}$ is V-generic for $\mathbb{P}$. Indeed, Let $G \subseteq \mathbb{T}_{\vec{\mathcal{W}}}$ be V-generic and let $H = (\underset{\sim}{H})_G$. Assume that $\langle \alpha_n \mid n < \omega\rangle$ is the Prikry sequence produced by G, and denote by $p_n = p_{\langle\alpha_0,\dots,\alpha_n\rangle}$, then

$$H = \{q \in \mathbb{P} \mid \exists n < \omega\ q \leq p_n\}$$

Note that, $\alpha_{n+1} > \alpha_n$ and by construction $p_{n+1} = f_{\langle\alpha_0,\dots,\alpha_n\rangle}(\alpha_{n+1}) \in \mathbb{P}/p_n$, hence the p_n's are increasing in the order of $\mathbb{P}$ and H is a filter. Let $D \subseteq \mathbb{P}$ be dense open. We proceed by a density argument, let $\langle t, T\rangle \in \mathbb{T}_{\vec{\mathcal{W}}}$ then D is dense open above p_t and therefore $D \cap \mathbb{P}/p_t \in U_{p_t}$. It is not hard to check from the definition that $f_t^{-1}[D \cap \mathbb{P}/p_t] \in W_t$. It follows that $succ_T(t) \in W_t$, fix any $\alpha \in f_t^{-1}[D \cap \mathbb{P}/p_t] \cap succ_T(t)$. Consider the condition $\langle t^\frown \alpha, T_{t^\frown \alpha}\rangle \geq \langle t, T\rangle$. By density, there is $\langle s^\frown \alpha_{n_0}, S\rangle \in G$ such that $p_{s^\frown \alpha_{n_0}} = f_s(\alpha_{n_0}) \in D$. By the definition of H we conclude that $p_{s^\frown \alpha_{n_0}} \in H \cap D$ and H is a V-generic filter for $\mathbb{P}$.

Let $\underset{\sim}{H}^*$ be a $\mathbb{T}_{\vec{\mathcal{W}}}$-name for the $B(\mathbb{P})$-generic corresponding to $\underset{\sim}{H}$. Now the projection is defined as follows:

$$\pi(x) = \inf\{b \in B(\mathbb{P}) \mid x \Vdash b \in \underset{\sim}{H}^*\}$$

Clearly π is order preserving and dense in $B(\mathbb{P})$. To see that condition (2) holds, is just an abstract argument, take $b \leq \pi(x)$, then $\neg(x \Vdash b^c \in H^*)$, otherwise $b^c \geq \pi(x) \geq b$. Hence there is an extension $x' \geq x$ such that $x' \Vdash b^c \notin \underset{\sim}{H}^*$, since $\underset{\sim}{H}^*$ is an ultrafilter it follows that $b \in \underset{\sim}{H}^*$. so $\pi(x') \leq b$. $\qquad\square$

Remark. If $D_p(\mathbb{P})$ can be extended to U_p only densely often, then we still get a projection.

The following theorem claims that in some sense, this is the only way to get a projection.

Theorem 13. *Let $\mathbb{P}$ be a σ-distributive forcing of size κ. The following are equivalent:*

- *There is a sequence $\vec{\mathcal{U}}$ of κ-complete ultrafilters and a projection $\pi : \mathbb{T}_{\vec{\mathcal{U}}} \to B(\mathbb{P})$.*
- *For every $p \in \mathbb{P}$, $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter U_p.*

Proof. If $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter, use claim 12. For the other direction, let $\pi : T_{\vec{\mathcal{U}}} \to B(\mathbb{P})$ be a projection, denote $\mathbb{T}_{\vec{\mathcal{U}}} = \mathbb{T}$. Without loss of generality, we can assume that $\mathbb{P} = \kappa$, and $\leq_\mathbb{P}$ is an order on κ.

Let $q \in \mathbb{P}$ and $p = \langle s, T\rangle \in \mathbb{T}$ such that $\pi(p) \geq q$ which exists since π is a projection. For every $D \subseteq \mathbb{P}$ dense and open subset above q, let

$$\bar{D} := \{b \in B(\mathbb{P}) \mid \exists a \in D.a \leq b\}$$

Then $\bar{D} \subseteq B(\mathbb{P})$ is dense open, and since π is a projection, $D' := \pi^{-1}[\bar{D}]$ is a dense open subset of $\mathbb{T}$ above p. By the strong Prikry property, there is a direct extension $p^* = \langle s, T_D \rangle \geq^* p$ and a natural number $n_D < \omega$ such that for all $\eta \in T^*$ such that $\mathrm{len}(\eta) = n_D$, $\pi(\langle s^\frown \eta, (T_D)_\eta \rangle) \in \bar{D}$, while the projection of any extension of $\langle s, T_D \rangle$ of smaller length is not in $\bar{D}$. We claim that there must be a single $n^* < \omega$ which is an upper bound to the set

$$\{ n_D \mid D \subseteq \mathbb{P} \text{ dense open above } q \}$$

Otherwise, there is a sequence of dense open subsets D_m above q for which

$$\sup_{m < \omega} n_{D_m} = \omega.$$

The forcing $\mathbb{P}$ is σ-distributive, thus

$$D^* = \bigcap_{m < \omega} D_m$$

is still dense and open above q. Consider n_{D^*} and T_{D^*}. Any extension p' of length n_{D^*} from T_{D^*} will satisfy $\pi(p') \in \bar{D}^*$ and in particular it will be in $\bar{D}_m$ for all m. But let m be so large that $n_{D_m} > n_{D^*}$. This is a contradiction to the definition of n_{D_m}.

Let us fix such n^*. Next we consider the iterated ultrapower of length ω using the ultrafilters in $\vec{\mathcal{U}}$.

Let $k = \mathrm{len}\, s$ (the stem of p) and let us denote $s = \langle \delta_0^*, \ldots, \delta_{k-1}^* \rangle$. Consider the iteration corresponding to $\vec{\mathcal{U}}$ above s, and denote $\delta_{k+n}^* = \delta_n$.

By theorem 10, $\langle \delta_n^* \mid n < \omega \rangle$ is a tree Prikry generic sequence for the forcing $j_\omega(\mathbb{T})$ over the model M_ω and by claim 11, this generic filter will contain the condition $j_\omega(p)$. Denote by $H_\omega \subseteq j_\omega(B(\mathbb{P}))$ the M_ω-generic filter generated by the Prikry sequence in $M_\omega[\langle \delta_n^* \mid n < \omega \rangle]$.

Working in M_{n^*-1}, let

$$F = \{ x \in j_{n^*-1}(\mathbb{P}) \mid \exists T, j_{n^*-1}(\pi)(\langle \langle \delta_0^*, \ldots, \delta_{k+n^*-1}^* \rangle, T \rangle) \geq x \}.$$

$F \in M_{n*-1}$ and it is a subset of $j_{n^*-1}(\mathbb{P}) = j_{n^*-1}(\kappa)$. In particular for every $x \in F$, $j_{n^*-1,\omega}(x) = x$. Since for every T, $j_{n^*-1,\omega}(\langle \langle \delta_0, \ldots, \delta_{k+n^*-1} \rangle, T \rangle)$ is a member of the generic filter which is generated by the sequence $\langle \delta_n^* \mid n < \omega \rangle$, we conclude that $F \subseteq H_\omega$. Note that $F \in M_{n^*}$, as M_{n^*-1} and M_{n^*} agree on subsets of $j_{n^*-1}(\kappa)$. It follows that $j_{n^*,\omega}(F) = F \in M_\omega$. Thus, there must be a single condition $f \in H_\omega$ forcing $F \subseteq \dot{H}_\omega$. This can be the case only if f is stronger that all elements of F. find any $f^* \in \mathbb{P}$ such that $f^* \geq f$. We conclude that for every dense open set $D \subseteq \mathbb{P}$ above q, $f \in j_\omega(\bar{D})$ and since D is dense open in $\mathbb{P}$, $f^* \in j_\omega(D)$.

Let us define:

$$U_q = \{ A \subseteq \mathbb{P} \mid f^* \in j_\omega(A) \}$$

U_q is a κ-complete ultrafilter (since $\mathrm{crit}\, j_\omega = \kappa$) and for all dense open $D \subseteq \mathbb{P}$ above q, $D \in U_q$. $\square$

Remark. In the previous proof we have defined the filter U_p to be

$$U_q = \{ A \subseteq \mathbb{P} \mid f^* \in j_\omega(A) \}$$

where $f^* \in \mathbb{P}$ was a condition forcing $F \subseteq \dot{H}_\omega$, $\dot{H}_\omega$ being a canonical name for the generic filter of $j_\omega(\mathbb{P})$. In M_{n^*}, we will have F bounded in the critical point of $j_{n^*,\omega}$ and therefore $j_{n^*,\omega}(F) = F$. By elementarity of $j_{n^*,\omega}$, there is a condition $q^* \in j_{n^*}(\mathbb{P})$ forcing that $F \subseteq \dot{H}_{n^*}$ where $\dot{H}_{n^*}$ is the canonical name for the generic filter of $j_{n^*}(\mathbb{P})$. So we may use q^* in order to define

$$U_q = \{ A \subseteq \mathbb{P} \mid q^* \in j_{n^*}(A) \}$$

This new definition indicates that if there is a projection from $\mathbb{T}_{\vec{\mathcal{U}}}$ onto $\mathbb{P}$ then there will be a Rudin-Keisler projection of the sequence of ultrafilters $\vec{\mathcal{U}}$ on an ultrafilter extending the filter of dense open subsets of $\mathbb{P}$.

Definition 14. Let $\vec{\mathcal{U}}$ be a tree of κ-complete ultrafilters and let W be a κ-complete ultrafilter. We say that $W \leq_{RK} \vec{\mathcal{U}}$ if there is $\vec{\eta} \in \kappa^{<\omega}$ and $n < \omega$ such that

$$W \leq_{RK} (\mathcal{U}_{\vec{\eta}})_n$$

.

Theorem 15. *Let $\vec{\mathcal{U}}$ be a tree of κ-complete ultrafilters and let $\mathbb{P}$ be σ-distributive forcing of cardinality κ.*

If $\mathbb{T}_{\vec{\mathcal{U}}}$ projects onto $B(\mathbb{P})$ then for every $p = \langle \delta_0, \ldots, \delta_{k-1}, T \rangle \in \mathbb{T}_{\vec{\mathcal{U}}}$ there is a κ-complete ultrafilter U_p which extends $D_{\pi(p)}(\mathbb{P})$ that contain p and $U_p \leq_{RK} \vec{\mathcal{U}}$.

Proof. The proof is just the continuation of the discussion following the proof of theorem 13, recall the definition of U_p

$$U_p = \{A \subseteq \mathbb{P} \mid q^* \in j_{n^*}(A)\}$$

There exists a function $g : [\kappa]^{n^*} \to \mathbb{P}$ such that $j_{n^*}(g)(\delta_k, \ldots, \delta_{n^*+k}) = q^*$. We claim that $U_p = g_*((\mathcal{U}_{\langle \delta_0, \ldots, \delta_{k-1} \rangle})_{n^*+1})$. Let $A \subseteq \mathbb{P}$, then

$$A \in U_p \iff j_{n^*}(g)(\delta_k, \ldots, \delta_{n^*+k}) \in j_{n^*}(A) \iff$$

$$\iff \langle \delta_k, \ldots, \delta_{n^*+k} \rangle \in j_{n^*}(g^{-1}[A]) \iff g^{-1}[A] \in (\mathcal{U}_{\langle \delta_0, \ldots, \delta_{k-1} \rangle})_{n^*+1}$$

$\square$

4. PROJECTIONS OF FORCINGS

The following simple lemma indicates that the difficulty of extending the dense open filter for different forcing notions is related to the existence of projections from other forcing notions.

Lemma 16. *Let $\pi \colon \mathbb{P} \to \mathbb{Q}$ be a projection of forcing notions and let κ be a regular cardinal. If there is a κ-complete ultrafilter that extends $\mathcal{D}_p(\mathbb{P})$, then there is a κ-complete ultrafilter that extends $\mathcal{D}_{\pi(p)}(\mathbb{Q})$.*

Proof. Let $\mathcal{U}$ be a κ-complete ultrafilter that extends $\mathcal{D}_p(\mathbb{P})$. Let:

$$\pi^*(\mathcal{U}) = \{A \subseteq \mathbb{Q} \mid \pi^{-1}(A) \in \mathcal{U}\}.$$

It is clear that $\pi^*(\mathcal{U})$ is a κ complete ultrafilter. For any dense open set $D \in \mathcal{D}_{\pi(p)}(\mathbb{Q})$, the fact that π is a projection ensures that $\pi^{-1}(D) \in D_p(\mathbb{P})$. Thus, $D \in \pi^*(\mathcal{U})$. $\square$

For the definition of λ-strategically closed forcings see [6]. The proof of the following lemma is a variant of theorem 14.1 in [6].

Lemma 17 (Folklore). *Let $\mathbb{P}$ be κ-strategically closed forcing notion of size $\leq \lambda$. There is a projection from $\mathrm{Col}(\kappa, \lambda)$ onto $\mathbb{P}$.*

The relevant case for our purpose is the case $\kappa = \lambda$. In this case, $\mathrm{Col}(\kappa, \kappa) \cong \mathrm{Add}(\kappa, 1)$. Thus, if $\mathbb{P}$ is a κ-strategically closed forcing of size κ then there is a projection from the Cohen forcing $\mathrm{Add}(\kappa, 1)$ onto $B(\mathbb{P})$.

Note that the other direction of lemma 17 is also true, namely that if there is a projection $\pi : \mathrm{Add}(\kappa, 1) \to B(\mathbb{P})$, then $\mathbb{P}$ must also be κ-strategically closed.

We conclude that questions about the existence of ultrafilters that extend the dense open filter of κ-strategically of cardinality κ closed forcing notions are equivalent to the same question about the Cohen forcing.

For $<\kappa$-strategically closed forcing notions the situation is more involved.

Definition 18 (Jensen). Let κ be an inaccessible cardinal. A *Jensen Square* on κ is a sequence $\langle C_\alpha \mid \alpha \in D \rangle$, such that

(1) D is a club consisting of only limit ordinals.
(2) C_α is a club at α.
(3) $\operatorname{otp} C_\alpha < \alpha$.
(4) If $\beta \in \operatorname{acc} C_\alpha$, then $\beta \in D$ and $C_\beta = C_\alpha \cap \beta$.

Note that if there is a Jensen square on κ then κ is not a Mahlo cardinal. The following lemma was proven by Velleman [21, Theorem 1].

Lemma 19. *Let κ be an infinite cardinal. If there is a Jensen square on κ then every $<\kappa$-strategically closed forcing is κ-strategically closed.*

There is a standard forcing for adding Jensen square at a cardinal κ, $\mathbb{S}_\kappa$.

Definition 20. The conditions of $\mathbb{S}_\kappa$ are pairs of the form $\langle \mathcal{C}, d \rangle$, such that
(1) $d \subseteq \kappa$ is closed and bounded (with last element) consisting only of limit ordinals.
(2) $\mathcal{C}$ is a function, $\operatorname{dom} \mathcal{C} = d$.
(3) For every $\alpha \in d$, $\mathcal{C}(\alpha)$ is a club at α, $\operatorname{otp} \mathcal{C}(\alpha) < \alpha$.
(4) $\forall \beta \in \operatorname{acc} \mathcal{C}(\alpha)$, $\beta \in d$ and $\mathcal{C}(\beta) = \mathcal{C}(\alpha) \cap \beta$.

For $\langle \mathcal{C}, d \rangle, \langle \mathcal{C}', d' \rangle \in \mathbb{S}_\kappa$, $\langle \mathcal{C}, d \rangle \le \langle \mathcal{C}', d' \rangle$ if $d = d' \cap (\max(d)+1)$ and $\mathcal{C} = \mathcal{C}' \restriction d$.

There are many variations of this forcing, some of them can be found in [7].

Lemma 21 (Folklore). *Let κ be a regular cardinal then $\mathbb{S}_\kappa$ is $<\kappa$-strategically closed.*

Proof. Let us define a strategy σ first. $\sigma(\langle \rangle) = \langle \emptyset, \emptyset \rangle$. Assume that
$$\langle \langle C_i, D_i \rangle, \langle E_i, F_i \rangle \mid i < \alpha \rangle$$
is defined and played according to σ and let us define
$$\sigma(\langle \langle C_i, D_i \rangle, \langle E_i, F_i \rangle \mid i < \alpha \rangle) = \langle C_\alpha, D_\alpha \rangle$$
Denote by $d_i = \max(D_i)$. If α is limit, let $d_\alpha = \sup_{i < \alpha} d_i$. Then $\langle C_\alpha, D_\alpha \rangle$ is defined if and only if d_α is a singular cardinal, in which case
$$D_\alpha = (\cup_{i < \alpha} D_i) \cup \{d'_\alpha\}$$
For every $i < \alpha$, $C_\alpha \restriction D_i = C_i$ and
$$C_\alpha(d_\alpha) = \bigcup_{i < \alpha} C_i(d_i)$$
If $\alpha = \beta + 1$, let d_α be an ordinal of cofinality ω above $\max(F_\beta)$ and $\langle x_n \mid n < \omega \rangle$ be a cofinal sequence in d_α such that $x_0 > d_\beta$. Define
$$D_\alpha = F_\beta \cup \{d_\alpha\}$$
Also $C_\alpha \restriction F_\beta = E_\beta$ and
$$C_\alpha(d_\alpha) = C_\beta(d_{\beta-1}) \cup \{x_n \mid n < \omega\}$$
Obviously, by the inductive construction C_α is coherent. It is not hard to see that $\operatorname{otp}(C_\alpha(d_\alpha))) = \omega \cdot \alpha$. So the strategy σ_λ starts by jumping above $\omega \cdot \lambda$, then uses σ. This guarantees that always $\operatorname{otp}(C_\alpha(d_\alpha)) < d_\alpha$ and that d_α for limit α is always singular. $\qquad \square$

In general, $|\mathbb{S}_\kappa| = \kappa^{<\kappa}$. Thus, for strongly inaccessible cardinals κ, $|\mathbb{S}_\kappa| = \kappa$ and it fits to the framework of this paper. For Mahlo cardinal κ, $\mathbb{S}_\kappa$ is not κ-strategically closed (otherwise, it would be possible to construct Jensen square sequence in the ground model). Thus, for Mahlo cardinal κ, $\mathbb{S}_\kappa$ is not isomorphic to a complete subforcing of $\operatorname{Add}(\kappa, 1)$.

