The Effective Cone of Moduli Spaces of Sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$

by

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To my family,

I wouldn’t be the same without each and every one of you.

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SUMMARY

Let ξ be a stable Chern character on $\mathbb{P}^1 \times \mathbb{P}^1$, and let $M(\xi)$ be the moduli space of Gieseker semistable sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ with Chern character ξ. In this paper, we provide an approach to computing the effective cone of $M(\xi)$. We find Brill-Noether divisors spanning extremal rays of the effective cone using resolutions of the general elements of $M(\xi)$ which are found using the machinery of exceptional bundles. We use this approach to provide many examples of extremal rays in these effective cones. In particular, we completely compute the effective cone of the first fifteen Hilbert schemes of points on $\mathbb{P}^1 \times \mathbb{P}^1$.

CHAPTER 1

INTRODUCTION

In this paper, we provide an approach to computing extremal rays of the effective cone of moduli spaces of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, we show that this approach succeeds in computing the entire effective cone on the first fifteen Hilbert schemes of points.

The effective cone of a scheme is an important invariant which controls much of the geometry of the scheme (1). For Mori dream spaces, it determines all of the birational contractions of the space (2). However, in general, determining the effective cone of a scheme is a very difficult question. There has been progress computing the effective cone for certain moduli spaces.

Moduli spaces of sheaves on a fixed surface are one kind of moduli space that has been extensively studied. In this setting, the geometry of the underlying variety can be used to study the moduli space. In the past decade, Bridgeland stability has motivated a program to compute the effective cones of these moduli spaces by corresponding the edge of the effective cone with the *collapsing wall* of Bridgeland stability. This approach has proved successful in general on *K3 surfaces* (3), *Enriques surfaces* (4), *Abelian surfaces* (5), and $\mathbb{P}^2$ (6).

The proof in the last case varies greatly from the proofs in the other cases as it is a surface of negative Kodaira dimension. However, the proof relies heavily on properties that are unique to $\mathbb{P}^2$.

This paper provides the general framework to potentially extend these results to del Pezzo and Hirzebruch surfaces and explicitly works out the framework on $\mathbb{P}^1 \times \mathbb{P}^1$. The increased ranks of the Picard group and the derived category in the case of $\mathbb{P}^1 \times \mathbb{P}^1$ compared to $\mathbb{P}^2$ make the proofs and results significantly harder to obtain.

These difficulties force us to add two new ingredients to the method. The first new addition is putting the choice of an exceptional collection in the context of the work of Rudakov et al. on coils and helices (e.g. (24), (53), (10), etc.). Certain special properties of exceptional collections on $\mathbb{P}^2$ that were used are no longer needed once the choice is put in these terms. The second addition is providing a way to link neighboring extremal rays to show that there are no missing extremal rays in between them. This addition is needed as the Picard rank of the moduli spaces are now higher than two, and it will be essential for expanding these results to other surfaces.

Let ξ be a Chern character of postive integer rank on $\mathbb{P}^1 \times \mathbb{P}^1$. Then there is a nonempty moduli space $M(\xi)$ that parametrizes S-equivalence classes of semistable sheaves with that Chern character on $\mathbb{P}^1 \times \mathbb{P}^1$ iff ξ satisfies a set of Bogomolov type inequalities given by Rudakov in (7). It is an irreducible (8), normal (8), projective variety (9). We show that these spaces are $\mathbb{Q}$-factorial [Prop. 3.0.3] and, furthermore, are Mori dream spaces [Thm. 3.0.4].

We construct effective Brill-Noether divisors of the form

$$D_V = \{U \in M(\xi) : h^1(U \otimes V) \neq 0\}.$$

We create an algorithmic method to produce these divisors. Conjecturally, this method produces a set of divisors spanning the effective cone.

Conjecture. *The method laid out in this paper produces a set of effective divisors spanning the effective cone for $M(\xi)$ for all ξ above Rudakov's surface.*

One reason for this conjecture is that the method computes the entire effective cone of the first fifteen Hilbert schemes of points on $\mathbb{P}^1 \times \mathbb{P}^1$ (which is as many as we applied it to). In the last chapter of the paper, we explicitly compute the effective cone of $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]}$ for $n \leq 16$ as well as several instances of types of extremal rays that show up in infinite sequences of n, and we give a rank two example. Even if this method fails to fully compute the effective cone of every moduli space, it does give a method to produce effective divisors on these moduli spaces.

The proofs of the steps of the method follow from constructing birational maps or Fano fibrations to simpler, Picard rank one, spaces and analyzing them to find our extremal divisors. Given a birational map or Fano fibration, giving an divisor on an edge of the effective cone follows directly. The difficult part of the process is constructing the maps.

The moduli spaces that we map to are moduli spaces of Kronecker modules, $Kr_V(m, n)$. The way we construct the map is to find a resolution of the general element of $M(\xi)$ containing a Kronecker module and then to forget the rest of the resolution. The key then is to find resolutions of the general element of the moduli space that contain Kronecker modules. On $\mathbb{P}^1 \times \mathbb{P}^1$, we have a powerful tool for finding resolutions of sheaves in the form of a generalized Beilinson spectral sequence (10). Using this method for finding a resolution will make it clear that the extremal divisors we construct are Brill-Noether divisors as they will be defined in terms of the jumping of ranks of cohomology groups appearing in the spectral sequence. Using that spectral sequence, finding resolutions with Kronecker modules is reduced to finding the right collections of exceptional bundles spanning the derived category.

We find the elements of the right collections by studying Rudakov's classification of stable bundles over the hyperbola of Chern characters ζ with the properties

$$\chi(\zeta^*, \xi) = 0 \text{ and } \Delta(\zeta) = \frac{1}{2}.$$

Rudakov's necessary and sufficient inequalities for a Chern character to be stable each came from an exceptional bundle on $\mathbb{P}^1 \times \mathbb{P}^1$, and the right collections of exceptional bundles are determined by which exceptional bundles have the sharpest inequalities over this curve. Say that (E_α, E_β) is an exceptional pair of bundles that have the sharpest inequalities over that curve. Then the resolution we get for the general $U \in M(\xi)$ might look like

$$0 \to E_{\alpha}^{*}(K)^{m_3} \bigoplus F_0^{*m_2} \to F_{-1}^{*m_1} \bigoplus E_{\beta}^{*m_0} \to U \to 0.$$

Using this resolution, we get the maps we need. There are several cases to be dealt with. On $\mathbb{P}^2$, there was only two cases. The new cases are a phenomenon that will persist on other surfaces and are not unique to $\mathbb{P}^1 \times \mathbb{P}^1$.

We summarize the approach in the most common case. The resolution of the general object of the moduli space in this case will look like the example resolution above. Then the resolution has four objects and four maps. The map $W : F_0^{*m_2} \to F_{-1}^{*m_1}$ gives the required Kronecker module. We map to the Kronecker moduli space corresponding to it, $f : M(\xi) \dashrightarrow Kr_V(m,n)$. Constructing the Brill-Noether divisor in this case is slightly tricky because the bundle whose corresponding divisor spans the extremal ray is not obviously cohomologically orthogonal to the general object of the moduli space. That orthogonality is established using properties of the Kronecker modules in the resolution of a bundle whose corresponding divisor spans the extremal ray and in the resolution of the general object.

1.1 Organization

In Chap. 2, we extensively lay out the necessary background definitions and theorems. In Chap. 3, we prove two new properties of $M(\xi)$. In Chap. 4, we define primary orthogonal Chern characters via controlling exceptional pairs. In Chap. 5, we use the controlling pairs and a generalized Beilinson spectral sequence to resolve the general object of our moduli space which constructs effective divisors on our space. In Chap. 6, we use these resolutions to construct maps from our moduli space to spaces of Kronecker modules that provide the dual moving curves we need. In Chap. 7, we use these results to compute extremal rays of the effective cone of $M(\xi)$. Finally, in Chap. 8, we compute the effective cone of for $n \leq 16$, provide some recurring examples of types of corners, and work out a rank two example.

CHAPTER 2

PRELIMINARIES

In this chapter, we cover background material in three broadly defined areas: the minimal model program, moduli spaces, and derived categories.

We first introduce the minimal model program (MMP). We approach the MMP from the perspective of cycle theory. Following a brief introduction to cycles focusing on curves and divisors, we discuss characteristic classes to provide a method of producing cycles and providing numerical invariants of sheaves. After concluding our discussion of characteristic classes with the statement of the Hirzebruch-Riemann-Roch theorem, we transition into a discussion of Mori theory and the minimal model program. The MMP states an approach to understanding a scheme using certain convex cones. The goal of this thesis is to give a method to calculate one of these cones, called the *effective cone*, for a certain class of schemes.

After introducing the MMP, we proceed to our second topic: moduli spaces of sheaves (on $\mathbb{P}^1 \times \mathbb{P}^1$). We introduce moduli spaces using the examples of Hilbert schemes of points. After discussing some properties of these, we define stability in order to generalize them. Specifically, we give theorems on the existence and properties of these generalizations as well as discussing their Picard groups. To conclude this part, we state a version of the interpolation problem and briefly introduce Kronecker modules. Solving the interpolation problem on these schemes is the method we will use to calculate the effective cone on these spaces.

Following the discussion of moduli spaces, we transition to presenting derived categories, which are our third broad topic. Our presentation starts with the notion of exceptional bundle. We use them as a springboard to discuss the derived category of a scheme and its implications for the geometry of the scheme. Using the setting of derived categories, we introduce Bridgeland stability conditions on a category as a tool for understanding the underlying scheme. Finally, we state the conjectured relationship between the viewpoints of the MMP and Bridgeland stability. This part will give us the tools that we need to solve the interpolation problem on the moduli spaces we introduced previously.