Let us remark that in models of the form $L[E]$, there is a partial square sequence in the ground model which is defined on all singular cardinals. In those cases the forcing that shoots a club through the singular cardinals clearly adds a Jensen square for κ.

The following lemma shows that adding a Jensen square to κ is maximal between all $<\kappa$-strategically closed forcing notions.

Lemma 22. $\mathbb{S}_\kappa \cong \mathbb{S}_\kappa \times \mathrm{Add}(\kappa, 1)$. *In particular, for every $<\kappa$-strategically closed forcing $\mathbb{P}$ of cardinality κ, there is a projection from $\mathbb{S}_\kappa$ onto $B(\mathbb{P})$.*

Proof. Let us define a dense embedding $\pi : \mathbb{S}_\kappa \to \mathbb{S}_\kappa \times \mathrm{Add}(\kappa, 1)$, for every $A \subseteq \kappa$ and $\alpha < \mathrm{otp}(A)$ let $A(\alpha)$ be the α-th element of A in it's natural enumeration. Define $E_\omega = \{\alpha + \omega \mid \alpha < \kappa\}$ and for $\alpha \in E_\omega$ let $\alpha^- = \max(Lim(\alpha))$ be the maximal limit ordinal below α. Let $\langle \mathcal{C}, d \rangle \in \mathbb{S}_\kappa$, define $\pi(\langle \mathcal{C}, d \rangle) = \langle \langle \mathcal{C}', d \rangle, f \rangle$ such that:

(1) $dom(f) = \gamma_d$, where $\gamma_d = \mathrm{otp}(d \cap E_\omega)$.
(2) For $i < \gamma_d$, define
$$f(i) = 1 \iff ((d \cap E_\omega)(i))^- + 1 \in \mathcal{C}((d \cap E_\omega)(i))$$
(3) $dom(\mathcal{C}') = d$.
(4) $\mathcal{C}'$ is defined inductively. For $\alpha \in d$ let $\beta_\alpha = \max(Lim(\mathcal{C}(\alpha)) \cap \alpha)$ and assume that $\mathcal{C}'(\beta)$ is defined coherently for every $\beta < \alpha$.
   (a) If $\alpha \in d \cap E_\omega$, then $\beta_\alpha \leq \alpha^-$ and define
$$\mathcal{C}'(\alpha) = \mathcal{C}'(\beta_\alpha) \cup \big([\beta, \alpha^-] \cap \mathcal{C}(\alpha)\big) \cup \{\gamma - 1 \mid \gamma \in [\alpha^- + 2, \alpha) \cap \mathcal{C}(\alpha)\}$$
   (b) If $\alpha \notin d \cap E_\omega$ and $\beta_\alpha = \alpha$ let
$$\mathcal{C}' = \bigcup_{\gamma \in \mathrm{acc}(\mathcal{C}(\alpha)) \cap \alpha} \mathcal{C}'(\gamma)$$
   (c) $\alpha \notin d \cap E_\omega$ and $\beta_\alpha < \alpha$ let
$$\mathcal{C}'(\alpha) = \mathcal{C}'(\beta_\alpha) \cup \big([\beta_\alpha, \alpha] \cap \mathcal{C}(\alpha)\big)$$

Let us prove first that $\langle \langle \mathcal{C}', d \rangle, f \rangle \in \mathbb{S}_\kappa \times \mathrm{Add}(\kappa, 1)$. Obviously, $f \in \mathrm{Add}(\kappa, 1)$, it is routine to check that $\langle \mathcal{C}', d \rangle \in \mathbb{S}_\kappa$, show by induction that,
$$\mathrm{otp}(\mathcal{C}(\alpha)) = \mathrm{otp}(\mathcal{C}'(\alpha)), \ \mathrm{acc}(\mathcal{C}(\alpha)) = \mathrm{acc}(\mathcal{C}'(\alpha))$$
and that condition $(3), (4)$ of definition 20 hold. The induction step use the fact that by removing at most one ordinal below a limit point of the set does not change the order type and does not change limit points of the set.

To see that $\pi" \mathbb{S}_\kappa$ is dense in $\mathbb{S}_\kappa \times \mathrm{Add}(\kappa, 1)$, let
$$p = \langle \langle \mathcal{N}, d \rangle, f \rangle \in \mathbb{S}_\kappa \times \mathrm{Add}(\kappa, 1)$$
Extend p if necessary to $\langle \langle \mathcal{N}', d' \rangle, f' \rangle$ so that $dom(f') = \mathrm{otp}(d' \cap E_\omega)$. This is possible since f can be defined arbitrarily on missing points of its domain and γ_d can be increased by extending $\langle \mathcal{N}, d \rangle$ at successor steps of d from the set E_ω in a coherent way just as in lemma 21. To see that $\langle \langle \mathcal{N}', d' \rangle, f' \rangle \in \pi" \mathbb{S}_\kappa$, define $\langle \mathcal{C}, d' \rangle$ recursively. Assume $\alpha \in d' \cap E_\omega$ and $\alpha = (d' \cap E_\omega)(i)$. If $f'(i) = 0$ define
$$\mathcal{C}(\alpha) = \mathcal{C}(\beta_\alpha) \cup \big([\beta, \alpha^-] \cap \mathcal{N}(\alpha)\big) \cup \{\gamma + 1 \mid \gamma \in (\alpha^-, \alpha) \cap \mathcal{N}(\alpha)\}$$
If $f'(i) = 1$ define
$$\mathcal{C}(\alpha) = \mathcal{C}(\beta_\alpha) \cup \big([\beta, \alpha^-] \cap \mathcal{N}(\alpha)\big) \cup \{\gamma + 1 \mid \gamma \in (\alpha^-, \alpha) \cap \mathcal{N}(\alpha)\} \cup \{\alpha - \omega + 1\}$$
If $\alpha \notin d' \cap E_\omega$ and $\alpha = \beta_\alpha$ define
$$\mathcal{C}(\alpha) = \bigcup_{\gamma \in \mathrm{acc}(\mathcal{N}(\alpha)) \cap \alpha} \mathcal{C}(\gamma)$$

Finally if $\alpha \notin d' \cap E_\omega$ and $\beta_\alpha < \alpha$ let

$$\mathcal{C}(\alpha) = \mathcal{C}(\beta_\alpha) \cup \big([\beta_\alpha, \alpha] \cap \mathcal{N}(\alpha)\big)$$

It is routine to check that π is an embedding.

For the second part, assume that $\mathbb{P}$ is a $<\kappa$-strategically closed forcing, let G be generic for $\mathbb{S}_\kappa$, then $V[G] = V[G'][H]$ where G' is another generic for $\mathbb{S}_\kappa$ and H is $V[G']$-generic for $Add(\kappa, 1)$. In $V[G']$, since $\mathbb{S}_\kappa$ is $<\kappa$ strategically closed, there are no new plays of $\mathbb{P}$ of length less than κ, indicating that $\mathbb{P}$ stays $<\kappa$-strategically closed in $V[G']$. Since in $V[G']$ there is a square sequence, use 19 to conclude that $\mathbb{P}$ is κ-strategically closed in $V[G']$. Thus by 17, there is $\pi : Add(\kappa, 1) \to \mathbb{P} \in V[G']$ a projection. Let us turn this projection to a projection in V of $\mathbb{S}_\kappa \times Add(\kappa, 1)$. Let $\tilde{\pi}$ be a $\mathbb{S}_\kappa$-name such that $\Vdash_{\mathbb{S}_\kappa} \tilde{\pi} : Add(\kappa, 1) \to B(\mathbb{P})$ is a projection. Consider the set

$$D = \{\langle p, q \rangle \in \mathbb{S}_\kappa \times Add(\kappa, 1) \mid \exists a \in \mathbb{P}.p \Vdash \tilde{\pi}(q) = a\}$$

It is dense in $\mathbb{S}_\kappa \times Add(\kappa, 1)$. For every $\langle p, q \rangle \in D$, define $\pi_*(\langle p, q \rangle) = a$ for the unique $a \in \mathbb{P}$, such that $p \Vdash \tilde{\pi}(q) = a$. It is a straightforward verification to see that $\pi_* : D \to \mathbb{P}$ is a projection. $\square$

The following lemma shows that $\mathbb{S}_\kappa$ is not maximal among the κ-distributive forcing notions. For a fat stationary set $S \subseteq \kappa$, let $Club(S)$ be the forcing that shoots a club through S using closed and bounded conditions. By [2], if $\kappa^{<\kappa} = \kappa$, then $Club(S)$ is κ-distributive if and only if S is fat stationary set.

Lemma 23. *Let $S \subseteq T \subseteq \kappa$ be fat stationary sets. If the set of all $\alpha \in T \setminus S$ such that $T \cap \alpha$ contains a club at α is stationary, then $T \setminus S$ stays stationary in $V^{Club(T)}$ and in particular there is no projection from $Club(T)$ to $Club(S)$.*

Remark. After adding a single Cohen set to κ, there is a partition of κ into κ many disjoint fat stationary sets. Thus, the structure of the κ-distributive forcing notions of size κ might be complicated in general, even when κ is a large cardinal.

Proof. Let $C \subseteq T$ be a V-generic club for $Club(T)$. Assume that S is not stationary in $V[C]$ and let $\underset{\sim}{B}$ be a name such that some $p \in Club(T)$ forces that $\underset{\sim}{B}$ is a club disjoint from $T \setminus S$. Let $\langle M_i \mid i < \kappa \rangle$ be an increasing and continuous chain of elementary substructures of $H(\theta)$ for some large enough θ such that:

(1) $p, \underset{\sim}{B}, S, T, Club(T) \in M_0$.
(2) $|M_i| < \kappa$.
(3) $x_i := M_i \cap \kappa \in \kappa$.
(4) and $^{x_i}M_i \subseteq M_{i+1}$.

Consider the club $\{\alpha \mid x_\alpha = \alpha\}$. There is $\alpha < \kappa$ such that $x_\alpha = \alpha \in T \setminus S$ and there is a closed unbounded set $D \subseteq T \cap \alpha$. Let us construct an increasing sequence of conditions $\langle p_i \mid i < \theta \rangle$ such that:

(1) $p_0 = p$ and $p_i \in M_\alpha$.
(2) $[p_{i+1} \setminus \max(p_i)] \cap D \neq \emptyset$.
(3) there is $\max(p_i) < y_i \in M_\alpha$ such that $p_{i+1} \Vdash y_i \in \underset{\sim}{B}$.

that $\langle p_i \mid i < j \rangle$ is defined and let $\eta = \sup(\max(p_i) \mid i < j)$. If j is limit and $\eta = \alpha$, define $\theta = j$ and stop. Otherwise, there is $r < \alpha$ such that $\langle p_i \mid i < j \rangle \subseteq M_r$ thus in M_{j+1}. By closure of D, $\eta \in D$, hence it is safe to define

$$p_j = \cup_{i<j} p_i \cup \{\eta\} \subseteq T$$

which is definable in M_α. For the successor step, assume $p_i \in M_\alpha$ is define. Work inside M_α and let p'_{i+1} be a condition deciding a value $y_i \in \underset{\sim}{B}$ above $\max(p_i)$. Since D is unbounded, there is $z \in D \setminus \max(p'_{i+1})$ then $p_{i+1} = p'_{i+1} \cup \{z\} \in M_\alpha$ is as wanted. Finally, $\cup_{i<\theta} p_i \cup \{\alpha\} \in Club(T)$ must force that $\alpha \in \underset{\sim}{B} \cap (T \setminus S)$ which is a contradiction. $\square$

5. IMPLICATIONS

In this section we will show that certain large cardinals weaker than κ-compacts already imply an existence of a κ-complete ultrafilter extending the filters $\mathcal{D}_p(\mathbb{P})$.

Let us deal first with $<\kappa$-strategically closed forcing notion of size κ.

Recall that a cardinal κ is called *superstrong* if and only if there is an elementary embedding $j: V \to M$ such that $\mathrm{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$.

While $j(\kappa)$ is always a strong limit cardinal, and inaccessible in M, it need not be regular in V. Actually, $\kappa^+ \leq \mathrm{cf}(j(\kappa)) \leq 2^\kappa$, for the first such cardinal, see [19]. However, if $\mathrm{cf}(j(\kappa)) > \lambda$, then $^\lambda j(\kappa) \subseteq M$.

Theorem 24. *Suppose that there is an elementary embedding $j: V \to M$ such that* $\mathrm{crit}\, j = \kappa$ *and* $^{2^\kappa} j(\kappa) \subseteq M$. *Let $\mathbb{P}$ be a $<\kappa$-strategically closed forcing notion of size* κ.

Then for every $p \in \mathbb{P}$ there is a κ-complete ultrafilter that extends $\mathcal{D}_p(\mathbb{P})$.

Proof. Assume without loss of generality that $\mathbb{P} = \kappa$. Fix $p \in \mathbb{P}$. Denote 2^κ by λ. Clearly, $\lambda < j(\kappa)$, since $\mathcal{P}(\kappa) \subseteq M$ and $j(\kappa)$ is a measurable in M.
Let $\langle D_\alpha \mid \alpha < \lambda \rangle$ be an enumeration of all subsets of $\mathbb{P}$ in V which are dense above p and open. In M, let Σ be a winning strategy for the game on $j(\mathbb{P})$ of length $\lambda + 1$. Such Σ exists since $j(\mathbb{P})$ is $<j(\kappa)$-strategically closed.

Let us pick by induction a sequence of conditions $p_\alpha \in j(\mathbb{P}) = j(\kappa)$, $\alpha < \lambda$, such that $\forall \alpha < \beta$, $p_\alpha \leq p_\beta$ and $p_{\alpha+1} \in j(D_\alpha)$.

First, let $p_0 = p$. Each condition p_α is played by Player I according to Σ and q_α is played by Player II, to be a condition stronger than p_α in $j(D_\alpha)$.

While the sequence $\langle j(D_\alpha) \mid \alpha < \lambda \rangle$ might not be in M, the sequence $\langle p_\alpha, q_\alpha \mid \alpha < \lambda \rangle$ is in M, since $^\lambda j(\kappa) \subseteq M$, and it is a play which is played according to the strategy Σ. Therefore, it has an upper bound $\tilde{p}$ which is stronger than all the conditions p_α, q_α, $\alpha < \lambda$. By construction, $\tilde{p} \in \bigcap_{\alpha < \kappa^+} j(D_\alpha)$.

Finally,
$$U = \{X \subseteq \kappa \mid \tilde{p} \in j(X)\}$$
will be as desired. $\qquad\square$

The assumption of the theorem cannot be optimal since $V_{\kappa+2} \subseteq M$, and thus it is true in M as well that for every $<\kappa$-strategically closed forcing notion $\mathbb{P}$, there is a κ-complete ultrafilter that extends its dense open filter. Thus, by reflection, the conclusion holds for many cardinals below κ as well.

Next, we turn to the class of κ-distributive forcings. The upper bound in this case is a 1-extendable cardinal:

Definition 25. A cardinal κ is called 1-extendible if there is a non-trivial elementary embedding $j: V_{\kappa+1} \to V_{\lambda+1}$ such that $crit(j) = \kappa$.

Proposition 26. *If κ is 1-extendible then for every κ-distributive forcing $\mathbb{P}$ of size κ and every $p \in \mathbb{P}$, the filter $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter.*

Proof. Code $\langle \mathbb{P}, \leq_\mathbb{P} \rangle$ and an order of κ. Thus we can assume without loss of generality that $\langle \mathbb{P}, \leq_\mathbb{P} \rangle \in V_{\kappa+1}$. Recall that $D_p(\mathbb{P}) := \{D \subseteq \mathbb{P} \mid D$ is dense open above $p\}$. Then $D_p(\mathbb{P}) \subseteq V_{\kappa+1}$ and it is definable from $\langle \mathbb{P}, \leq_\mathbb{P} \rangle$. Since $j(\kappa) = \lambda$, and $V_{\kappa+1} \models \kappa$ is inaccessible cardinal, by elementarity $V_\lambda \models \lambda$ is an inaccessible cardinals, and therefore λ is inaccessible cardinal in V. In particular, V_λ is closed under $< \lambda$ sequences and the set $j''D_p(\mathbb{P}) = \{j(D) \mid D \in D_p(\mathbb{P})\} \in V_{\lambda+1}$. By elementarity $j(\mathbb{P})$ is λ-distributive, and since $|j''D_p(\mathbb{P})| = |D_p(\mathbb{P})| = 2^k < \lambda$, we conclude that $\cap_{D \in D_p(\mathbb{P})} j(D)$ is dense open in $j(\mathbb{P})$. In particular it is non empty and we can fix any $p^* \in \cap_{D \in D_p(\mathbb{P})} j(D)$. Now in V we can define
$$F = \{X \subseteq \mathbb{P} \mid p^* \in j(X)\}$$

As in previous arguments, this F is a κ-complete ultrafilter extending $D_p(\mathbb{P})$ □

We deal here with the following weakening of κ-compactness:

For every κ-distributive forcing notion of cardinality κ, the filter of its dense open subsets can be extended to a κ-complete ultrafilter.

In this context, the major difference between κ being κ-compact and κ being 1-extendible is that we do not need to extend *every* κ-complete filter on κ. For our purposes we are only interested in extending filters which are definable using a parameter which is a subset of κ. This distinction leads to the realm of subcompact cardinals. Subcompact cardinals were defined by R. Jensen:

Definition 27. A cardinal κ is called *subcompact* if for every $A \subseteq H(\kappa^+)$, there are $\rho < \kappa$, $B \subseteq H(\rho^+)$ and an elementary embedding

$$j \colon \langle H(\rho^+), \in, B \rangle \to \langle H(\kappa^+), \in, A \rangle$$

with critical point ρ, such that $j(\rho) = \kappa$.

The following strengthening was introduced by I. Neeman and J. Steel [18]:

Definition 28. κ is called Π_1^1-*subcompact* if for every $A \subseteq H(\kappa^+)$ and for every Π_1^1-statement Φ, if $\langle H(\kappa^+), \in, A \rangle \models \Phi$ then there are $\rho < \kappa$ and $B \subseteq H(\rho^+)$ such that $\langle H(\rho^+), \in, B \rangle \models \Phi$ and there is an elementary embedding

$$j : \langle H(\rho^+), \in, B \rangle \to \langle H(\kappa^+), \in, A \rangle$$

with critical point ρ, such that $j(\rho) = \kappa$.

The third author showed in [13] the following:

Theorem 29. *If κ is Π_1^1-subcompact, then it is a κ-compact cardinal i.e. every κ-complete filter over κ extends to a κ-complete ultrafilter.*
On the other hand, if κ is κ-compact then $\square(\kappa)$ and $\square(\kappa^+)$ fails.

The failure of square at two consecutive cardinals seem to have very high consistency strength, which made the conjecture that κ-compactness is equiconsistent with Π_1^1-subcompactness plausible. However, a recent work of Larson and Sargsyan, [15], casts doubt on this heuristic by showing that the consistency strength of the failure of two consecutive squares at ω_3 and ω_4 is below a Woodin limit of Woodin cardinals.

Let us start with the following observation:

Proposition 30. *Let κ be a subcompact cardinal such the filter $F_{\mathbb{Q}}$ of dense open subsets of $\mathbb{Q}$ extends to a κ-complete ultrafilter over κ, for every κ-distributive poset $\mathbb{Q}$ of size κ.*
Then κ is a limit of cardinals with the same extension property.

Proof. Clearly it is enough to deal with posets which are partial orders on the set κ.

For every such $\mathbb{Q}$, fix a κ-complete ultrafilter $F_{\mathbb{Q}}^*$ over κ which extends $F_{\mathbb{Q}}$. The ultrafilter $F_{\mathbb{Q}}^*$ is a subset of $P(\kappa) \subseteq H(\kappa^+)$. Thus, one can code the set:

$$\{ \langle \mathbb{Q}, F_{\mathbb{Q}}^* \rangle \mid \mathbb{Q} \subseteq \kappa, \text{ is } \kappa\text{-distributive} \}$$

by a subset A of $H(\kappa^+)$ (for example, we can set $A = \bigcup_{\mathbb{Q}} \{ \mathbb{Q} \} \times F_{\mathbb{Q}}^*$).

Now by the definition of subcompactness, with parameter A, there are $\rho < \kappa$ and $B \subseteq \rho^+$ and an elementary embedding

$$j \colon \langle H(\rho^+), \in, B \rangle \to \langle H(\kappa^+), \in, A \rangle$$

with critical point ρ, such that $j(\rho) = \kappa$.

Since the set of all ρ-distributive posets is definable in $H(\rho^+)$, and j is elementary, the set B is a code of the set:

$$\{\langle \mathbb{Q}, F_{\mathbb{Q}}^* \rangle \mid \mathbb{Q} \subseteq \rho \text{ is } \rho\text{-distributive}\},$$

and in particular for every ρ-distributive poset of size ρ, $\mathbb{Q}$, there is a ρ-complete filter extending $F_{\mathbb{Q}}$. $\qquad\square$

Let us consider now the lightface version of Definition 28:

Definition 31. A cardinal κ is called *lightface Π_1^1-subcompact* if for every Π_1^1-statement Φ, if $\langle H(\kappa^+), \in \rangle \models \Phi$ then there is $\rho < \kappa$ such that $\langle H(\rho^+), \in \rangle \models \Phi$ and there is an elementary embedding

$$j : \langle H(\rho^+), \in \rangle \to \langle H(\kappa^+), \in \rangle$$

with critical point ρ, such that $j(\rho) = \kappa$.

The definition does not allow us to add parameters from $H(\kappa^+)$ to the formula Φ, and thus this large cardinal property is witnessed by a countable set of elementary embeddings.

The next proposition is similar to 29:

Proposition 32. *Let κ be a lightface Π_1^1-subcompact. Then every κ-distributive forcing $\mathbb{P}$ of size κ, and every $p \in \mathbb{P}$, the filter $D_p(\mathbb{P})$ can be extended to a κ-complete ultrafilter.*

Proof. Assume otherwise, and let Φ be the statement that there is $\mathbb{P} = \langle \kappa, \leq_{\mathbb{P}} \rangle \in H(\kappa^+)$ and no ulrafilter extending $D_p(\mathbb{P})$. Φ is of the form

$$\underbrace{\exists \mathbb{P}}_{First\ order} \underbrace{\forall U}_{Second\ order} \underbrace{\mu(\mathbb{P}, U)}_{First\ order}$$

Using AC, such a formula can be expressed as a Π_1^1 formula [8, P. 153, Lemma 7.2]. Note that Φ is defined with no parameters, hence by a lightface Π_1^1-subcompactness of κ, there is $\rho < \kappa$ an elementary embedding

$$j : H(\rho^+) \to H(\kappa^+)$$

with critical point ρ such that $j(\rho) = \kappa$, such that $H(\rho^+) \models \Phi$.

Therefore there is $\mathbb{P}_\rho$ which is a counterexample of a forcing of size ρ which is ρ-distributive such that there is no κ-complete filter extending the filter $D_p(\mathbb{P}_\rho)$ for some $p \in \mathbb{P}_\rho$. Let us enumerate all dense open subsets of $\mathbb{P}_\rho$ above p by $\langle D_i \mid i < 2^\rho \rangle$. The sequence $\langle j(D_i) \mid i < 2^\rho \rangle$ is in $H(\kappa^+)$, since $2^\rho < \kappa$. By elementarity, $j(\mathbb{P}_\rho)$ is $j(\rho)$- distributive and therefore $\bigcap_{i < \rho^+} j(D_i) \neq \emptyset$, so let x be an element in the intersection. Then

$$\{X \subseteq \mathbb{P}_\rho \mid x \in j(X)\}$$

is a ρ-complete ultrafilter extending $D_p(\mathbb{P}_\rho)$ — a contradiction to the choice of $\mathbb{P}_\rho$. $\qquad\square$

To see that the notion of a lightface Π_1^1-subcompact is strictly weaker than Π_1^1-subcompact we have the following proposition:

Proposition 33. *Let κ be Π_1^1-subcompact. Then κ is a limit of lightface Π_1^1-subcompact cardinals.*

Proof. Suppose that κ is a Π_1^1-subcompact cardinal. Let Φ be a Π_1^1 statement (with no parameters). If Φ holds in $H(\kappa^+)$ let

$$j_\Phi : \langle H(\rho_\Phi^+), \in \rangle \to \langle H(\kappa^+), \in \rangle$$

witness the reflection. Otherwise, let B_Φ be a subset of $H(\kappa^+)$ witness the negation of Φ. Let $T \subseteq \omega$ be the set of all Gödel number of true Π_1^1-formulas in $H(\kappa^+)$, and let k_Φ be the Gödel number of Φ.

For each Φ such that $k_\Phi \in T$, $j_\Phi \subseteq H(\rho_\Phi^+) \times H(\kappa^+) \subseteq H(\kappa^+)$. There are only countably many such formulas Φ, and thus we can code all those elementary embeddings as a single subset $A_T \subseteq H(\kappa^+)^1$.

Similarly, we can gather all the sets B_Φ for Φ such that $k_\Phi \notin T$, into a single subset of $H(\kappa^+)$, B. Since the truth values of first order formulas is Δ_1^1, we can take a Π_1^1-formula Λ with parameter k, using the predicate B such that $\Lambda(k, B)$ if and only if k is the Gödel number of a Π_1^1-formula Φ and B_k is a counterexample doe Φ.

Let A be a set coding T, A_T and B.

There is a universal Π_1^1-formula $\Psi(y)$ where y is a first order free variable such that for every regular cardinal β, every Π_1^1 statement ϕ,

$$H(\beta) \models \phi \Longleftrightarrow H(\beta) \models \Psi(k)$$

for some natural number k which is the Gödel numbering of formulas [8, p. 272, Lemma 1.9]. In the language of the model $\langle H(\kappa^+), \in, A \rangle$ we can formulate the statement $\alpha(A)$ "For every Π_1^1-statement Φ, $k_\Phi \in T$ implies $\Psi(k_\Phi)$ and $k_\alpha \notin T$ implies $\Lambda(B, k)$". Now, $\langle H(\kappa^+), \in, A \rangle \models \alpha(A)$, apply Π_1^1-subcompactness to A, there are $\rho < \kappa$, $B \subseteq H(\rho^+)$ and an elementary embedding:

$$j \colon \langle H(\rho^+), \in, B \rangle \to \langle H(\kappa^+), \in, A \rangle$$

such that $\mathrm{crit}(j) = \rho$, $j(\rho) = \kappa$ and $\langle H(\rho^+), \in, B \rangle \models \alpha(B)$. Let us show that ρ is lightface Π_1^1-subcompact and we will be done. Let ζ be a Π_1^1-statement, such that $\langle H(\rho^+), \in \rangle \models \zeta$, then k_ζ is coded in B_1 and by elementarity of j also in A_1, hence $\langle H(\kappa^+), \in \rangle \models \zeta$. So there is an embedding j_ζ coded by A. In particular for $\rho_\zeta < j(\rho)$,

$$\langle H(\kappa^+), \in, A \rangle \models \{x \mid \langle k_\zeta, x, j_\zeta(x) \rangle \in A\} = H(\rho_\zeta^+)$$

by elementarity of j, there is $\rho_\zeta' < \rho$ such that

$$\langle H(\rho^+), \in, B \rangle \models \{x \mid \langle k_\zeta, x, j_\zeta(x) \rangle \in B\} = H(\rho_\zeta'^+).$$

It must be that $\rho_\zeta' = j(\rho_\zeta') = \rho_\zeta$, since the critical point is ρ. For every $x \in H(\rho_\zeta^+)$ there is a unique y such that $\langle k_\zeta, x, y \rangle \in B$, define $i_\zeta(x) = y$. So

$$i_\zeta : \langle H(\rho_\zeta^+), \in \rangle \to \langle H(\rho^+), \in \rangle$$

We claim that i_ζ is elementary, and that $i_\zeta(\rho_\zeta) = \rho$. This will follow after we show that $j \circ i_\zeta = j_\zeta$. Indeed, $\langle k_\zeta, x, i_\zeta(x) \rangle \in B$ and be elementarity $\langle k_\zeta, j(x), j(i_\zeta(x)) \rangle \in A$ but $j(x) = x$ since $\rho_\zeta^+ < \rho$ and therefore $\langle k_\zeta, x, j(i_\zeta(x)) \rangle = \langle k_\zeta, x, j_\zeta(x) \rangle$ in particular $j(i_\zeta(x)) = j_\zeta(x)$. $\qquad\square$

6. Lower bound

In this section we deal with the forcing notion for shooting a club through the stationary set of singular ordinals below κ, i.e.

$$Q = \{a \subseteq \kappa \mid |a| < \kappa, a \text{ is closed and each member of } a \text{ is singular}\}$$

ordered by end-extension.

This forcing is $< \kappa$-strategically closed. In our framework, κ is strongly inaccessible and thus this forcing is of cardinality κ.

Our aim will be to show the following:

Theorem 34. *Let us assume that there is a κ-complete ultrafilter which extending $\mathcal{D}_\emptyset(Q)$.*

Then either there is an inner model for $\exists \lambda, o(\lambda) = \lambda^{++}$, or $o^{\mathcal{K}}(\kappa) > \kappa^+$.

---

[1]indeed, A_T is an element of $H(\kappa^+)$.

We split the proof into three parts. First, we will derive some unconditional claims that follow from the existence of such an ultrafilter. Then, we will focus in the case that there is no inner model with a measurable λ of Mitchell order λ^{++}, and discuss the structure of the indiscernibles that follows from the hypothesis of the theorem. Finally, we will combine those two paths and obtain a robust way to extract some of the indiscernibles, from which we are going to get strength.

6.1. **Combinatorial consequences.** In order to prove Theorem 34, we will start with a sequence of lemmas, establishing the existence of a certain elementary embedding with some useful properties.

Let F be a κ-complete ultrafilter which extends $\mathcal{D}_\emptyset(Q)$. Consider the corresponding elementary embedding $j_F \colon V \to \mathrm{Ult}(V, F) \simeq M_F$. Let $a = [id]_F$. Then

$$a \in \bigcap\{j_F(D) \mid D \subseteq Q \text{ is dense open }\}$$

Let a^* be a closed set of ordinals with minimal value of $\max(a^*)$, such that there is an ultrafilter $U = \{X \subseteq Q \mid a^* \in j_U(X)\}$ extending the filter of dense open subsets of Q. Equivalently, $a^* \in j_U(D)$, for every $D \subseteq Q$ dense open.

Fix such a^* and let U be a witnessing ultrafilter. So $[id]_U = a^*$, by [12, Lemma 1.6], [3, Proposition 2.5].

Lemma 35. *For every $\xi < \max(a^*)$, there is a dense open D such that $a^* \cap (\xi + 1) \notin j_U(D)$.*

Proof. Otherwise, let $\xi < \max(a^*)$ be the least ordinal such that $a' = a^* \cap (\xi + 1)$ belongs to $j_U(D)$ for all $D \subseteq Q$ dense open. Let

$$U' = \{X \subseteq Q \mid a' \in j_U(X)\}.$$

The ultrafilter U' is below U in the Rudin-Kiesler order. One way to illustrate that is to pick a function $g \colon Q \to \kappa$ such that $j_U(g)(a^*) = \xi + 1$ and define the function $f(p) = p \cap g(p)$. Then $j_U(f) = a'$.

Let $k \colon M_{U'} \to M_U$ be the elementary embedding defined by $k([f]_{U'}) = j_U(f)(a')$. By standard arguments, $k \circ j_{U'} = j_U$.

Consider in $M_{U'}$ the element $b = [id]_{U'}$ (note that $U_b = U'$). By the properties of k, $k(b) = a' = a^* \cap (\xi + 1)$, hence $\max(b) \leq k(\max(b)) \leq \xi < \max(a^*)$. To see that b contradicts the minimality of a^*, note that for every dense open D, $k(b) = a' \in k(j_{U'}(D))$ and by elementarity of k, $b \in j_{U'}(D)$. □

Lemma 36. *Let $\eta < \max(a^*)$ and $q \subseteq \eta$, $q \in j(Q)$. For every dense open set $D \subseteq Q$, the condition $(a^* \setminus \eta) \cup q$ is in $j(D)$.*

Proof. Otherwise, let D_1, $\eta < \max(a^*)$ and q' be such that

$$a^* \setminus \eta \cup q' \notin j(D_1)$$

By minimality of a^*, there is $D_2 \subseteq Q$ dense and open such that $a^* \cap \eta \notin j(D_2)$.

Let D^* be the set of all conditions $p \in Q$ such that there is $\eta < \max(p)$, $p \cap \eta \in D_1$ and moreover for every condition $q \in Q$ with $\max(q) \leq \eta$, $q \cup (p \setminus \eta) \in D_2$.

We claim that D^* is dense open.

Let us show that D^* is open. Let $p_1 \in D^*$ and let $p_1 \leq p_2$. Take $\eta < \max(p_1) \leq \max(p_2)$ witnessing $p_1 \in D^*$ then

$$p_1 \cap \eta = p_2 \cap \eta \in D_1$$

and if $\max(q) \leq \eta$ then

$$q \cup (p_1 \setminus \eta) \leq q \cup (p_2 \setminus \eta) \in D_2$$

since D_2 is open. Thus $p_2 \in D^*$.

Let us show that D^* is dense. Let $p \in Q$ be any condition, find $p \leq p_1 \in D_1$. denote $\max(p_1) = \eta$ and note that $2^\eta < \kappa$. Let us enumerate all $q \in Q$ with $\max(q) \leq \eta$, $\langle q_i \mid i < 2^\eta \rangle$ and let D_2^i be the collection of all conditions $r \in \mathbb{P}$ such that $r \setminus \eta \cup q_i \in D_2$. For every i, D_2^i is dense open. $\mathbb{P}$ is κ-distributive and thus $\bigcap_{i < 2^\eta} D_2^i$ is dense. Let us pick a condition $p_2 \geq p_1$ in this intersection. Clearly, $p_2 \in D^*$.

Let us claim that $a^* \notin j(D^*)$, and conclude the proof. For any $\xi < \max(a^*)$, if $\xi \leq \eta$ then $a^* \cap \eta \notin j(D_1)$ and if $\xi > \eta$ then let $p = a^* \cap (\eta, \xi)$ then $q' \cup p \notin j(D_2)$ since $q' \cup a^* \geq q' \cup p$. thus $a^* \notin j(D^*)$ contradiction the choice of a^*. $\square$

We conclude that for every η, $a^* \setminus \eta \in j(D)$ for all $D \in V$, dense open. In particular, we may assume that $\min(a^*) > \kappa$. Although, $a^* \setminus \kappa$ does not necessarily generates U, we take $b = [id]_{U_{a^* \setminus \kappa}}$, where

$$U_{a^* \setminus \kappa} = \{X \subseteq Q \mid a^* \setminus \kappa \in j_U(X)\}$$

and b will be as wanted, since $\max(b) = \max(a^*)$ but also $\min(b) \geq \kappa$. To see this, assume otherwise that $b \cap \kappa \neq \emptyset$. Let $k : M_{U_b} \to M_U$, then $\mathrm{crit}(k) \geq \kappa$ and $k(b) = a^* \setminus \kappa$ so $b \cap \kappa = k(b \cap \kappa) \subseteq a^* \setminus \kappa$, contradiction.

Continuing, we would like to derive some more information about the size of $\max(a^*)$.