2.1 Cycle Theory

We begin with an introduction to cycle theory. One good reference for this material is the recent book by Eisenbud and Harris (11).

2.1.1 Cycle Groups

For this entire chapter, let X be a smooth quasiprojective variety of dimension n.

Definition 2.1.1. The *group of algebraic cycles*, denoted $Z(X)$, is the free Abelian group generated by reduced irreducible subschemes of X.

An element of $Z(X)$ is a *cycle*.

Every subscheme Y of X has a class, $[Y]$, in $Z(X)$ defined by

$$[Y] = \sum m_Y(Y_i)Y_i$$

where the sum is taken over all reduced irreducible components, Y_i, of Y and $m_Y(Y_i)$ is the multiplicity of Y at the generic point of Y_i.

The cycle group is graded by the dimension of reduced irreducible subschemes. The k-th graded piece, $Z_k(X)$, is the free Abelian group generated by reduced irreducible subschemes of X of dimension k. A scheme is of *pure dimension* d if its class is contained entirely in $Z_d(X)$. Although grading by dimension is intuitive, we generally will prefer to grade the cycle group by codimension as it will eventually allow a graded ring structure to be introduced. Focusing on the codimension, the free Abelian group generated by reduced irreducible subschemes of X of codimension $n-k$ is $Z^{n-k}(X)$. Then $Z(X) = \sum_{k=0}^{\dim(X)} Z_k(X) = \sum_{k=0}^{\dim(X)} Z^{n-k}(X)$. In particular, in this paper we will be interested in $Z_{n-1}(X)$ and $Z_1(X)$.

Definition 2.1.2. $Z_{n-1}(X)$ is the *group of Weil divisors.*

Definition 2.1.3. $Z_1(X)$ is the *group of curves.*

Example 2.1.1. Let's work out the cycle group of $\mathbb{P}^1 \times \mathbb{P}^1$. First, the reduced irreducible subvarieties of codimension 2 on $\mathbb{P}^1 \times \mathbb{P}^1$ are exactly the closed points of the space so we have

$$Z^2(\mathbb{P}^1 \times \mathbb{P}^1) = \left\langle [x] : x \in \mathbb{P}^1 \times \mathbb{P}^1 \text{ is a closed point} \right\rangle.$$

Next, the reduced irreducible subvarieties of codimension 1 on $\mathbb{P}^1 \times \mathbb{P}^1$ are each determined by a single irreducible bihomogeneous polynomial in $k[x_0, x_1] \otimes_k k[y_0, y_1]$ so we have

$$Z^1(\mathbb{P}^1 \times \mathbb{P}^1) = \langle [f] : f \in k[x_0, x_1] \otimes_k k[y_0, y_1] \text{ is bihomogenous and irreducible} \rangle.$$

Finally, the only reduced irreducible subvariety of codimension 0 is the space itself so we have

$$Z^0(\mathbb{P}^1 \times \mathbb{P}^1) = \left\langle \left[\mathbb{P}^1 \times \mathbb{P}^1\right] \right\rangle.$$

2.1.2 Push-forward and Pull-back of Cycles

The cycle group interacts nicely with maps of varieties. Given a morphism of projective smooth varieties $\phi : X \to Y$ that is flat and of constant relative dimension, define the *pullback map*, ϕ^*, on subvarieties Z of Y by the equation

$$\phi^*([Z]) = [\phi^{-1}(Z)].$$

If ϕ is also proper, similarly define the pushforward map, ϕ_*, on subvarieties Z of X by the equation

$$\phi_*([Z]) = \left\{ \begin{array}{ll} 0 & \text{if } \dim(\phi(Z)) < \dim(Z) \\ \deg_{\mathbb{C}(\phi(Z))}(\mathbb{C}(Z)) \cdot [\phi(Z)] & \text{otherwise} \end{array} \right\}.$$

By linearity both of these maps extend to cycle maps, $\phi^* : Z(Y) \to Z(X)$ and $\phi_* : Z(X) \to Z(Y)$, respectively.

2.1.3 Equivalence Relations on Cycles

Although the cycle group respects maps of varieties, other difficulties present themselves when working with it. For example, as we can saw in the example, the cycle group and its graded pieces are often quite large. Their size can make them hard to work with so we would like to find useful quotients of them. We get a quotient group of the cycle group by modding out by an equivalence relation on cycles. There are three common equivalence relations put on the cycle group: rational equivalence, algebraic equivalence, and numerical equivalence.

Definition 2.1.4. Two cycles Y, Z are *rationally equivalent*, denoted $Y \equiv Z$, if there is a cycle A on $\mathbb{P}^1 \times X$ such that $Y - Z = A_{[0:1]} - A_{[1:0]}$.