Lemma 37. *For any $f : \kappa \to \kappa$ and any $\tau < \max(a^*)$, $j_U(f)(\tau) < \max(a^*)$.*

Proof. Assume otherwise, then there is f, τ, witnessing the negation. By lemma 35, there is a dense open set D such that $a^* \cap (\tau + 1) \notin j_U(D)$. Consider the set D^* of all conditions $p \in D$ such that for every $\xi < \max(p)$, if $p \cap (\xi + 1) \notin D$ then $f(\xi) < \max(p)$. Then D^* is dense since for every p_0, we take $q \in D$ above p_0, the set $\{f(\xi) \mid \xi < \max(q)\}$ is bounded by some $\max(q) \leq \alpha < \kappa$, then

$$p_0 \leq q \leq q^* := q \cup \{\alpha\} \in D^*$$

since if $\xi < \max(q^*)$ and $q^* \cap (\xi + 1) \notin D$, then $\xi < \max(q)$ as $q^* \cap (\max(q) + 1) = q \in D$ and D is open. so $f(\xi) < \alpha = \max(q^*)$. Also D is open since is $p \in D^*$ and $p \leq p_1$, then $p_1 \in D$ (since D is open and $p \in D$), but also for every $\xi < \max(p_1)$, id $p_1 \cap (\xi + 1) \notin D$, then $\xi < \max(p)$. Thus $f(\xi) < \max(p) \leq \max(p_1)$.

It follows that $a^* \in j_U(D^*)$, but this is a contradiction since $\tau < \max(a^*)$,

$$a^* \cap (\tau + 1) \notin j_U(D) \text{ and } j_U(f)(\tau) \geq \max(a^*)$$

$\square$

Lemma 38. $\kappa^+ \leq \mathrm{cf}^V \max(a^*) \leq 2^\kappa$.

Remark. Note that $2^\kappa > \kappa^+$ already implies, by Mitchell [16], that $o(\kappa) \geq \kappa^{++}$, since κ is a measurable. Thus, assuming our anti-large cardinal hypothesis, we get $\mathrm{cf} \max(a^*) = \kappa^+$.

Proof. First let us show that $\mathrm{cf}^V \max(a^*) \geq \kappa^+$.

Otherwise, let $\langle \zeta_\delta \mid \delta < \delta^* \rangle$ be cofinal at $\max(a^*)$, $\delta^* \leq \kappa$. For every $\delta < \delta^*$, there is a dense open set $D_\delta \in V$ such that $a^* \cap (\zeta_\delta + 1) \notin D_\delta$.

Let D_* be the set of all condition $p \in D$ such that there is $\xi < \max p$ such that $p \cap \xi \in \bigcap_{\delta < \min p} D_\delta$. Clearly, D_* is dense open. Let us show that $a^* \notin j_U(D_*)$. Indeed, we assume that $\min a^* > \kappa$ and therefore if $a^* \in j_U(D_*)$ then there is some $\xi < \kappa^*$ such that $a^* \cap \xi \in j_U(D_\delta)$ for all $\delta < \delta^* \leq \kappa$, which contradicts our assumption.

Let us show now that $\mathrm{cf}^V(\max(a^*)) \leq 2^\kappa$. Indeed, let us fix some elementary submodel H of sufficiently large $H(\theta)$ of cardinality 2^κ that contains a^* and for every $D \subseteq Q$, $j(D) \in H$. It follows that for every $D \subseteq Q$ dense open in V, the

minimal ordinal $\rho < \max(a^*)$ such that $a^* \cap \rho \in j(D)$ belongs to H. In particular, $\sup(\max(a^*) \cap H) = \max(a^*)$, by the minimality of $\max(a^*)$. Since $|H \cap \max(a^*)| \leq 2^\kappa$, we conclude that $\mathrm{cf}(\max(a^*)) \leq 2^\kappa$. $\qquad\square$

Next, we would like to get a parallel of Claim 12. Since we only assume the existence of an ultrafilter extending $\mathcal{D}_\emptyset(Q)$, we have to be a bit more careful. We could use the homogeneity of Q and derive an extension of $\mathcal{D}_p(Q)$ for all p, but we would like to get a relatively concrete representation of the generic, which would be useful during the proof.

Since $|Q| = \kappa$, there is a bijection $f : \kappa \to Q$. Denote $\delta_{a^*} = j(f^{-1})(a^*)$ and define
$$\mathcal{W} = \{X \subseteq \kappa \mid \delta_{a^*} \in j(X)\}$$
then W is a κ-complete ultrafilter on κ, $U \equiv^{\mathrm{RK}} \mathcal{W}$. $M_\mathcal{W} = M_U$ and $[g]_U \mapsto [g \circ f]_\mathcal{W}$ is the unique isomorphism between the two ultrapowers.

Lemma 39. *Let $\langle \kappa_n \mid n < \omega \rangle$ be a generic Prikry sequence for $\mathcal{W}$. Then $\bigcup_{n<\omega} a_n$ is a generic club for Q where $a_n = f(\kappa_n)$. Moreover, there is $N < \omega$ such that for every $N \leq n < \omega$, $\max(a_n) < \min(a_{n+1})$.*

Proof. Let $\pi \colon Q \to \kappa$ be such that $\kappa = [\pi]_U = [\pi \circ f]_\mathcal{W}$ be the projection to normal. In V, define the set
$$A = \{\alpha < \kappa \mid \forall \beta < \pi(f(\alpha)),\ \max(f(\beta)) < \min(f(\alpha))\},$$
then $A \in \mathcal{W}$. To see this note that
$$M_U \models \forall \beta < \kappa = j(\pi)(a^*),\ \max(j(f)(\beta)) < \min(a^*)$$
since $j(f)(\beta) = f(\beta) < \kappa \leq \min(a^*)$. Now $a^* = [id]_U$, to see this, note that
$$j(\pi)(a^*) = j(\pi \circ f)(\delta_{a^*}),\ \min(a^*) = \min(j(f)(\delta_{a^*}))$$
thus $\delta_{a^*} \in j(A)$ and $A \in \mathcal{W}$. Let $\langle \kappa_n \mid n < \omega \rangle$ be a Prikry sequence for W. Then there is N such that for every $N \leq n < \omega$, $\kappa_n < \pi(\kappa_{n+1})$ and $\kappa_n \in A$. By the definition of A it follows that $\max(a_n) < \min(a_{n+1})$. Denote by $p_n = a_0 \cup \cdots \cup a_n \in \mathbb{P}$, then for every $n \geq N$, $p_n \leq p_{n+1}$. We claim that $C_G = \bigcup_{n<\omega} a_n$ is a generic club though the singulars of V. To see this, let $D \subseteq \mathbb{P}$ be dense open, then by claim 36, $a^* \in j(D)$ and for every $\xi < \max(a^*)$, $q \subseteq \xi$, $a^* \setminus \xi \cup q \in j(D)$, this property reflects on a set in $\mathcal{W}$ i.e.
$$B = \{\alpha < \kappa \mid \forall \xi < \max(f(\alpha)) \forall q \subseteq \xi,\ f(\alpha) \setminus \xi \cup q \in D\} \in \mathcal{W}$$
and therefore there is $N \leq M < \omega$ such that for every $n \geq M$, $\kappa_n \in B$ and so $p_n \in D$. $\qquad\square$

We denote $C(Q) = \bigcup_{n<\omega} a_n$ to be the V-generic club for Q.

The idea is that properties of a^* reflect in some sense to the generic club $C(Q)$. This will be useful later, when we encounter some more delicate properties of a^*, using Mitchell's analysis of indiscernibles.

6.2. **Mitchell's indiscernibles.** Recall that $\mathcal{K}$ is the Mitchell's core model, under the anti-large cardinal hypothesis, $\neg\exists\lambda, o(\lambda) = \lambda^{++}$.

For the convenience of the reader, we include here the statements of the basic definitions and results which we are going to use in the course of the proof, which we cite from [17].

Definition 40. (1) Let U be a measure, then $\mathrm{crit}(U)$ is the measurable κ such that U is a measure over κ.

(2) Let $\vec{U}$ be a sequence of measures and let $\gamma' < \gamma$ in $\mathrm{dom}(\vec{U})$, denote by $coh_{\gamma',\gamma} = f$ for the least function in the well ordering of $\mathcal{K} = L[\vec{U}]$ such that $\gamma' = [f]_{\vec{U}_\gamma} \in \mathrm{Ult}(\mathcal{K}, \vec{U}_\gamma)$.

(3) A system of indiscernibles for $\mathcal{K}$ is a sequence $\mathcal{C}$ such that:
   (a) $\mathrm{dom}(\mathcal{C}) \subseteq \mathrm{dom}(\vec{U})$ and $\forall \gamma \in \mathrm{dom}(\mathcal{C})$, $\mathcal{C}_\gamma \subseteq \mathrm{crit}(\vec{U}_\gamma)$.
   (b) For every $f \in \mathcal{K}$, there is a finite sets $a \subseteq On$ such that for every $\gamma \in \mathrm{dom}(\vec{U})$:

$$\forall \nu \in \mathcal{C}_\gamma \setminus \sup(a \cap \mathrm{crit}(\vec{U}_\gamma)).\forall X \in f\,"(\nu \times \{\mathrm{crit}(\vec{U}_\gamma)\})$$
$$\nu \in X \leftrightarrow X \cap \mathrm{crit}(\vec{U}_\gamma) \in \vec{U}_\gamma$$

(4) A sequence $\mathcal{C}$ of indiscernibles for $\mathcal{K}$ is said to be $h-coherent$ if $h \in \mathcal{K}$ is a function and:
   (a) $\forall \nu \in \cup_{\gamma \in \mathrm{dom}(\mathcal{C})} \mathcal{C}_\gamma$, there is a unique $\xi \in h"\nu$ such that $\nu \in \mathcal{C}_\xi$.
   (b) If $\nu \in \mathcal{C}_\gamma \cap \mathcal{C}_{\gamma'}$ where $\gamma \neq \gamma'$ and $\gamma \in h"\nu$, then $\mathrm{crit}(\vec{U}_{\gamma'}) \in \mathcal{C}_{\gamma''}$ for some $\gamma'' < \gamma$ with $\mathrm{crit}(\vec{U}_{\gamma''}) = \mathrm{crit}(\vec{U}_\gamma)$.
   (c) If $\nu \in \mathcal{C}_\gamma$, $\gamma_\nu = coh\gamma',\gamma(\nu)$ for $\gamma' < \gamma$, and $\gamma' \in h"\nu$, then $\mathcal{C}_{\gamma_\nu} = \mathcal{C}_{\gamma'} \cap (\nu \setminus \nu')$ where ν' is the least such that $\gamma \in h"\nu'$

(5) Let x be any set and h a function. Then set $h"(x; \mathcal{C})$ is the smallest set X such that $x \subseteq X$ and $X = h"[X \cup (\bigcup_{\gamma \in X} \mathcal{C}_\gamma)]$.

(6) Suppose that $\mathcal{C}$ is a g-coherent system of indiscernibles. Define:
   (a) $S^{\mathcal{C}}(\gamma, \xi) = \min(\mathcal{C}_\gamma \setminus \xi + 1)$.
   (b) $S^{\mathcal{C}}_*(\gamma, \xi) = \min(\bigcup_{\gamma' \geq \gamma} \mathcal{C}_\gamma \setminus \xi + 1)$.
   (c) If X is any set, and $\gamma \in \mathrm{dom}(\mathcal{C}) \cap X$. An accumulation point of $\mathcal{C}_\gamma$ in X is an ordinal $\nu \in X$ such that for every $\gamma' \in X \cap \gamma \cap g"\nu$, the

$$\bigcup \{\mathcal{C}_{\gamma''} \mid \gamma'' \geq \gamma', \mathrm{crit}\,\vec{U}_{\gamma''} = \mathrm{crit}\,\vec{U}_\gamma\}$$

is unbounded in ν. Let $a^{\mathcal{C},X}(\gamma, \xi)$ is the least accumulation point of $\mathcal{C}_\gamma$ in X above ξ.

Theorem 41 (Mitchell's Covering Lemma). *Assume there is no inner model with $\exists \lambda. o(\lambda) = \lambda^{++}$. Let κ, κ' be a $\mathcal{K}$-cardinal such that $\kappa' \geq \max\{\kappa, o(\kappa)\}$. Also let X be a set such that $\kappa \not\subseteq X = Y \cap \mathcal{K}_{\kappa'}$ where $Y \prec_1 H(\kappa'^+)$. Then there is $\rho < \kappa$, $h \in \mathcal{K}$ and $\mathcal{C}$ such that:*

(1) *$\mathcal{C}$ is an h-coherent system of indiscernibles for $\mathcal{K}$.*
(2) *$\mathrm{dom}(C) \subseteq X$ and $\cup_\gamma \mathcal{C}_\gamma \subseteq X$.*
(3) *$X = h"(X \cap \rho; \mathcal{C})$ and hence $X \subseteq h"(\rho; \mathcal{C})$.*
(4) *For every $\nu \in X \cap \kappa$, either $\nu \in h"[X \cap \nu]$, or $\nu \in \mathcal{C}_\gamma$ for some γ in which case there is $\xi \in X \cap \nu$ such that either $\nu = S^{\mathcal{C}}(\gamma, \xi) = S^{\mathcal{C}}_*(\gamma, \xi)$ or there is $\gamma < \gamma' \in h"[X \cap \nu]$ such that $\nu = a^{\mathcal{C},X}(\gamma', \xi)$.*
(5) *If X' is another set satisfying is another set satisfying the assumption of the theorem then there is a finite set $a \subseteq On$ such that for every $\xi, \gamma \in X \cap X'$ such that $a \cap \mathrm{crit}(\vec{U}_\gamma) \subseteq \xi$ and $\xi > \max\{\rho_X, \rho_{X'}\}$ then:*

$$S^{\mathcal{C}}(\gamma, \xi) = S^{\mathcal{C}'}(\gamma, \xi)$$
$$S^{\mathcal{C}}_*(\gamma, \xi) = S^{\mathcal{C}'}_*(\gamma, \xi)$$
$$a^{\mathcal{C},X}(\gamma, \xi) = a^{\mathcal{C}',X'}(\gamma, \xi)$$

whenever either is defined.

A *Covering model* is a set X satisfying the assumptions of the theorem 41. In the discussion ahead, we will not distinguish between a model and its set of ordinals.

Thus, we will freely take elementary substructures in some model of ZFC, that do not contain all ordinals below some ζ and call them covering models.

The elementary embedding $j_U \restriction \mathcal{K} \colon \mathcal{K} \to \mathcal{K}^{M_U}$ is an iterated ultrapower of $\mathcal{K}$ by its measures. Let us denote the iteration by $\langle j_{\alpha,\beta} \mid \alpha \leq \beta \leq l^* \rangle$ where $j_{\alpha,\beta} \colon \mathcal{K}_\alpha \to \mathcal{K}_\beta$. We can assume that the iteration is normal i.e. $\langle \lambda_i \mid i < l^* \rangle$ is increasing where $\lambda_i = \mathrm{crit}(j_{i,i+1})$. Hence $\lambda_0 = \kappa$. Let $\langle \kappa_\alpha \mid \alpha \leq \alpha^* \rangle$ be the strictly increasing list of images of κ under this iteration. In particular, $\kappa_0 = \kappa$ and $\kappa_{\alpha^*} = j_U(\kappa)$, and $\alpha^* \leq l^*$.

Lemma 42. *Let $\eta < \alpha^*$ and let η_0 be the least ordinal such that $\kappa_\eta \leq \lambda_{\eta_0}$. Also let $\xi_\eta < l^*$ be such that $j_{\xi_\eta}(\kappa) = \kappa_\eta$ Then $j_{\xi_\eta,\eta_0}(\kappa_\eta) = \kappa_\eta$*

Proof. By elementarity, $N_0 := \mathcal{K}_{\xi_\alpha} \models \kappa_\eta$ is measurable. Let us define an internal iteration of the measures of N_0, $i \colon N_0 \to N_0^*$, $\langle i_{\alpha,\beta} \mid \alpha \leq \beta \leq \theta^* \rangle$ defined as follows:

At limit steps we simply take a direct limit. At successor step $\gamma + 1$, assume that $i_{0,\gamma} \colon N_0 \to N_\gamma$ is defined and U_β is a measure of ζ_β for $\beta < \gamma$ are the measures applied at stage β.

Let
$$\bar{\zeta}_\gamma = \sup_{\beta < \gamma}(\zeta_\beta + 1).$$

We split into cases:

- If $\mathrm{cf}^{N_0}(\gamma) > \kappa$ or γ is successor ordinal, consider the first measurable $\zeta_\gamma \geq \bar{\zeta}_\gamma$ in N_γ and apply $U(\zeta_\gamma, 0)$.
- If $\mathrm{cf}^{N_0}(\gamma) \leq \kappa$ and γ is a limit ordinal, we take the least N_γ-measurable $\zeta = \zeta_\gamma$, such that for some ρ, the set $\{\beta < \gamma \mid i_{\beta,\gamma}(U_\beta) = U(\zeta, \rho)\}$ is bounded in γ, assuming that there is one. If there is no such ζ, take $\gamma = \theta^*$ and halt.

Let us claim that the elementary embedding $j_{\xi_\eta,l^*} \colon N_0 \to \mathcal{K}^{M_U}$ can be completed to N^*. Indeed, in the comparison process between the models N^* and $\mathcal{K}^{M_U}$, the model N^* will not move since measurable cardinal in N^* are critical points of steps of the iteration of cofinality at most κ and M_U is closed under κ-sequences.

Hence there is an iteration $\langle \sigma_{\alpha,\beta} \mid \alpha \leq \beta \leq \rho^* \rangle$ such that $\sigma_{\rho^*} \circ j_{\xi_\eta,l^*} = i$. We are only interested in the part of the iteration which have critical points below κ_η, and the iteration $\sigma \circ j_{\xi_\eta,l^*}$ is equivalent to a normal one. Let β_0 be the least such that $\mathrm{crit}(i_{\beta_0,\beta_0+1}) \geq \kappa_\eta$, then there is γ_0 such that $\sigma_{\gamma_0} \circ j_{\eta_0,\xi_\eta} = i_{\beta_0}$. Since i_{β_0} is an internal iteration of N_0 with critical points below κ_η which is measurable in N_0, $i(\kappa_\eta) = \kappa_\eta$.

Hence
$$\kappa_\eta \leq j_{\eta_0,\xi_\eta}(\kappa_\eta) \leq \sigma_{\gamma_0}(j_{\eta_0,\xi_\eta}(\kappa_\eta)) = \kappa_\eta.$$
We conclude that $j_{\xi_\eta,\eta_0}(\kappa_\eta) = \kappa_\eta$. $\qquad\square$

Corollary 43. $\{\kappa_\alpha \mid \alpha \leq \alpha^*\} \subseteq \{\lambda_i \mid i \leq l^*\} \cup \{j_U(\kappa)\}$.

Proof. Assume that $\kappa_\alpha \notin \{\lambda_i \mid i \leq l^*\}$, let us show that $\kappa_\alpha = j_U(\kappa)$. Let ξ_α to be the least such that $j_{\xi_\alpha}(\kappa) = \kappa_\alpha$. Consider α_0 to be the minimal such that $\kappa_\alpha \leq \lambda_{\alpha_0}$. If $\alpha_0 = l^*$, then we are done. Otherwise we actually get the conclusion by using the assumption that $\kappa_\alpha < \lambda_{\alpha_0} = \mathrm{crit}(j_{\alpha_0,l^*})$. Clearly, $\xi_\alpha \leq \alpha_0$, otherwise, since $\mathrm{crit}(j_{\alpha_0,\xi_\alpha}) = \lambda_{\alpha_0} > \kappa_\alpha$ (again, this is clear in case $\alpha_0 = l^*$), $j_{\xi_\alpha,\alpha_0}(\kappa_\alpha) = \kappa_\alpha = j_{\xi_\alpha,\alpha_0}(j_{\alpha_0}(\kappa))$, hence $\kappa_\alpha = j_{\alpha_0}(\kappa)$, contradiction the minimality of ξ_α. By lemma 42, $j_{\alpha_0,\xi_\alpha}(\kappa_\alpha) = \kappa_\alpha$, hence
$$j_U(\kappa) = j_{l^*,\alpha_0}(j_{\alpha_0,\xi_\alpha}(j_{\xi_\alpha}(\kappa))) = j_{\alpha_0,l^*}(j_{\xi_\alpha,\alpha_0}(\kappa_\alpha)) = j_{\alpha_0,l^*}(\kappa_\alpha) = \kappa_\alpha$$

$\qquad\square$

Claim 44. *If $\kappa_\alpha \leq \delta < \kappa_{\alpha+1}$ then there is $h \in ({}^\kappa\kappa)^\mathcal{K}$ such that $\delta \leq j_U(h)(\kappa_\alpha) < \kappa_{\alpha+1}$.*

Proof. Assume $\kappa_\alpha \leq \delta < \kappa_{\alpha+1}$ decompose the iteration

$$j_U \upharpoonright \mathcal{K} = j_{\xi_\alpha+1,l^*} \circ j_{\xi_\alpha+1,\xi_{\alpha+1}} \circ j_{\xi_\alpha,\xi_\alpha+1} \circ j_{\xi_\alpha}$$

where

$$j_{\xi_\alpha} : \mathcal{K} \to \mathcal{K}_{\xi_\alpha}, \ \mathrm{crit}(j_{\xi_\alpha}) = \kappa$$
$$j_{\xi_\alpha,\xi_\alpha+1} : \mathcal{K}_{\xi_\alpha} \to \mathcal{K}_{\xi_\alpha+1}, \ \mathrm{crit}(j_{\xi_\alpha,\xi_\alpha+1}) = \kappa_\alpha, \ \text{and } j_{\xi_\alpha,\xi_\alpha+1}(\kappa_\alpha) = \kappa_{\alpha+1}$$
$$j_{\xi_\alpha+1,\xi_{\alpha+1}} : \mathcal{K}_{\xi_\alpha+1} \to \mathcal{K}_{\xi_{\alpha+1}}, \mathrm{crit}(j_{\xi_\alpha+1,\xi_{\alpha+1}}) = \lambda_{\xi_\alpha+1}$$
$$j_{\xi_\alpha+1,l^*} : \mathcal{K}_{\xi_\alpha+1} \to \mathcal{K}^{M_U}, \ \mathrm{crit}(j_{\xi_\alpha+1,l^*}) = \kappa_{\alpha+1}$$

First consider only the iteration $j_{\xi_\alpha+1}$, there is $f \in ({}^\kappa\kappa)^\mathcal{K}$ such that

$$j_{\xi_\alpha+1}(f)(\lambda_{i_1}, \ldots, \lambda_{i_n}) = \delta$$

where $\lambda_{i_1}, \ldots, \lambda_{i_n} \leq \kappa_\alpha$. Now let us define $h : \kappa \to \kappa$ by

$$h(\alpha) = \sup(f(\vec{\xi}) \mid \vec{\xi} \in [\alpha+1]^n)$$

$h \in \mathcal{K}$ as it is definable. It follows that $\delta \leq j_{\xi_\alpha+1}(h)(\kappa_\alpha) < \kappa_{\alpha+1}$. Further iteration might move $j_{\xi_\alpha+1}(h)(\kappa_\alpha)$, but not past $\kappa_{\alpha+1}$. Indeed, by lemma 42,

$$\kappa_{\alpha+1} \leq j_{\xi_\alpha+1,\xi_{\alpha+1}}(\kappa_{\alpha+1}) \leq j_{\xi_\alpha+1}(\kappa_{\alpha+1}) = \kappa_{\alpha+1}$$

Hence $\kappa_{\alpha+1} = j_{\xi_\alpha+1,\xi_{\alpha+1}}(\kappa_{\alpha+1})$. It follows that

$$j_U(h)(\kappa_\alpha) = j_{\xi_\alpha+1,l^*}(j_{\xi_\alpha+1}(h)(\kappa_\alpha)) = j_{\xi_\alpha+1}(h)(\kappa_\alpha) =$$
$$= j_{\xi_\alpha+1,\xi_{\alpha+1}}(j_{\xi_\alpha+1}(h)(\kappa_\alpha)) < j_{\xi_\alpha+1,\xi_{\alpha+1}}(\kappa_{\alpha+1}) = \kappa_{\alpha+1}$$

$\square$

There is a close connection between the critical points of the iteration j_U and indiscernibles of covering models from Mitchell's covering lemma.

Lemma 45. *Let $N = h^N$"$(\rho; \mathbb{C}^N)$ be a covering model where $\mathbb{C}^N$ is a h^N-coherent system of indiscernibles for $\mathcal{K}_{M_U}$ where $h^N \in \mathcal{K}^{M_U}$ is a Skolem function. Suppose that $\kappa_{\gamma_0} \in N$ for some $\gamma_0 < \alpha^*$.*

Then for all but finitely many $c \in \cup\{\mathbb{C}_\gamma^N \mid \mathrm{crit}(\mathbb{C}_\gamma^N) = \kappa_{\gamma_0}\}$, $c \in \{\kappa_\alpha \mid \alpha \leq \gamma_0\}$.

Proof. Suppose otherwise. Let $\langle \delta_n \mid n < \omega \rangle$ be an increasing sequence in

$$\cup\{\mathbb{C}_\gamma^N \mid \mathrm{crit}(\mathbb{C}_\gamma^N) = \kappa_{\gamma_0}\} \setminus \{\kappa_\alpha \mid \alpha \leq \alpha^*\}$$

Set

$$\alpha_n = \max(\{\alpha \leq \alpha^* \mid \kappa_\alpha < \delta_n\}).$$

By Claim 44 there is $f_n : \kappa \to \kappa$ in $\mathcal{K}$ increasing such that

$$\delta_n < j_U(f_n)(\kappa_{\alpha_n}) < \kappa_{\alpha_n+1}.$$

Consider $\{f_n \mid n < \omega\}$. While this set might not be a member of $\mathcal{K}$, we are above to bound it. Let $\langle t_\xi \mid \xi < \kappa^+ \rangle$ be the canonical enumeration in $\mathcal{K}$ of $({}^\kappa\kappa)^\mathcal{K}$. For every $n < \omega$, let ξ_n be the unique ordinal such that $f_n = t_{\xi_n}$. Both κ and $\kappa^+ = (\kappa^+)^\mathcal{K}$ are regular in V (here we are using the covering theorem, and the measurability of κ in V). So, there is $a \subseteq \kappa^+, a \in \mathcal{K}, |a| < \kappa$ which covers $\{\xi_n \mid n < \omega\}$. To find such a set, let $\xi = \sup_n \xi_n < \kappa^+$. Let $p : \kappa \to \xi$ be a bijection in K. Since $\mathrm{cf}^V \kappa = \kappa > \omega$, $\sup p^{-1}(\xi_n) = \beta' < \kappa$. Then take $a = p" \beta' \in K$.

Define a function $f : \kappa \to \kappa$ in $\mathcal{K}$ as follows:

$$f(\nu) = \sup\{t_\xi(\nu) \mid \xi \in a\},$$

for every $\nu < \kappa$. Then, for every $n < \omega, \nu < \kappa$,

$$\kappa > f(\nu) > f_{\xi_n}(\nu).$$

Now, in the ultrapower, for every $n < \omega$,

$$\kappa_{\alpha_n+1} > j_U(f)(\kappa_{\alpha_n}) > \delta_n.$$

Let $\delta^* = \sup_{n<\omega} \delta_n \leq \kappa_{\gamma_0}$. If $\delta^* = \kappa_{\gamma_0}$, then the function $j_U(f)\,"\,\nu$ in is $\mathcal{K}^{M_U}$. Note that $\delta_n \in j_U(f)\,"\,\kappa_{\alpha_n}$. For high enough n, this will contradicts Definition 40, 3b and the indiscernibility of δ_n's. If $\delta^* < \kappa_{\alpha+1}$, then it is also indiscernible and by definition 40, 4c, the δ_n's are part of the indiscernibles for δ^*. Then we again reach a contradiction to 40, 3b. $\qquad\square$

Lemma 46. *For every $\alpha < \alpha^*$, $\kappa_{\alpha+1}$ is regular in M_U.*

Proof. Otherwise, it is singular in M_U, denote by $\lambda = \mathrm{cf}^{M_U}(\kappa_{\alpha+1}) < \kappa_{\alpha+1}$.

Work in M_U, let $H \prec H(\theta^+)$ be an elementary submodel for some high enough θ, closed to λ sequences, such that $|H| < \kappa_{\alpha+1}$. Apply Mitchell's covering lemma 41, find a covering model $H \cap \mathcal{K} \subseteq N$ of cardinality less than $\kappa_{\alpha+1}$. It is of the form $h^N\,"(\delta^N, \mathbb{C}^N)$, where $\delta^N < \kappa^*$, $\mathbb{C}^N$ is a h^N-coherent system of indiscernibles for $\mathcal{K}_{M_U}$ and $h^N \in \mathcal{K}^{M_U}$ is a Skolem function. We can assume also that $\lambda \subseteq H$. The indescernibles for $\kappa_{\alpha+1}$ in N are unbounded in $\kappa_{\alpha+1}$. On the other hand, all but finitely many indiscernables for $\kappa_{\alpha+1}$ are among $\{\kappa_\beta \mid \beta \leq \alpha\}$. This is a contradiction. $\qquad\square$

Consider $\kappa^* = \sup(\max(a^*) + 1 \cap \{\kappa_\alpha \mid \alpha \leq \alpha^*\})$. Then there is $\alpha^{**} < \alpha^*$ such that $\kappa^* = \kappa_{\alpha^{**}}$.

$\alpha^{**} \geq \kappa$.

In particular, the length of the sequence $\langle \kappa_\alpha \mid \alpha \leq \alpha^* \rangle$ is at least κ, and hence, $o^{\mathcal{K}}(\kappa) \geq \kappa$. We will not pursuit that direction here, as the next lemma gives a strictly stronger result.

Lemma 47. $\kappa^* = \max(a^*)$.

Proof. Otherwise, $\kappa^* < \max(a^*) < \kappa_{\alpha^{**}+1}$. By claim 36, for every D dense open, $a^* \setminus \kappa^* \in j(D)$. Also, by minimality of a^*, there is a dense open set D_0 such that $a^* \cap (\kappa^* + 1) \notin D_0$.

Let $h\colon \kappa \to \kappa$ be such that $j(h)(\kappa^*) \geq \max(a^*)$ which exists by claim 44. Consider

$$C = \{\alpha < \kappa \mid \forall \beta < \alpha,\ h(\beta) < \alpha\},$$

the club of all closure points of h.

Let D be the dense open set of all conditions $p \in Q$ such that there are

$$\eta < \eta' < \max p$$

such that $p \cap \eta \in D_0$ and $\eta' \in C$. Let us claim that $a^* \notin j(D)$, and thus obtain a contradiction. Indeed, the least η such that $a^* \cap \eta \in D_0$ is above $\kappa^* + 1$ and the next element of $j(C)$ above $\kappa^* + 1$ is at least $\max a^*$. $\qquad\square$

Claim 48. *For every $\alpha < \alpha^*$, and a function $f\colon \kappa_\alpha \to \kappa_\alpha$ in $\mathcal{K}^{M_U}$, there is a function $g \in \mathcal{K}$ such that $j(g)(\zeta) \geq f(\zeta)$ for all $\zeta < \kappa_\alpha$, except for a bounded error.*

Proof. Fix in $\mathcal{K} = \mathcal{K}^V$ a sequence $\langle h_\tau \mid \tau < \kappa^+ \rangle$ of functions such that for every $\tau < \tau' < \kappa^+$ the following hold in $\mathcal{K}$:

(1) $h_\tau \colon \kappa \to \kappa$,
(2) $h_\tau < h_{\tau'}$ mod bounded,
(3) for every $g\colon \kappa \to \kappa$ there is $\rho < \kappa^+$ such that $g < h_\rho$ mod bounded.

Note that $2^\kappa = \kappa^+$ in $\mathcal{K}$, hence it is easy to construct such a sequence. Apply the iteration $j_U \upharpoonright \mathcal{K}$ to the list $\langle h_\tau \mid \tau < \kappa^+ \rangle$. Let us denote by $\mathcal{K}'$ the iterated ultrapower of $\mathcal{K}$, and $i\colon \mathcal{K} \to \mathcal{K}'$ the iteration, so that $i(\kappa) = \kappa_\alpha$, and the critical point of the rest of the iteration is $\geq \kappa_\alpha$.

Note that $i " \kappa^+$ is cofinal at $i(\kappa^+)$. Moreover, $\kappa^+ = (\kappa^+)^{\mathcal{K}}$, by the anti-large cardinal assumptions made. Hence $\langle i(h_\tau) \mid \tau < \kappa^+ \rangle$ will be dominating family of functions from κ_α to κ_α in $\mathcal{K}'$. As the critical point of the rest of the iteration is high enough, $j(h_\tau) \upharpoonright \kappa_\alpha = i(h_\tau)$. $\qquad\square$

6.3. **Isolating the indiscernibles.** Recall that $\langle \kappa_\beta \mid \beta \leq \alpha^{**} \rangle$ is the sequence of images of κ under the iterated ultrapower $j_U \upharpoonright \mathcal{K}$. In particular, each $\kappa_{\beta+1}$ is the image of κ_β under the ultrapower embedding using a measure over κ_β, $\kappa_0 = \kappa$ and $\kappa_{\alpha^{**}} = \kappa^*$.

The following lemma provides a sufficient condition for the main theorem of this section:

Lemma 49. Let $A(\eta) = \{\kappa_\gamma \mid \kappa_\gamma < \eta\} \cap \mathrm{acc}(a^*)$.

If there function $t \in (\kappa^* \kappa^*)^{\mathcal{K}^{M_U}}$ and $\gamma < \kappa^*$ such that $A(\kappa^*) \setminus \gamma = C_t \cap \mathrm{acc}(a^*) \setminus \gamma$, then $o^{\mathcal{K}^{M_U}}(\kappa^*) \geq (\kappa^*)^+$,

Proof. Assume otherwise that $o^{\mathcal{K}^{M_U}}(\kappa^*) < (\kappa^*)^+$. Using Claim 48, we find some $t^* \in \mathcal{K}$ such that $j_U(t^*)$ dominates t. Find disjoint sets $\langle X_i \mid i < o^{\mathcal{K}}(\kappa)\rangle$ such that $X_i \in U(\kappa, i)$. Since $o^{\mathcal{K}}(\kappa) < \kappa^+$ there is a bijection $\pi : o^{\mathcal{K}}(\kappa) \to \kappa$. Define $g : \kappa \to \kappa$ by $g(\nu) = \pi(i)$ for the unique i such that $\nu \in X_i$. Let us argue that

$$(\star) \quad A^* := \{\nu < \kappa \mid g(\nu) < \nu\} \in \cap_{\xi < o^{\mathcal{K}}(\kappa)} U(\kappa, \xi).$$

Let $\xi < o^{\mathcal{K}}(\kappa)$, then in the ultrapower $Ult(\mathcal{K}, U(\kappa, \xi))$, $j_{U(\kappa,\xi)}(g)$ is defined similarly using $j_{U(\kappa,\xi)}(\pi) : j_{U(\kappa,\xi)}(o^{\mathcal{K}}(\kappa)) \to j_{U(\kappa,\xi)}(\kappa)$ and the sequence

$$j_{U(\kappa,\xi)}(\langle X_i \mid i < o^{\mathcal{K}}(\kappa)\rangle) = \langle X_i' \mid i < j_{U(\kappa,\xi)}(o^{\mathcal{K}}(\kappa))\rangle$$

Note that $\kappa \in j_{U(\kappa,\xi)}(X_\xi) = X'_{j_{U(\kappa,\xi)}(\xi)}$ hence

$$j_{U(\kappa,\xi)}(g)(\kappa) = j_{U(\kappa,\xi)}(\pi)(j_{U(\kappa,\xi)}(\xi)) = j_{U(\kappa,\xi)}(\pi(\xi)) = \pi(\xi) < \kappa$$

which is what we needed.