If Y and Z are divisors, this notion is historically called *linear equivalence.*

Definition 2.1.5. Two cycles Y, Z are *algebraically equivalent*, denoted $Y \equiv_{alg} Z$, if there is a cycle A on $C \times X$ such that $Y - Z = A_p - A_q$ where C is an algebraic curve and p and q are closed points of C.

Definition 2.1.6. Two cycles Y, Z of pure dimension d are *numerically equivalent* if $[Y \cap W] = [Z \cap W]$ for all cycles W with pure dimension $n - d$ which intersects both generically transversally.

Being generically tranverse intuitively just means that Y and Z intersect in a nice way. Formally, subvarieties Y and Z of X are *generically transverse* if

$$\operatorname{codim}_p(Y) + \operatorname{codim}_p(Z) = \operatorname{codim}_p(X)$$

at a general point p of each component of $Y \cap Z$.

Definition 2.1.7. Two general cycles are *numerically equivalent*, denoted $Y \equiv_{num} Z$, if each of their pure dimensional pieces are numerically equivalent.

Note that rationally equivalent implies algebraically equivalent implies numerically equivalent. The implications in the other direction only hold under stronger hypotheses, which do hold in many common cases.

2.1.4 Quotient Cycle Groups and a Product Structure

Using these relations, we define three quotient groups of the cycle group. Note that each of these equivalence relations preserves the grading on the cycle group so these quotient groups are again graded.

Definition 2.1.8. The *Chow group* is the cycle group modded out by rational equivalence, denoted $A(X)$.

Definition 2.1.9. The *Néron-Severi group* is the cycle group modded out by algebraic equivalence, denoted $\mathrm{NSC}(X)$.

Definition 2.1.10. The *numerical cycle group* is the cycle group modded out by numerical equivalence, denoted $\mathrm{Num}(X)$.

Note that ϕ_* and ϕ^* are compatible with these quotients so they descend to maps between the quotient groups.

These quotient groups have an additional ring structure that we could not add to the original cycle group for technical reasons. We want the ring structure to have the property that

$$[\mathsf{Y}] \cdot [\mathsf{Z}] = [\mathsf{Y} \cap \mathsf{Z}]$$

if Y and Z intersect "nicely".

Theorem 2.1.11 ((12),Prop. 8.3)**.** *There is a unique product structure on the Chow (Néron-Severi/ numerical cycle) group turning it into the Chow (Néron-Severi/numerical cycle) ring with the property that if* Y *and* Z *are generically transverse then we have* $[\mathsf{Y}] \cdot [\mathsf{Z}] = [\mathsf{Y} \cap \mathsf{Z}]$*.*

For convenience and historical reasons, the codimension one graded pieces of these groups have special names.

Definition 2.1.12. For the Chow group, $\mathsf{A}^1(\mathsf{X})$ is known as the *Picard group*, denoted $\mathrm{Pic}(\mathsf{X})$.

Definition 2.1.13. For the Néron-Severi cycle group, $\mathrm{NSC}^1(\mathsf{X})$ is known as the *Néron-Severi group*, denoted $\mathrm{NS}(\mathsf{X})$.

Definition 2.1.14. For the numerical cycle group, $\mathrm{Num}^1(\mathsf{X})$ is the Néron-Severi group mod torsion, confusingly also denoted $\mathrm{NS}(\mathsf{X})$.

Example 2.1.2. We return to our example of $\mathbb{P}^1 \times \mathbb{P}^1$. On this surface all our equivalence relations coincide. For the zero dimensional cycles, all points are equivalent so we have

$$\mathsf{A}^2(\mathbb{P}^1 \times \mathbb{P}^1) = \langle [\text{point}] \rangle \cong \mathbb{Z}.$$

For one dimensional cycles, any two irreducible curves that are cut out by two polynomials of the same bi-degree are equivalent so we have

$$A^1(\mathbb{P}^1 \times \mathbb{P}^1) = \{[\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)] : a, b \in \mathbb{Z}\} \cong \mathbb{Z}^{\bigoplus 2}.$$

As there was only one two dimensional reduced irreducible subvariety, we have

$$A^0(\mathbb{P}^1 \times \mathbb{P}^1) = \left\langle \left[\mathbb{P}^1 \times \mathbb{P}^1\right] \right\rangle \cong \mathbb{Z}.$$

As a group, we then have that

$$A(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z} \bigoplus \mathbb{Z}^2 \bigoplus \mathbb{Z}.$$

Then the ring structure is given by the chart below.