By $(\star)$, we can deduce that that $\forall \alpha < \alpha^{**}$, $j_U(g)(\kappa_\alpha) < \kappa_\alpha$. In particular

$$(\star\star) \quad M_U \models j_U(g) \text{ is regressive on } \mathrm{acc}(a^*) \cap j_U(C_{t^*})$$

Using our hypothesis again, $\mathrm{acc}(a*) \cap j_U(C_{t*})$ consists of the indiscernibles of κ^*. In particular, if $\langle \alpha_n \mid n < \omega \rangle$ is a sequence of ordinals below α^{**}, such that $j_U(g)(\kappa_{\alpha_n})$ is fixed, then $j_U(g)(\sup \kappa_{\alpha_n})$ is strictly higher.

Let $\langle a_n \mid n < \omega \rangle$ be a Prikry sequence for U obtained by lemma 39 and let $C(\mathcal{Q}) = \cup_{n < \omega} a_n$ be the generic club induced for $\mathcal{Q}$. By reflecting $(\star\star)$, we get that for every $n \geq n_0$, g is regressive on $\mathrm{acc}(a_n) \cap C_{t*}$. Hence in $V[C(\mathcal{Q})]$, g is regressive on a final segment of $C(\mathcal{Q}) \cap C_{t*}$ which is a club in $V[C(\mathcal{Q})]$. Since κ remains regular in the generic extension $V[C(\mathcal{Q})]$, there is a stationary subset $S \subseteq C_{t*} \cap \mathrm{acc}\, C(\mathcal{Q})$ of ordinals of countable cofinality, on which g is fixed. But, in particular, there is a continuous copy of $\omega + 1$ that consists of elements of S—a contradiction. $\qquad\square$

Note that if $o^{\mathcal{K}^{M_U}}(\kappa^*) \geq (\kappa^*)^+$, then $o(\kappa) > \kappa^+$. Indeed, $\kappa^* = \max(a^*) < j_U(\kappa)$. It follows that there is $\xi < l^*$ such that $\kappa^* = \mathrm{crit}(j_{\xi,\xi+1})$. This means that $o^{\mathcal{K}_\xi}(\kappa^*) > o^{\mathcal{K}^{M_U}}(\kappa^*) \geq (\kappa^*)^+$. By elementarity, $o^{\mathcal{K}}(\kappa) > \kappa^+$.

So, in order to conclude the proof, we need to prove that the hypothesis of Lemma 49 holds.

Lemma 50. *For every $\eta \leq \alpha^{**}$, such that $\kappa_\eta \in \mathrm{acc}(a^*)$, there are $t_\eta \in (\kappa_\eta \kappa_\eta)^{\mathcal{K}^{M_U}}$ and γ_η such that $A(\eta) \setminus \gamma_\eta = (C_{t_\eta} \cap a^* \cap \kappa_\eta) \setminus \gamma_\eta$.*

Proof. First note that if η is a limit ordinal and t is any function from κ_η to κ_η in $\mathcal{K}^{M_U}$, then for every sufficiently large $\alpha < \eta$, $\kappa_\alpha \in C_t$. This is true, by the arguments of the proof of Claim 44 — each such function t is obtained by plugging into a an j-image of a function in $\mathcal{K}$, finitely many ordinals below κ_η, and restricting it to κ_η.

We prove by induction in $\eta \leq \alpha^{**}$ such that $\kappa_\eta \in \mathrm{acc}(a^*)$.

Assume inductively that the claim holds for all $\eta' < \eta$. Since a^* is closed, $\kappa_\eta \in a^*$, thus κ_η is singular in M_U. Let us denote by $\lambda = \mathrm{cf}^{M_U}(\kappa_\eta) < \kappa_\eta$ and split into cases:

Case 1: Assume that $\lambda > \omega$. Since κ_η is measurable in $\mathcal{K}^{M_U}$ and singular in M_U there is a Prikry-Magidor sequence in $\langle c_i \mid i < \lambda \rangle \in M_U$ witnessing the singularity of κ_η. We can cover $\{c_i \mid i < \lambda\}$ with a covering model N for κ_η of cardinality less than κ_η such that all the c_i's are indiscernibles for κ^* is N. By lemma 45, for all but finitely many indiscernibles for κ_η, $c_i \in \{\kappa_\gamma \mid \gamma < \eta\}$. By removing a bounded piece if necessary, we can assume that $c_i = \kappa_{\eta_i}$, for some ordinals η_i.Since both $a^* \cap \kappa_\eta$ and $\{\kappa_{\eta_i} \mid i < \lambda\}$ are clubs in κ_η inside M_U, and the cofinality of κ_η is $\lambda > \omega$, we my also assume that each κ_{η_i} is a limit point of a^*. Apply the inductive hypothesis to each of the points κ_{η_i} and obtain a function $t^{\eta_i} \colon \kappa_{\eta_i} \to \kappa_{\eta_i}$ in $\mathcal{K}^{M_U}$ such that for some $\nu_i < \kappa_{\eta_i}$,

$$C_{t^{\eta_i}} \cap \mathrm{acc}(a^*) \setminus \nu_i = A(\eta_i) \setminus \nu_i.$$

By Mitchell's covering lemma, 41, $N = h^N{}''(\delta^N, \mathbb{C}^N)$, where $\delta^N < \kappa_\eta$, $\mathbb{C}^N$ is a sequence of indiscernibles and $h^N \in \mathcal{K}^{M_U}$ is a Skolem function.

In order to find a single function that works for κ_η we will prove that we can choose these functions t^{η_i} so that they are definable in the covering model N.

First let us argue that $A(\eta_i)$ is definable in M_U from the parameters κ_{η_i} and a^*, up to an initial segment, and this definition is uniform.

Lemma 51. *Let $\xi \leq \alpha^{**}$ and suppose that $\langle c_i \mid i < \lambda \rangle \in M_U$ be an increasing sequence, cofinal in κ_ξ.*

Let N' be a covering model for κ_ξ with $\{c_i \mid i < \lambda\} \subseteq N'$. Suppose that $\langle c_i \mid i < \lambda \rangle$ are indiscernibles in N' for κ_ξ. Then $c_i \in \{\kappa_\beta \mid \beta < \xi\}$, for all but finitely many i's.

Proof. Similar to Lemma 45. $\qquad\square$

Now we can formulate the crucial property of subsets of κ_ξ in M_U: $(*)(B)$

(1) $B \subseteq \kappa_\xi \cap \mathrm{acc}(a^*)$.
(2) For every covering model N' for κ_ξ there is $\rho < \kappa_\xi$ such that for every indiscernible $c > \rho$ for κ_ξ in N', if c is a limit point of a^*, then $c \in B$.
(3) For every sequence $\langle c_i \mid i < \theta \rangle \in M_U$ of elements of B, cofinal in κ_ξ, there is a covering model N' for κ_ξ and an ordinal $\theta' < \theta$ such that $\langle c_i \mid \theta' \leq i < \theta \rangle$ are indiscernibles for κ_ξ in N'.

Lemma 52. $(*)(A(\xi))$ *holds.*

Proof. Requirement (1) is clear. Requirement (2) follows from Lemma 51. Let us show requirement (3).

Indeed, for every function $g \colon \kappa_\xi^{<\omega} \to \kappa_\xi$ in $\mathcal{K}^{M_U}$ there is a function $f \in \mathcal{K}$ such that

$$g(\bar{x}) = j(f)(\rho_0, \ldots, \rho_{m-1}, \bar{x}),$$

for some fixed $\rho_0, \ldots, \rho_{m-1} < \kappa_\xi$. This follows from the iterated ultrapower representation. Thus, every κ_α which is larger than $\max(\rho_0, \ldots, \rho_{m-1})$ would be a closure point of this function. In particular, taking g to be the Skolem function of any covering model N' for the sequence $\langle c_i \mid i < \theta \rangle$ and taking θ' to be the least index in which $c_i > \max(\rho_0, \ldots, \rho_{m-1})$. we conclude that each of the elements c_i must be an indiscernible, by Theorem 41. $\qquad\square$

Lemma 53. *If $(*)(B_1)$ and $(*)(B_2)$ hold, then B_1 agrees with B_2 on a final segment, i.e. there is $\nu < \kappa_\xi$ such that $B_1 \setminus \nu = B_2 \setminus \nu$.*

Proof. Suppose otherwise. By symmetry, let us assume that there is a cofinal in κ_ξ sequence $\{e_i \mid i < \theta\} \in B_1 \setminus B_2$, in M_U. By the first clause of $(*)(B_1)$, each e_i is a limit point of a^*.

By $(*)(B_1)(3)$, there will be a covering model N' for κ_ξ with $\{e_i \mid i < \theta\} \subseteq N'$ such that for some $\theta' < \theta$, $\langle e_i \mid \theta' \le i < \theta \rangle$ are indiscernibles for κ_ξ in N'. Apply now $(*)(B_2)(2)$ to N' and $\langle e_i \mid \theta' \le i < \theta \rangle$. We will have then that a final segment of $\langle e_i \mid \theta' \le i < \theta \rangle$ is in B. Contradiction. $\qquad\square$

Claim 54. *If there is a function $t \in (^{\kappa_\xi}\kappa_\xi)^{\mathcal{K}^{M_U}}$ and some $\gamma < \kappa_\xi$ such that $A(\xi) \setminus \gamma = C_t \cap \mathrm{acc}(a^*) \setminus \gamma$ then there is a uniformly definable function in M_U, $t^\xi \colon \kappa_\xi \to \kappa_\xi \in \mathcal{K}^{M_U}$, with parameters κ_ξ, a^*, such that for some $\mu < \xi$, $(C_{t^\xi} \cap \mathrm{acc}(a^*) \cap \kappa_\eta) \setminus \mu = A(\eta) \setminus \mu$.*

Proof. By assumption, t satisfies the above equality, and by the previous claim, we let t^ξ be the least function t in the order of $\mathcal{K}^{M_U}$ such that $(*)(C_t \cap \mathrm{acc}(a^*))$ holds. This is formulated in M_U using the parameters κ_ξ and a^*. $\qquad\square$

Back to κ_{η_i}'s, by the induction hypothesis and by claim 54, fix the function t^{η_i} which is definable with parameters κ_{η_i}, a^*.

Lemma 55. *Assume that N_0 is a covering model for κ_η and $h^{N_0} \in \mathcal{K}^{M_U}$ the associated Skolem function. Consider $\tilde{h}^{N_0} \colon \kappa_\eta \to \kappa_\eta \in \mathcal{K}^{M_U}$ defined as follows:*

$$\tilde{h}^{N_0}(\rho) = \sup(\{h^{N_0}(\vec{\xi}) \mid \vec{\xi} \in [\rho + 1]^{<\omega} \text{ and } h^{N_0}(\vec{\xi}) < \kappa^*\}).$$

Suppose $\eta' < \eta$ is such that $\kappa_{\eta'}, a^* \in N_0$ and $t^{\eta'}$ is definable as above.
Then for all but boundedly many $\nu < \kappa_{\eta'}$, $\tilde{h}^{N_0}(\nu) \ge t^{\eta'}(\nu)$.

Proof. We use the elementarity of N_0 and the definability of $t^{\eta'}$ to conclude that $t^{\eta'} \in N_0 \cap \mathcal{K}^{M_U}$. Note that $t^{\eta'} = h^{N_0}(\vec{c})$, for a finite sequence of N_0-indiscernibles $\vec{c} \le \kappa_{\eta'}$. By the construction of the covering model N_0, we can find $t \in N_0$, $t \colon \kappa_\eta \to \kappa_\eta$ such that $t \restriction \kappa_{\eta'} = t^{\eta'}$ and $t = h^N(\vec{c})$ where $\vec{c}$ are all indiscernables strictly below $\kappa_{\eta'}$.

Hence by the definition of $\tilde{h}^{N_0}$, for every $\max(\vec{c}) \le \nu < \kappa_{\eta'}$, $t^{\eta'}(\nu) \le \tilde{h}^{N_0}(\nu)$. It follows then by the definition of $\tilde{h}^{N_0}$ that for all but boundedly many $\nu < \kappa_{\eta'}$, $\tilde{h}^{N_0}(\nu) \ge t^{\eta'}(\nu)$. $\qquad\square$

For every $i < \lambda$, apply lemma 55 to κ_{η_i} and the model N to find $\nu_i < \kappa_{\eta_i}$ such that for every $\nu_i \le \nu < \kappa_{\eta_i}$, $\tilde{h}^N(\nu) \ge t^{\eta_i}(\nu)$.
Then, by pressing down, and since $\lambda > \omega$, there will be a stationary $Z \subseteq \lambda$ and $\nu^* < \lambda$ such that for every $\nu, \nu^* \le \nu < \kappa_{\eta_\xi}$, $\xi \in Z$ the inequality $\tilde{h}^N(\nu) \ge t^{\eta_\xi}(\nu)$ holds.

Now, shrinking Z more if necessary, we will get $\nu^{**} < \kappa^*$ such that

$$C_{\tilde{h}^N} \cap \mathrm{acc}(a^*) \setminus \nu^{**} = A(\eta) \setminus \nu^{**}.$$

Case 2: Suppose that $\lambda = \omega$.

Once again, since $\kappa_\eta \in a^*$ we can find an increasing and cofinal sequence in κ_η, $\langle \kappa_{\eta_n} \mid n < \omega \rangle \in M_U$. Let us add points to this sequence. If $\kappa_{\eta_n} \in \mathrm{acc}(a^*)$, apply the induction hypothesis, find t^{η_n} and let $\nu_n < \kappa_{\eta_n}$ be minimal such that

$$C_{t^{\eta_n}} \cap \mathrm{acc}(a^*) \setminus \nu_n = A(\eta_n) \setminus \nu_n.$$

Find $\xi_n < l^*$ be such that $\mathrm{crit}(j_{\xi_n,\xi_n+1}) = \kappa_{\eta_n}$, then $t^{\eta_n} \in \mathcal{K}_{\xi_n}$. We can represent t^{η_n} in the iteration using some $f_n \in {}^{(\kappa}\kappa)^{\mathcal{K}}$ and some intermediate critical points $\lambda_1, \ldots, \lambda_m < \kappa_{\eta_n}$, $j_{\xi_n}(f_n)(\lambda_1, \ldots, \lambda_m) = t^{\eta_n}$. Let

$$\max((\{\kappa_\alpha \mid \eta_{n-1} < \alpha < \eta_n\} \cap \{\lambda_1, \ldots, \lambda_m\}) \cup \{\kappa_{\eta_{n-1}}\}) = \kappa_{\eta_{n,1}}$$

By minimality of ν_n, there is $\eta_{n,2} < \eta_n$ such that $\kappa_{\eta_{n,2}} \leq \nu_n \leq \kappa_{\eta_{n,2}+1}$. If $\eta_{n,2} \leq \eta_{n,1}$ then add $\eta_{n,1}$ to the sequence and set $\eta^{(1)} = \eta_{n,1}$. Otherwise, add $\kappa_{\eta_{n,2}}$ to the sequence and set $\eta^{(1)} = \kappa_{\eta_{n,2}}$. If $\kappa_{\eta_n} \notin \mathrm{acc}(a^*)$, denote by

$$\nu_n = \sup(a^* \cap \kappa_{\eta_n}) < \kappa_{\eta_n}$$

There is $\eta' < \eta_n$ such that $\kappa_{\eta'} \leq \nu_n < \kappa_{\eta'+1}$ and there is a function $t^{\eta_n} \in {}^{(\kappa_{\eta_n}}\kappa_{\eta_n})^{\mathcal{K}_{\xi_n}}$ such that $\nu_n \leq t^{\eta_n}(\kappa_{\eta'})$. Indeed, by lemma 44, there is $f \in {}^{(\kappa}\kappa)^{\mathcal{K}}$ such that $j_{\xi+1}(f)(\kappa_{\eta'}) \geq \nu_n$, where $\xi < \xi_n$ is the step of the iteration such that $\kappa_{\eta'}$ is a critical point. Then we can set $t^{\eta_n} = j_{\xi_n}(f)$. Let $\eta^{(1)} = \eta'$.

In any case, if $\eta^{(1)} \leq \eta_{n-1}$ we are done. Otherwise, we move to $\kappa_{\eta^{(1)}}$ and repeat the above. After finitely many steps, defining $\eta^{(k)} < \eta^{(k-1)} < \cdots < \eta^{(1)} < \eta_n$ we reach η_{n-1}. After adding these new points, we obtain a sequence still of order type ω. Without loss of generality, this was the sequence $\langle \kappa_{\eta_n} \mid n < \omega \rangle$ that we started with. During the construction we have defined a sequence of functions $\langle t^{\eta_n} \mid n < \omega \rangle$, such that $t^{\eta_n} \in {}^{(\kappa_{\eta_n}}\kappa_{\eta_n})^{\mathcal{K}^{M_U}}$ and by closure $\langle t^{\eta_n} \mid n < \omega \rangle \in M_U$. Clearly, $t^{\eta_n} \in \mathcal{K}_{\xi_n}$. Let $\xi^* = \sup \xi_n$, then $\mathrm{crit}(j_{\xi^*,l^*}) \geq \kappa_\eta$.

Claim 56. *There is $\phi \in {}^{(\kappa}\kappa)^{\mathcal{K}}$ such that $\forall n < \omega$, and every $\kappa_{\eta_{n-1}} \leq \nu < \kappa_{\eta_n}$*
$$t^{\eta_n}(\nu) < j_{\xi_n}(\phi)(\nu)$$

Proof. By construction of the sequence $\langle \kappa_{\eta_n} \mid n < \omega \rangle$, either $\kappa_{\eta_n} \notin \mathrm{acc}(a^*)$ in which case there is $f_n \in \mathcal{K}$ such that $t^{\eta_n} = j_{\xi_n}(f_n)$ (no parameters needed). If $\kappa_{\eta_n} \in \mathrm{acc}(a^*)$, then by the construction of the sequence κ_{η_n}, there is a function $f_n \in \mathcal{K}$ and critical points

$$\lambda_1 < \cdots < \lambda_k < \kappa_{\eta_{n-1}} < \theta_1 < \cdots < \theta_m < \kappa_{\eta_{n-1}+1} \leq \kappa_{\eta_n}$$

such that

$$t^{\eta_n} = j_{\xi_n}(f_n)(\lambda_1, \ldots, \lambda_k, \kappa_{\eta_{n-1}}, \theta_1, .., \theta_m).$$

Since $\theta_m < \kappa_{\eta_{n-1}+1}$, by lemma 44, there is $b_n \in {}^{(\kappa}\kappa)^{\mathcal{K}}$ such that

$$\theta_m < j_{\xi_{n-1}}(b_n)(\kappa_{\eta_{n-1}}) \leq j_{\xi_n}(b_n)(\kappa_{\eta_{n-1}}) \leq j_U(b_n)(\kappa_{\eta_{n-1}}) < \kappa_{\eta_n}.$$

In $\mathcal{K}$, define $\phi_n : \kappa \to \kappa$ by

$$\phi_n(\alpha) = \sup\{f_n(\vec{\rho})(\xi) \mid \vec{\rho} \in [b_n(\alpha)]^{<\omega} \cap \mathrm{dom}(f_n) \wedge \xi \leq \alpha\} + 1.$$

Then for every $\kappa_{\eta_{n-1}} \leq \nu < \kappa_{\xi_n}$,

$$j_{\xi_n}(f_n)(\vec{\lambda}, \kappa_{\eta_{n-1}}, \vec{\theta})(\nu) \leq \sup\{j_{\xi_n}(f_n)(\vec{\xi})(\xi) \mid \vec{\xi} \in [j_{\xi_n}(b_n)(\nu)]^{<\omega} \wedge \xi \leq \nu\}.$$

Hence $t^{\eta_n}(\nu) = j_{\xi_n}(f_n)(\vec{\lambda}, \kappa_{\eta_{n-1}}, \vec{\theta})(\nu) < j_{\xi_n}(\phi_n)(\nu)$. We proceed as in lemma 45. Suppose that $\langle d_i \mid i < \kappa^+ \rangle$ is an enumeration of ${}^{(\kappa}\kappa)^{\mathcal{K}}$ and that $\phi_n = d_{\mu_n}$ There is a set $a \subseteq \kappa^+$ such that $a \in \mathcal{K}$, $|a| < \kappa$ and $\{\mu_n \mid n < \omega\} \subseteq a$. Define in $\mathcal{K}$, $\phi : \kappa \to \kappa$ by

$$\phi(\alpha) = \sup\{d_i(\alpha) \mid i \in a\}$$

Since κ is regular in $\mathcal{K}$, ϕ is well defined and for every $n < \omega$, ϕ dominates ϕ_n everywhere. By elementarity of j_{ξ_n}, ϕ will be as desired $\square$

Denote by $t^\eta = j_U(\phi) \restriction \kappa_\eta \in \mathcal{K}^{M_U}$. Note that $t^\eta \restriction \kappa_{\eta_n} = j_{\xi_n}(\phi)$. Let us prove that t^η is as wanted:

Claim 57. *There is $\gamma_\eta < \kappa_\eta$ such that*

$$(C_{t^\eta} \cap \mathrm{acc}(a^*) \cap \kappa_\eta) \setminus \gamma_\eta = A(\eta) \setminus \gamma_\eta.$$

Proof. As we claimed before, $\{\kappa_\gamma \mid \gamma_\eta \leq \gamma < \eta\}$ is a weak Prikry-Magidor sequence for $\mathcal{K}^{M_U}$ and C_{t^η} is a club in $\mathcal{K}^{M_U}$, there is γ_η such that $\{\kappa_\gamma \mid \gamma_\eta \leq \gamma < \eta\} \subseteq C_{t^\eta}$. This proves the inclusion from right to left. For the other direction, assume that $\delta \in C_{t^\eta} \setminus \kappa_{\gamma_\eta}$ such that $\delta \notin \{\kappa_\gamma \mid \gamma_\eta \leq \gamma < \eta\}$, let us argue that $a^* \cap \delta$ is bounded below δ. Fix any $n < \omega$ such that $\kappa_{\eta_n} < \delta < \kappa_{\eta_{n+1}}$. We split into cases. If $\kappa_{\eta_{n+1}} \notin \mathrm{acc}(a^*)$, then

$$\sup(a^* \cap \delta) \leq \sup(a^* \cap \kappa_{\eta_{n+1}}) = \nu_n \leq t^{\eta_{n+1}}(\kappa_{\eta_n})$$

By claim 56, $t^{\eta_{n+1}}(\kappa_{\eta_n}) < j_{\xi_n}(\phi)(\kappa_{\eta_n}) = t_\eta(\kappa_{\eta_n})$. Since $\kappa_{\eta_n} < \delta \in C_{t_\eta}$, we conclude that $\sup(a^* \cap \delta) < \delta$ and δ is not a limit point of a^*.

If $\kappa_{\eta_{n+1}} \in \mathrm{acc}(a^*)$, then by the construction of κ_{η_n} we have that $C_{t^{\eta_{n+1}}} \cap \mathrm{acc}(a^*) \setminus \kappa_{\eta_n} = A(\eta_{n+1}) \setminus \kappa_{\eta_n}$. By assumption, $\delta \notin \{\kappa_\alpha \mid \gamma_\eta \leq \alpha < \eta\}$, hence $\delta \notin A(\eta_{n+1})$. Since $\kappa_{\eta_n} < \delta$, it follows that $\delta \notin C_{t^{\eta_{n+1}}} \cap \mathrm{acc}(a^*)$. □

This conclude that proof of lemma 49, and the proof of theorem 34. □

It is possible to try to proceed further and to deal with the situation when $o(\kappa^*) = (\kappa^*)^+$. If, as a result, κ^* remain regular (which is typical forcing situation) then a^* must be bounded in κ^*, since no regular cardinal can be in a^*, and so we are basically in the situation considered above.

However, κ^* can change cofinality — there are forcing construction in which it changes cofinality to ω. In this case a finer analysis of indiscernibles seems to be needed, and Mitchell's accumulation points may appear.

Our conjecture is that the result above is not optimal and it can be strengthened.

7. COMPACTNESS FOR MASTERABLE FORCING NOTIONS

In this section we will isolate a subclass of forcing notions that consistently include many important forcing notions (such as all the complete subforcings of $\mathrm{Add}(\kappa, 1)$ and more), such that it is possible to force from a measurable cardinal that for any forcing $\mathbb{P}$ in this class, there is a κ-complete ultrafilter extending $\mathcal{D}(\mathbb{P})$.

Lemma 58. *Let $\mathbb{Q}$ be a κ-distributive forcing of size κ.*
Suppose that there is a generic elementary embedding

$$j \colon V^{\mathbb{Q}} \to M$$

with $\mathrm{crit}\, j = \kappa$. *Then, in M, there is a single condition $m \in j(\mathbb{Q})$ which is stronger than $j(p)$ for any condition p in the generic filter for $\mathbb{Q}$.*

Proof. Without loss of generality we can assume that $\mathbb{Q} = \kappa$, i.e. the set of conditions of the forcing $\mathbb{Q}$ is just κ. Let $G \subseteq \mathbb{Q}$ be the generic filter. By elementarity, $M = M'[j(G)]$, where $\forall p \in G$, $j(p) \in j(G)$. Note that since $G \subseteq \kappa$ and $\mathrm{crit}(j) = \kappa$, $G = j(G) \cap \kappa \in M$. By the distributivity of $\mathbb{Q}$ over V and by elementarity of j, $j(\mathbb{Q})$ is also $j(\kappa)$-distributive over M', hence $G \in M'$. In particular, the set

$$D = \{q \in j(\mathbb{Q}) \mid ((\forall p \in G)(q \geq j(p))) \vee ((\exists p \in G)(q \perp j(p)))\}$$

is dense open in M'. Clearly, any condition $m \in j(G)$ from this set will witness the validity of the lemma, since $j(G) \supseteq j\text{''}G = G$. □

Define now a subclass of κ-distributive forcing of size κ.

Definition 59. A forcing notion $\mathbb{Q}$ is called *masterable* if
  (1) $\mathbb{Q}$ is a κ-distributive forcing of size κ,
  (2) there is a forcing notion $\underset{\sim}{\mathbb{R}} \in V^Q$ such that

(a) In $V^{\mathbb{Q}*\mathbb{R}}$, there is an elementary embedding
$$j \colon V^{\mathbb{Q}} \to M$$
with $\operatorname{crit} j = \kappa$.

(b) $\mathbb{Q} * \underset{\sim}{\mathbb{R}}$ contains a dense subset of size $\leq \kappa$ and $\mathbb{Q} * \underset{\sim}{\mathbb{R}}$ is $<\kappa$-strategically closed.

Let $\mathcal{N}_\kappa$ denotes the class of all masterable forcing notions.

Claim 60. *$\mathcal{N}_\kappa$ is closed under complete subforcings.*

Proof. Assume $\mathbb{Q}$ is a complete subforcing of $\mathbb{P} \in \mathcal{N}_\kappa$. Then $|\mathbb{Q}| \leq |\mathbb{P}| \leq \kappa$ and let $\underset{\sim}{\mathbb{R}}$ witness propery (2) for $\mathbb{P}$. Let $\underset{\sim}{\mathbb{R}}' = \mathbb{P}/G_{\mathbb{Q}} * \underset{\sim}{\mathbb{R}}$ where $\mathbb{P}/G_{\mathbb{Q}}$ is then quotient forcing. Now $\mathbb{Q} * \underset{\sim}{\mathbb{R}}' \simeq \mathbb{P} * \underset{\sim}{\mathbb{R}}$ and so condition (2) holds for $\mathbb{Q}$. $\qquad\square$

Theorem 61. *Assume GCH and let κ be a measurable cardinal.*
   Then there is a cofinality preserving forcing extension in which for any $\mathbb{Q} \in \mathcal{N}_\kappa$, there is a κ-complete ultrafilter $\mathcal{U}$ extending $\mathcal{D}_p(\mathbb{Q})$ for every $p \in \mathbb{Q}$.

Proof. Let $\mathbb{P}_\kappa$ be a Easton support iteration of length κ, $\langle \mathbb{P}_\alpha, \underset{\sim}{\mathbb{Q}}_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$. At each step, $\mathbb{Q}_\alpha$ is either the trivial forcing, if α is not inaccessible, or the lottery sum of all $<\alpha$-strategically closed forcing notions of size α (were the trivial forcing is included).

Let $G_\kappa \subseteq P_\kappa$ be a generic. We argue that the model $V[G_\kappa]$ is as desired.

Let $\mathbb{Q}$ be a forcing notion in $(\mathcal{N}_\kappa)^{V[G_\kappa]}$ and $p \in \mathbb{Q}$. Let U be a normal, κ-complete ultrafilter over κ. Let $j_1 \colon V \to N_1 \cong \operatorname{Ult}(V, U)$ be the ultrapower maps using U. Let $\kappa_1 = j_1(\kappa)$.

Let us extend, in $V[G_\kappa]$, the embedding j_1 to an elementary embedding
$$j_1^* \colon V[G_\kappa] \to N_1[G_{\kappa_1}].$$
Indeed, $j(\mathbb{P}_\kappa) = \mathbb{P}_\kappa * j(\mathbb{P})_{[\kappa, j(\kappa))}$. By picking the trivial forcing at κ, the rest of the iteration is κ^+-strategically closed in V (by the closure of N_1 to κ-sequences). The number of dense open sets of the tail forcing is κ^+ (as enumerated in V) and thus one can construct in $V[G_\kappa]$ an $N_1[G_\kappa]$-generic filter for the tail forcing $j(\mathbb{P})_{\kappa, j(\kappa)}$. Let G_{κ_1} be the generic filter for N_1.

By elementarity, $j_1^*(\mathbb{Q}) \in (\mathcal{N}_{\kappa_1})^{N_1[G_{\kappa_1}]}$, thus by condition (2) there is a $\underset{\sim}{\mathbb{R}}$ and a dense subset $X \subseteq j_1^*(\mathbb{Q}) * \underset{\sim}{R}$ such that $N_1[G_{\kappa_1}] \models |X| \leq \kappa_1$. By GCH, from the point of view of $V[G_\kappa]$, there are κ^+ dense open sets to meet in order to generate a generic filter for $j_1^*(\mathbb{Q})$. By condition (2), $j_1^*(\mathbb{Q}) * \underset{\sim}{\mathbb{R}}$ is $<\kappa_1$-strategically closed in $N_1[G_{\kappa_1}]$, again by closure of $N_1[G_{\kappa_1}]$ to κ sequences from $V[G_k]$, it is κ^+-strategically closed from the point of view of $V[G_\kappa]$. Hence, one can find a $N_1[G_{\kappa_1}]$-generic filter, $G_{j_1^*(\mathbb{Q})} * G_{\mathbb{R}} \in V[G_\kappa]$ with $j_1^*(p) \in G_{j_1^*(\mathbb{Q})}$. Since $j_1^*(\mathbb{Q})$ is masterable using the forcing $\underset{\sim}{\mathbb{R}}$ in the extension $N_1[G_{\kappa_1}][G_{j_1^*(\mathbb{Q})} * G_{\mathbb{R}}]$ there is an elementary embedding
$$k \colon N_1[G_{\kappa_1}][G_{j_1^*(\mathbb{Q})}] \to N^*$$
such that $\operatorname{crit} k = \kappa_1$. Let m be a condition in $k(j_1(\mathbb{Q}))$ such that m is stronger than $k(p)$ for all $p \in G_{j_1^*(\mathbb{Q})}$ which exists by applying lemma 58 to $j_1^*(\mathbb{Q})$. In $V[G_\kappa]$, define
$$\mathcal{U} = \{A \subseteq \mathbb{Q} \mid m \in k(j_1^*(A_\xi))\}$$
It is clear that $\mathcal{U}$ is a κ-complete ultrafilter that extends $\mathcal{D}_p(\mathbb{Q})$. $\qquad\square$

Corollary 62. *Consider $\mathcal{N}_\kappa$ of the model of the previous theorem $V[G_\kappa]$. Then*

(1) $\operatorname{Add}(\kappa, 1) \in \mathcal{N}_\kappa$, *and hence, by the claim above, all its complete subforcings are in $\mathcal{N}_\kappa$ (for example: adding a Suslin tree to κ, adding a non-reflecting stationary subset of a given stationary set etc.).*

(2) $Club(S) \in \mathcal{N}_\kappa$ for all $S \subseteq \kappa$ that contains all the singular cardinals and is of measure one in a normal measure over κ.

Proof. For (1), we wish to prove that $\mathbb{Q} = \mathrm{Add}(\kappa, 1) \in (N_\kappa)^{V[G_\kappa]}$. Let f be $V[G_\kappa]$-generic for $\mathrm{Add}(\kappa, 1)$, We will extend in $V[G_\kappa][f]$ the elementary embedding $j_U : V \to M_U$ to

$$j^* : V[G_\kappa][f] \to M_U[G_{\kappa_1}][f']$$

Then we can take $\underset{\sim}{\mathbb{R}}$ to be the trivial forcing in the definition of masterable. the generic G_{κ_1} will be made of G_κ followed by f as generic for Q_κ, then a $M[G_\kappa * f]$-generic filter for the rest of the forcing $P_{(\kappa, \kappa_1]}$, can be constructed in $V[G_k][f]$ using the strategic closure of of the forcing as we did in theorem 61. Also we can find the generic $f_{\kappa_1} \in V[G_\kappa][f]$ for $(\mathrm{Add}(\kappa_1, 1))^{M[G_{\kappa_1}]}$, and $f_{\kappa_1} \restriction \kappa = f$. Note that this is a condition in $\mathrm{Add}(\kappa_1, 1)^{M[G_{\kappa_1}]}$. Above this condition, we can construct the generic f_{κ_1} since again $\mathrm{Add}(\kappa_1, 1)^{M[G_{\kappa_1}]}$ as κ^+ many dense open subsets from the point of view of $V[G_\kappa][f]$ and is it k^+-closed since the model is closed under κ-sequences.

For (2), Let $S \subset \kappa$ be a stationary set that contains all the singular cardinals and let us assume that $S \in W$, for normal measure W over κ. We need to show that $Club(S) \in (\mathcal{N}_\kappa)^{V[G_\kappa]}$. Indeed, the forcing $Club(S)$ is $<\kappa$-strategically closed. Let $H \subseteq Club(S)$ be $V[G_\kappa]$-generic. Let us show that in $V[G_\kappa][H]$, the elementary embedding j_W, which corresponds to W, extends to an elementary embedding:

$$j'_W : V[G_\kappa] \to N_W[G'_{\kappa_1}],$$

where $\kappa_1 = j_W(\kappa)$, by taking the generic of $j(\mathbb{P}_\kappa) \restriction \kappa + 1$ to be $G_\kappa * H$ and extending it to a generic filter G'_{κ_1}, using the κ^+-strategically closure of the tail forcing in $V[G_\kappa][H]$.

Since $\mathrm{crit}\, j'_W = \kappa$, for any $p \in Club(S)$, $j'_W(p) = p$. Also, since $\kappa \in j'_W(S)$.

$$m = \{\kappa\} \cup \bigcup_{p \in H} p \in j'_W(Club(S)).$$

Using the same arguments as before, we can find an $N_W[G_{\kappa_1}]$-generic filter $H' \in V[G_\kappa][H]$ for $j'_W(Club(S))$ such that $m \in H'$. We conclude that the embedding j'_1 extends to an embedding:

$$j''_W : V[G_\kappa][H] \to N_1[G_{\kappa_1}][H'].$$

Therefore, we can take $\underset{\sim}{\mathbb{R}}$ to be the trivial forcing. $\square$

Note that in general $\mathrm{Add}(\kappa, 1)$ might not be masterable. For example, if we force above $L[U]$ with $\mathrm{Add}(\kappa, 1)$ the κ is no longer measurable.

Let us deduce now one more corollary that relates to the result of section 6.

Corollary 63. *Consider, in $V[G_\kappa]$, the forcing for adding a club through singulars and inaccessibles which are not Mahlo, i.e.*

$$\mathbb{Q} = \{a \subseteq \kappa \mid |a| < \kappa, a \text{ is closed and each member of}$$

$$a \text{ is either a singular cardinal or an inaccessible which is not a Mahlo}\}$$

ordered by end-extension. Then $\mathbb{Q} \in \mathcal{N}_\kappa$.