Intersection Pairing	[point]	$[\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)]$	$\left[\mathbb{P}^1 \times \mathbb{P}^1\right]$
[point]	0	0	[point]
$[\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c,d)]$	0	$(ad + bc)$ [point]	$[\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(c,d)]$
$\left[\mathbb{P}^1 \times \mathbb{P}^1\right]$	[point]	$[\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a,b)]$	$\left[\mathbb{P}^1 \times \mathbb{P}^1\right]$

2.1.5 Cycle Spaces and Cones

We have now reduced our cycle group to these useful quotient groups. When studying a graded piece of each of these rings, it is useful to use fractional coefficients in order to "scale" cycles. In order to allow this, we tensor these spaces with $\mathbb{Q}$ (tensoring by $\mathbb{R}$ is similar). Tensoring by $\mathbb{Q}$ removes any torsion and turns these groups into $\mathbb{Q}$ vector spaces.

Definition 2.1.15. The *Chow spaces* are $A_{\mathbb{Q},i}(X) = A_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Definition 2.1.16. The *Néron-Severi spaces* are $\mathrm{NS}_{\mathbb{Q},i}(X) = \mathrm{NS}_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Since $\mathrm{NSC}_i(X) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathrm{NS}_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$, we do not need to define a third space. Again, we primarily focus on the divisor case, $\mathrm{NS}_{\mathbb{Q},n-1}(X)$ and $A_{\mathbb{Q},n-1}(X)$. The rank of $A_{\mathbb{Q},n-1}(X)$, denoted $\rho(X)$, is called the Picard rank of X. We will now refer to $\mathrm{NS}_{\mathbb{Q},n-1}(X)$ unambiguously as the Néron-Severi space or as $\mathrm{NS}(X)$. As we are only interested in $\mathbb{Q}$-factorial schemes, every class in the Néron-Severi space is a $\mathbb{Q}$-Cartier divisor class as well. All the results of this subsection extend to normal $\mathbb{Q}$-factorial schemes.

Working in the Néron-Severi space $\mathrm{NS}(X)$, we study the spans of several different types of cycle. Each of these spans will be a *convex cone* by which we mean a set of rays originating from the origin whose cross-section is convex. First, we look at the span of effective divisors.

Definition 2.1.17. The span of the effective divsors is the *effective cone*, $\mathrm{Eff}(X)$.

It has the same dimension as the space. Its closure is the *pseudo-effective cone*, $\overline{\mathrm{Eff}}(X)$; its interior is the *big cone*. The big cone is spanned by the classes of *big* divisors. A Cartier divisor L is big if the dimension of the image of the map associated to $L^{\otimes m}$(see Sec. II.7 (13)) is the dimension of X for $m >> 0$.

Next, we examine the span of moving cycles.

Definition 2.1.18. A cycle is *moving* if cycles rationally equivalent to it do not have any fixed components and cover a dense open set of X.

In this case, we will be interested in curves as well as divisors.

Definition 2.1.19. The span of moving curves (divisors) is *the moving cone* of curves (divisors), and again it is a convex cone of full dimension, $\mathrm{Mov}(X)$ ($\mathrm{Mov}_{n-1}(X)$).

As a matter of convention, we will call $\mathrm{Mov}_{n-1}(X)$ the *movable cone* and $\mathrm{Mov}(X)$ the moving cone. Also, a moving divisor has nonempty linear system so we have that the movable cone is a subset of the effective cone,

$$\mathrm{Mov}_{n-1}(X) \subset \mathrm{Eff}(X).$$

One reason that we include the moving cone of curves in this discussion is its relation to the pseudo-effective cone of divisors. Given a moving curve C and effective D, we can conclude that

$$C \cdot D \geq 0$$

because we can move C to an equivalent curve which intersects D in a finite number of reduced points. In fact, this property precisely characterizes the pseudo-effective cone.

Theorem 2.1.20 ((14))**.** $\overline{\mathrm{Eff}}(X) = \{D \in \mathrm{NS}(X) : C \cdot D \geq 0, \forall C \in \mathrm{Mov}(X)\}$

Again restricting our attention to divisors on a projective $\mathbb{Q}$-factorial scheme (we now add these as standing assumptions on X in this chapter unless otherwise noted), there is one more type of divisor, and associated convex cone, that we want to define. A Cartier divisor L is called *very ample* if its associated map is an embedding. Although very ample is a geometrically clear definition, it is ill-behaved in the Néron-Severi space. Desiring an analog which behaves better, we define ample divisors. An intuitive definition of an *ample* divisor is a divisor L such that $L^{\otimes m}$ is very ample for all $m >> 0$.

Definition 2.1.21. The *ample cone* is the span of all cycles corresponding to ample Cartier divisors on X, $\mathrm{Amp}(X)$.

The ample cone is an open convex cone. It has full dimension in the space if X is projective. We define the *nef cone* to be the closure of the ample cone, $\overline{\mathrm{Amp}(X)} = \mathrm{Nef}(X)$. An ample divisor must have an equivalent divisor passing through every point and not have any fixed components, so we have that the ample cone is a subset of the movable cone,

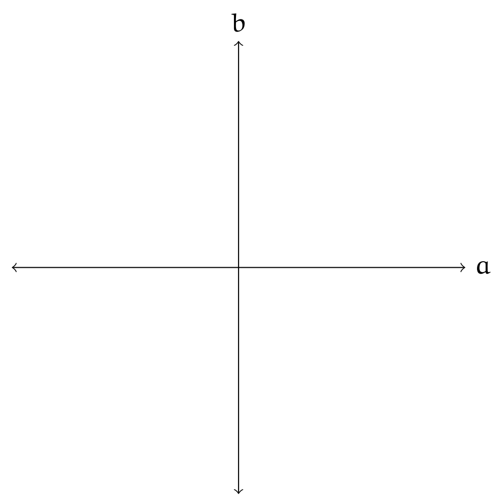
$$\mathrm{Amp}(X) \subset \mathrm{Mov}_{n-1}(X) \subset \mathrm{Eff}(X).$$

In this paper, we primarily focus on computing the effective cones of certain schemes.

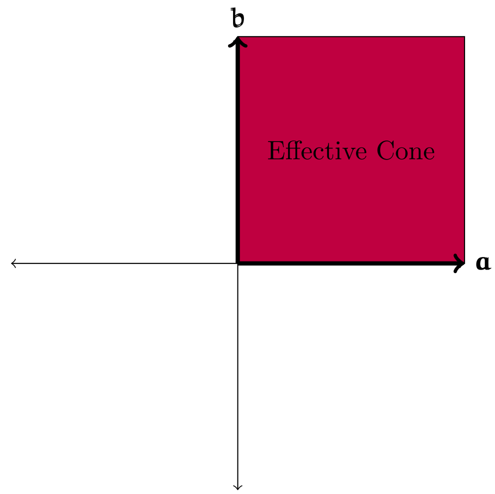
Example 2.1.3. Returning to $\mathbb{P}^1 \times \mathbb{P}^1$, we have

$$\mathrm{NS}(X) = \{[\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)] : a, b \in \mathbb{Z}\} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^{\oplus 2}.$$

We can draw this space as follows.



Then we know that any effective divisor, which in this case is also a curve, is cut out by a polynomial of bidegree (a, b) for some nonnegative $a, b \in \mathbb{Z}$ where at least one is positive. Thus the effective cone can be shaded in as follows.

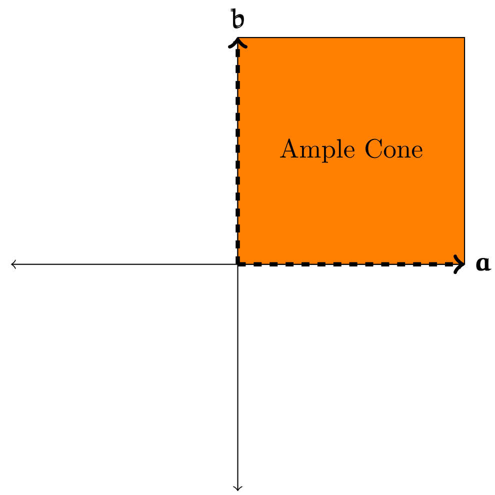


As this cone is already closed, we see that the pseudo-effective cone is equal to the effective cone. The big cone is the interior of these cones, which is the cone where both coordinates are positive.

A moving class has to be effective and the space is covered by divisors of type (a, b) for all nonnegative a and b with at least one positive so we have that the movable cone is equal to the effective cone.

Any line bundle $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ gives an embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ into $\mathbb{P}^{ab+a+b}$ if a and b are positive. Conversely, $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(a, b)$ does not give an embedding if either a or b is not positive. If either a or b

was zero and the other was positive, the image is $\mathbb{P}^1$; otherwise, a or b is negative and no power of the line bundle has global sections so there is no corresponding map. Thus, the ample cone is as follows.



From this picture, it is clear that the nef cone is equal to the effective cone.

2.2 Characteristic Classes

2.2.1 Chern Classes

One important way of describing an effective Cartier divisor is as the vanishing of a section of a line bundle. Formally, given $f \in H^0(L, U)$ for a line bundle L and open set $U \subset X$,

$$\mathrm{div}(f) = \{x \in X : f(x) = 0\}.$$

In fact, we can describe all Cartier divisors similarly, once we allow poles (this description extends to the codimension one graded pieces of the quotients). Again more precisely, given $f \in H^0(L, U)$ for a line bundle L,

$$\mathrm{div}(f) = \{x \in X : f(x) = 0\} - \{x \in X : f(x) = \infty\}$$

is a Cartier divisor on X. Conversely, given a Cartier divisor Y on X, there exists $f \in H^0(\mathcal{I}_Y^*, U)$ such that $Y = \operatorname{div}(f)$. By definition, picking different rational sections will give the same class in the Picard group (and hence in the Néron-Severi group) so this is well-defined. Characteristic classes generalize this construction in order to give cycles of all codimensions.