Proof. Let $G(\mathbb{Q})$ be a $V[G_\kappa]$-generic subset of $\mathbb{Q}$. Clearly, $\mathbb{Q}$ is a $<\kappa$-strategically closed forcing of cardinality κ. Let $\underset{\sim}{\mathbb{R}}$ be the forcing for adding a club through singulars over $V[G_\kappa, G(\mathbb{Q})]$. Again $\underset{\sim}{\mathbb{Q}} * \underset{\sim}{\mathbb{R}}$ is a $<\kappa$-strategically closed forcing of cardinality κ.

Let $G(\underset{\sim}{\mathbb{R}})$ be a generic subset of $\underset{\sim}{\mathbb{R}}$ over $V[G_\kappa, G(\mathbb{Q})]$. We shall argue that in $V[G_\kappa, G(\underset{\sim}{\mathbb{Q}}), G(\underset{\sim}{\mathbb{R}})]$ there is an elementary embedding

$$i : V[G_\kappa, G(\mathbb{Q})] \to M,$$

with $\mathrm{crit}(i) = \kappa$ and $({}^\kappa M) \cap V[G_\kappa, G(\mathbb{Q})] \subseteq M$.

Let U be a normal ultrafilter over κ in V and $j : V \to N$ the corresponding elementary embedding. Work in $V[G_\kappa, G(\mathbb{Q}), G(\underset{\sim}{\mathbb{R}})]$ and extend it to an elementary embedding

$$i : V[G_\kappa, G(\mathbb{Q})] \to N[G_{j(\kappa)}, G(j(\mathbb{Q}))]$$

as follows. Set $G_{j(\kappa)} \restriction \kappa = G_\kappa$. Now let $Q_\kappa = \mathbb{Q} * \underset{\sim}{\mathbb{R}}$ and take $G(\mathbb{Q}) * G(\underset{\sim}{\mathbb{R}})$ to be its generic subset.

Note that κ was a Mahlo cardinal in $V[G_\kappa, G(\mathbb{Q})]$, and hence, in $N[G_\kappa, G(\mathbb{Q})]$, but $G(\mathbb{R})$ destroys its Mahloness. We complete building $G_{j(\kappa)}$ using the strategic closure of the relevant forcing.

Let $G(j(\mathbb{Q}))$ starts with $\bigcup G(\mathbb{Q}) \cup \{\kappa\}$. κ is not Mahlo anymore, and so, can be added. Finally, complete building $G_{j(\mathbb{Q})}$ using the strategic closure of the forcing $j(\mathbb{Q})$ i.e. we have κ^+ many dense open sets to meet, the bad player starts with playing $G(\mathbb{Q}) \cup \{\kappa\}$ and then using the strategy we meet the rest of the dense open sets.

This completes the proof of $\mathbb{Q} \in \mathcal{N}_\kappa$. $\square$

8. Other examples

The next interesting examples should be of forcings of size κ, which are κ-distributive, but not $<\kappa$-strategically closed nor masterable.

Let start with two simple general observations.

Proposition 64. *Let $\kappa > \aleph_1, \eta < \kappa$ be a regular cardinals. Assume that for every $\lambda < \kappa$, $\lambda^{<\eta} < \kappa$. Suppose that $\langle Q, \leq_Q \rangle$ is an $\eta + 1$-strategicaly closed forcing notion. Then $\langle Q, \leq_Q \rangle$ preserves stationary subsets of κ which concentrate on cofinality η i.e. For any set S such that $S \subset \{\nu < \kappa \mid \mathrm{cf}(\nu) = \eta\}$ is stationary, $\Vdash_Q \dot{S}$ is stationary.*

Proof. Let $S \subseteq \{\nu < \kappa \mid \mathrm{cf}(\nu) = \eta\}$ be stationary. Suppose that for some generic subset $G(Q)$ of Q, S is non-stationary in $V[G(Q)]$. Let $C \subseteq \kappa$ be a club disjoint from S. Let $\underset{\sim}{C}$ be a Q-name for C.

Then, back in V there are $q \in G(Q)$ such that

$$q \Vdash (\underset{\sim}{C} \subseteq \kappa \text{ is a club and } S \cap \underset{\sim}{C} = \emptyset).$$

Fix a winning strategy σ for the Player I in plays of the length $\eta + 1$ for Q.

Pick now an elementary submodel N of H_θ, with θ large enough, such that

(1) $N \supseteq \eta + 1$ and $Q, S, \sigma, \underset{\sim}{C}, q \in N$.
(2) $\kappa > |N| \geq \eta$,
(3) $\sup(N \cap \kappa) \in S$,
(4) ${}^{<\eta}N \subseteq N$,

This is possible since we can construct an continuous and increasing sequence of models $\langle N_i \mid i < \eta \rangle$ satisfying $(1), (2)$, ${}^{<\eta}N_i \subseteq N_{i+1}$ and $\sup(N_i \cap \kappa) < \kappa$. Since η is regular and $\eta^{<\eta} = \eta$ we can construct such a sequence and $\cup_{i < \eta} N_i = N_0^*$. Then N_0^* satisfy $(1), (2), (4)$. We keep defining increasing and continuous models

$$\langle N_i^* \mid i < \kappa \rangle$$

satisfying $(1), (2)$ and at successor points also (4). In this definition we exploit the cardinal assumption that for every $\lambda < \kappa$, $\lambda^{<}\eta < \kappa$. The set

$$\{\sup(N_i^* \cap \kappa) \mid i < \kappa\}$$

is a club at κ thus there is $i < \kappa$ such that $\alpha = \sup(N_i^* \cap \kappa) \in S$. Note that the cofinality of α is η and therefore $(N_i^*)^{<\eta} \subseteq N_i^*$. Let $N = N_i^*$, then N satisfy $(1) - (4)$.

Let $\langle \xi_i \mid i < \eta \rangle$ be a cofinal sequence in $\sup(N \cap \kappa)$. By (4), every initial segment of it is in N.

Using σ it is easy to define an increasing sequence of conditions $\langle q_i \mid i \leq \eta \rangle$ in Q such that

(1) $q_0 = q$.
(2) $q_i \in N$, for every $i < \eta$.
(3) There is $\alpha_i \geq \xi_i$ such that $q_{i+1} \Vdash \dot{\alpha}_i \in \underset{\sim}{C}$.

Since $q_i \in N$, $\alpha_i \in N \cap \kappa$ such $\langle \alpha_i \mid i < \eta \rangle$ form an increasing and continuous sequence in α. Let $p_\eta = \sigma(\langle p_i q_i \mid i < \eta \rangle)$, then

$$q_\eta \Vdash \sup(N \cap \kappa) \in \underset{\sim}{C},$$

since it also forces that $\underset{\sim}{C}$ is closed.

This is impossible, since $\sup(N \cap \kappa) \in S$, Contradiction. $\qquad \square$

Proposition 65. *Let $\kappa > \aleph_1, \eta < \kappa$ be a regular cardinals. Assume that for every $\lambda < \kappa$, $\lambda^{<\eta} < \kappa$. Suppose that $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is a forcing notion that destroys stationarity of a subset of κ which concentrate on cofinality η. Then P is not masterable.*

Proof. Suppose otherwise. Then there is a forcing notion $\underset{\sim}{\mathbb{R}}$ such that $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ is $<\kappa$-strategically closed. In particular, $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ is $\eta + 1$-strategically closed.

By the previous proposition, then $\mathbb{P} * \underset{\sim}{\mathbb{R}}$ preserves stationary subsets of κ which concentrate on cofinality η.

This is impossible since already $\langle \mathbb{P}, \leq_{\mathbb{P}} \rangle$ is a forcing notion that destroys stationarity of some stationary subset $S \subseteq \kappa$ which concentrate on cofinality η, hence the witnessing club which is disjoint from S will also be present in extensions of $\mathbb{P} * \underset{\sim}{\mathbb{R}}$, Contradiction. $\qquad \square$

Now we deal with a particular example. Let S be a fat subset of κ such that

$$\{\nu < \kappa \mid \mathrm{cf}(\nu) = \eta\} \setminus S$$

is stationary.

Then, the forcing $Club(S)$ is κ-distributive (since S is fat). $Club(S)$ shoots a club through S and therefore distroys the stationarity of $\{\nu < \kappa \mid \mathrm{cf}(\nu) = \eta\} \setminus S$. It follows that $Club(S)$ is not $< -\kappa$-strateginaly closed (even not $\eta + 1$-strateginaly closed) and not masterable.

Note that if we force a Cohen function $f : \kappa \to \kappa$, then for every $\delta < \kappa$ the set

$$S_\delta^f = \{\nu < \kappa \mid f(\nu) = \delta\}$$

will be a fat stationary subset of κ such that for every regular $\eta < \kappa$, the set

$$S_\delta^f \cap \{\nu < \kappa \mid \mathrm{cf}(\nu) = \eta\}$$

is co-stationary. The next lemma shows that a similar method to the one used for masterable forcings, can be used to extend $D_p(Q)$ for this kind of fat stationary sets.

Lemma 66. *Let κ be measurable cardinal and assume GCH. There is a cofinality preserving extension $V[G_\kappa]$ in which the following holds:*

After forcing a Cohen function $f : \kappa \to \kappa$ with $\mathrm{Add}(\kappa, 1)^{V[G_\kappa]}$, for every $\delta < \kappa$ and $p \in Club(S_\delta^f)$, $D_p(Club(S_\delta^f))$ can be extended to a κ-complete ultrafilter.

Proof. Let us use the same Easton support iteration $\langle P_\alpha, Q_\beta \mid \alpha \leq \kappa, \beta < \kappa \rangle$ as for masterable forcing, where Q_β is the trivial forcing for accessible ordinals and the lottery sum over all $<\beta$-strategically closed forcings of size $\leq \beta$ for inaccessible β. Let $G_\kappa \subseteq P_\kappa$ be V-generic. We claim that the model $V[G_\kappa]$ is as wanted. Let f_κ be a $V[G_\kappa]$-generic function for $\mathrm{Add}(\kappa, 1)^{V[G_\kappa]}$. In $V[G_\kappa][f_\kappa]$ we shell extend $D_p(Club(S_\delta^{f_\kappa}))$ for some $\delta < \kappa$ and $p \in Club(S_\delta^{f_\kappa})$. First let $U \in V$ be some normal measure on κ,

$$j_1 : V \to M_1 \simeq Ult(V, U)$$

is the corresponding elementary embedding and

$$j_{1,2} : M_1 \to M_2 \simeq Ult(M_1, j_1(U))$$

is the second iteration. Denote by $j_2 = j_{1,2} \circ j_1$, $\kappa_i = j_i(\kappa)$ for $i = 1, 2$.

Secondly, by the same arguments as in 62, by picking $\mathrm{Add}(\kappa, 1)$ at $\underset{\sim}{Q_\kappa}$, we can construct the generic

$$\underbrace{G_\kappa * f_\kappa * G_{(\kappa,\kappa_1)}}_{G_{\kappa_1}} * f_{\kappa_1} \in V[G_\kappa][f_\kappa]$$

which is M_1-generic for $j(P_\kappa * \mathrm{Add}(\kappa_1)) = P_\kappa * \underset{\sim}{Q_\kappa} * P_{(\kappa,\kappa_1)} * \mathrm{Add}(\kappa_1, 1)$. Then the embedding $j_1 : V \to M_1$ lifts to

$$j_1^* : V[G_\kappa][f_\kappa] \to M_1[G_{\kappa_1}][f_{\kappa_1}]$$

Next, we claim that the forcing $\mathrm{Add}(\beta, 1) * Club(S_\beta^f)$ is $<\beta$-strategically closed when β is inaccessible. To see this, let $\lambda < \beta$, Then the good player can always play conditions of the form $\langle g, \dot{a} \rangle \in \mathrm{Add}(\beta, 1) * Club(S_\beta^f)$ where $\dot{a}$ is the canonical name for some closed set such that $\max(a) = \mathrm{dom}(g)$. The strategy is defined as follows,

$$\sigma_\lambda(\langle\langle g_i, \dot{a}_i \rangle, \langle f_i, \underset{\sim}{b_i} \rangle \mid i < \theta \rangle) = \langle g, \dot{a} \rangle$$

where for limit steps θ,

$$g = \bigcup_{i<\theta} g_i \cup \{\langle \nu, \delta \rangle\}, \ a = \cup_{i<\theta} a_i \cup \{\nu\}$$

ν being $\sup_{i<\theta}(\sup(\mathrm{dom}(g_i)))$. This will form an element of $\mathrm{Add}(\beta, 1) * Club(S_\beta^f)$ by the definition at successor points $\theta = \tau + 1$, in which case g will simply fill the missing points in $\mathrm{dom}(f_\tau)$ with some value different then δ up to $\sup(\mathrm{dom}(f_\tau))$, if there is a maximal element in $\mathrm{dom}(f_\tau)$ let $\nu = \max(f_\tau) + 1$ otherwise $\nu = \sup(\mathrm{dom}(f_\tau))$ and define $g(\nu) = \delta$. b_τ will be extended to a canonical name $\dot{a}$ according to g.

Using this strategically closure of the forcing

$$\mathrm{Add}(\kappa_1, 1) * Club(\underset{\sim}{S_\delta^{f_{\kappa_1}}})$$

and the usual arguments of number of dense open sets, in $V[G_\kappa][f_\kappa]$ we can find a $M_1[G_{\kappa_1}][f]$-generic club H for $Club(S_\delta^{f_{\kappa_1}})^{M_1[G_{\kappa_1}][f_\kappa]}$ with $j_1(p) \in H$. Let $C = \cup H \subseteq S_\delta^{f_{\kappa_1}}$ be the generic club.

Next we shell extend $j_{1,2} : M_1 \to M_2$ to

$$j_{1,2}^* : M_1[G_{\kappa_1}][f_{\kappa_1}][H] \to M_2[G_{\kappa_2}][f_{\kappa_2}][H'']$$

To do this, note that

$$j_{2,1}(P_{\kappa_1} * \mathrm{Add}(\kappa_1, 1) * Club(S_\delta^{f_{\kappa_1}})) = P_{\kappa_1} * \underset{\sim}{Q_{\kappa_1}} * P_{(\kappa_1,\kappa_2)} * \mathrm{Add}(\kappa_2, 1) * club(\underset{\sim}{S_\delta^{f_{\kappa_2}}}))$$

For $P_{\kappa_1} * Q_\kappa$ we take $G_{\kappa_1} * (\widetilde{f_{\kappa_1} * H})$. For the forcing $P_{(\kappa_1,\kappa_2)}$ we can find a generic $G_{(\kappa_1,\kappa_2)} \widetilde{\in} M_1[G_\kappa][f'][H]$ which is $M_2[G_{\kappa_1}][f'][H]$-generic for $P_{(\kappa_1,\kappa_2)}$. Finally, note that the condition

$$\langle f_{\kappa_1} \cup \{\langle \kappa_1, \delta \rangle\}, C \cup \{\kappa_1\} \rangle \in \mathrm{Add}(\kappa_2, 1) * Club(\underset{\sim}{S_\delta^{f_{\kappa_2}}}))$$

and once again by the strategically closure and GCH we can extend this condition to a generic $f_{\kappa_2} * H' \in M_1[G_{\kappa_1}][f_{\kappa_1}][H]$. So the embedding $j_{1,2} : M_1 \to M_2$ is lifted to

$$j_{1,2}^* : M_1[G_{\kappa_1}][f_{\kappa_1}][H] \longrightarrow M_2[G_{\kappa_1}][f_{\kappa_1}][H][G_{(\kappa_1,\kappa_2)})][f_{\kappa_2}][H']$$

By lemma 58 there is a condition $m \in H'$ such that for every $q \in H$ $j_U^*(p) \leq m$. In $V[G_\kappa][f]$, define

$$W = \{x \subseteq Club(S_\delta^f) \mid m \in j_{2,1}^*(j_1^*(X)\}$$

This κ-complete ultrafilter extends $D_p(Club(S_\delta^f))$. $\qquad \square$

9. Open Problems

The following question looks natural:

Question 1. What is the exact strength of the following assertion: For every κ-distributive forcing notion of size κ the filter of its dense open subsets can be extended to a κ-complete ultrafilter?

This question is twofold. We can ask what is the *consistency strength* of this assertion and we can also inquire which large cardinals imply it.

Let Q be the forcing for shooting a club through the singulars.

Question 2. Assume that $D(Q)$ can be extended to a κ-complete ultrafilter is it consistent that $\exists \lambda.\ o(\lambda) = \lambda^{++}$?

A natural candidate for a forcing for which extending the dense open filter to an ultrafilter might require a higher consistency strength is the forcing of adding a club through a fat stationary set $S \subseteq \kappa$.
However, as it was shown above, depending on the fat stationary set, it may require a measurable alone.
A. Brodsky and A. Rinot [5] give a different way to produce many fat stationary sets. They showed that $\square(\kappa)$ implies that κ can be partitioned into κ many disjoint fat stationary sets. In our context, κ is a measurable, and so $\square(\kappa)$ fails. It is likely that still in $L[E]$-type models there will be interesting fat sets.

The next question relates to theorem 13. Recall that an abstract Prikry type forcing, is a forcing notion $\langle \mathcal{Q}, \leq, \leq^* \rangle$ such that $\leq^* \subseteq \leq$, and the Prikry property holds:

For every statement in the forcing language σ, and any condition $q \in \mathcal{Q}$,

there is $q \leq^* q^* \in \mathcal{Q}$, such that q^* decide σ

To obtain interesting Prikry type forcing we usually require that the order $\leq^*$ has high closure or directness degree.

Question 3. Is there an abstract generalization of theorem 13 to Prikry type forcing? Namely, assume there is a projection from a Prikry type forcing $\mathcal{Q}$, for which $\leq^*$ is sufficiently closed or directed onto a distributive forcing $\mathbb{P}$. Can the filter $D_p(\mathbb{P})$ be extended to a κ-complete ultrafilter?

As we noted after the proof of Theorem 13, the current formulation does not quite give us an equivalence, as we do not know if the Prikry forcing can be projected onto a distributive forcing notion of size larger than κ.

Question 4. Is there a tree of measures on κ such that the corresponding tree Prikry forcing, projects onto a σ-distributive forcing notion of size $> \kappa$.

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