The first characteristic classes we will discuss are Chern classes. Let V be a rank m vector bundle on X. For the moment, assume V is globally generated.

Definition 2.2.1. The i*th Chern class* of V is the class in the Chow ring of the *degeneracy locus* of f_0, $\cdots$, and f_{m-i},

$$c_i(V) = [\{x \in X : f_0 \wedge \cdots \wedge f_{m-i} \text{ vanishes } x\}]$$

where the f_j are general elements of $H^0(V, X)$.

Note that picking different sections would again give something that is rationally equivalent by definition so this is well-defined. The wedge product of sections vanishing is equivalent to the sections being linearly dependent. By choosing general sections, we have guaranteed that the degeneracy locus is generically reduced and that every component of it has codimension exactly i [EH, Lemma 5.2]. Thus, the ith Chern class is a codimension i cycle. This codimension statement makes it clear that the Chern classes vanish once $i > n$.

Definition 2.2.2. The *Chern polynomial* of V is the class

$$C(V) = c_0(V) + c_1(V) + \cdots \in A(X).$$

We call this the Chern *polynomial* as it is a polynomial in the generators of the Chow ring. From this definition, it can also be shown that if we have a flat morphism of smooth varieties $\phi : Y \to X$ that

$$\phi^*(c(V)) = c(\phi^*(V))$$

and that if we have another vector bundle W on X, we have

$$c(V \bigoplus W) = c(V)c(W).$$

This direct sum formula forces the same formula to hold for short exact sequences in the sense that if

$$0 \to V \to U \to W \to 0$$

is exact we have

$$c(U) = c(W)c(V).$$

As with the intersection product, we take the properties of Chern classes we want on the "good" objects and show that it forces a unique set of Chern classes in general [EH, Thm. 5.3] so we can now remove the globally generated hypotheses on the vector bundle.

As a consequence of the good properties that we have enforced on the Chern classes, we have what is known as the *Splitting Principle.*

Theorem 2.2.3 (EH Thm. 5.11)**.** *Any identity among Chern classes of bundles that is true for bundles that are direct sums of line bundles is true in general.*

The proof follows by working on a filtration of your vector bundle with one dimensional quotients.

The splitting principle is a powerful tool for computing Chern classes. By the splitting principle, we can think of any rank m vector bundle as having a Chern polynomial of this form

$$c(V) = \prod_{i=1}^{m} c(L_i) = \prod_{i=1}^{m}(1+\alpha_i)$$

where $\alpha_i = c_1(L_i)$ and each L_i is a line bundle. An immediate consequence of this is that $c_i(V) = 0$ if $i > m$.

Using this formula, we can compute the Chern classes of any linear algebraic construction of bundles (e.g. direct sum, tensor, symmetric power, exterior power, etc.) in terms of the bundles' original Chern classes. For example, if V and W were direct sums of line bundles we would have immediately that

$$c(V \otimes W) = \prod_{i=1}^{m} \prod_{j=1}^{\mathrm{rk}(W)} (1 + \alpha_i + \beta_j) = 1 + (\mathrm{rk}(V)c_1(W) + \mathrm{rk}(W)c_1(V)) + \cdots .$$

By the splitting principle, this holds in general. Although we can calculate this formula, we see from this calculation that the Chern polynomial do not give a "nice" formula for the Chern polynomial of a tensor product (or most of the operations) in terms of the original Chern polynomials. In fact, it is an unfortunate byproduct of our definition that the Chern polynomial takes the "addition" operation of direct sum and turns it into multiplication in the Chow ring. In order to be precise, we have to define the Groethendieck ring of bundles.

2.2.2 The Chern Character

The *Grothendieck ring* of vector bundles on X, denoted $K(X)$, is the commutative ring whose underlying group is the free Abelian group of isomorphism classes of vector bundles on X modulo the relation that

$$[B] = [A] + [C]$$

if there exists a short exact sequence

$$0 \to A \to B \to C \to 0$$

with direct sum as the addition operation and tensor product as the multiplication operation. Then the Chern polynomial assigns an element of the Chow ring to each vector bundle, but this cannot be a group or ring homomorphism from $K(X)$ to $A(X)$ for several reasons, most obviously because $c(B) \neq c(A) + c(C)$ when those bundles sit in a short exact sequence. However, there is a way to almost make such a morphism possible.

Recall that we can treat V's Chern polynomial as being of the form $\prod_{i=1}^{m}(1 + \alpha_i)$. From basic algebra, we know that the standard way to pass between a product and a sum is to exponentiate or take a logarithm, so we formally do that.

Definition 2.2.4. The *Chern character* of V is

$$\mathrm{ch}(V) = \sum_{i=1}^{m} e^{\alpha_i}$$

where we can write out the exponential function using its power series expansion (which is, in fact, a finite sum in this case because the powers vanish above the dimension of X).

Definition 2.2.5. The *ith Chern character* is the sum of degree i part of this expression when it is expanded.

The Chern characters give the nice formula for direct sum that the Chern classes failed to give,

$$\mathrm{ch}(V \bigoplus W) = \sum_{i=1}^{m} e^{\alpha_i} + \sum_{j=1}^{\mathrm{rk}(W)} e^{\beta_j} = \mathrm{ch}(V) + \mathrm{ch}(W).$$

This formula comes from the formula $c(V \bigoplus W) = \prod_{i=1}^{m}(1+\alpha_i) \cdot \prod_{j=1}^{\mathrm{rk}(W)}(1+\beta_j)$. Similarly for the tensor product, we have

$$\mathrm{ch}(V \otimes W) = \sum_{i=1}^{m} \sum_{j=1}^{\mathrm{rk}(W)} e^{\alpha_i+\beta_j} = \sum_{i=1}^{m} \sum_{j=1}^{\mathrm{rk}(W)} e^{\alpha_i} e^{\beta_j} = \mathrm{ch}(V) \cdot \mathrm{ch}(W)$$

since $c(V \otimes W) = \prod_{i=1}^{m} \prod_{j=1}^{\mathrm{rk}(W)}(1+\alpha_i+\beta_j)$.

The Chern character now assigns to each isomorphism class of vector bundles a rational number times an element of the Chow ring (not just an element of the Chow ring due to the power series expansion) which descends to an assignment on $K(X)$ by the splitting principle and the fact that it turns direct sums into a sum. Thus, the Chern character is a ring map

$$\mathrm{ch} : K(X) \to A(X) \otimes \mathbb{Q}.$$

These definitions can be naturally extended to all coherent sheaves using locally free filtrations. In that more general context, we have the following theorem by Groethendieck.

Theorem 2.2.6. *If X is a smooth projective variety, then the map*

$$\mathrm{ch} : K(X) \otimes \mathbb{Q} \to A(X) \otimes \mathbb{Q}$$

is an isomorphism of rings.

2.2.3 Logarithmic Invariants

The Chern character is perfect when you wish to consider the direct sum of bundles. However, if you wish to consider the tensor product of two bundles, the Chern character does not give nice formulas for each graded piece, the exception being in degree zero. In degree zero, we have that $\mathrm{ch}_0(V) = \mathrm{rk}(V)$ which leads to

$$\mathrm{ch}_0(V \otimes W) = \mathrm{rk}(V \otimes W) = \mathrm{rk}(V) \cdot \mathrm{rk}(W).$$

However, in degree one, we have that $\mathrm{ch}_1(V) = c_1(V)$ which leads to

$$\mathrm{ch}_1(V \otimes W) = c_1(V \otimes W) = \mathrm{rk}(V) \cdot c_1(W) + \mathrm{rk}(W) \cdot c_1(V) = \mathrm{ch}_0(V) \cdot \mathrm{ch}_1(W) + \mathrm{ch}_0(W) \cdot \mathrm{ch}_1(V).$$

We would like another reformulation of the Chern invariants that works nicely with the tensor product instead of with the direct sum. These new invariants are the *logarithmic invariants* as defined by Drezet in [D].

Definition 2.2.7. The *logarithmic Chern character* of a vector bundle is the formal expression $\log(\mathrm{ch}(V))$.

Definition 2.2.8. Similarly, the *ith logarithmic invariant* of a vector bundle is the ith part of $\log(\mathrm{ch}(V))$, denoted $\Delta_i(V) \in A^i(V) \otimes \mathbb{Q}$.

In other words,

$$\log(\mathrm{ch}(V)) = \sum_{i=0}^{n} \Delta_i(V).$$

These invariants have the nice property with respect to tensor that we wanted.

Theorem 2.2.9. *(1) For a line bundle L, $\Delta_0 = 1$, $\Delta_1 = c_1(L)$, and $\Delta_i = 0$ for all $i > 1$.*

(2) $\Delta_i(V \otimes W) = \Delta_i(V) + \Delta_i(W)$ for $1 \leq i \leq n$.

(3) $\Delta_i(V \otimes L) = \Delta_i(V)$ for $2 \leq i \leq n$ for a line bundle L.

(4) $\Delta_i(V^*) = (-1)^i \Delta_i(V)$ if $1 \leq i \leq n$.

As we plan to work on surfaces, we name Δ_i for $i \leq 2$. Δ_0 is just the log of the 0-th Chern character (recall that the 0-th Chern character is the rank when working with a locally free sheaf) so we will not name it.

Definition 2.2.10. Δ_1 is the *slope* of a vector bundle, denoted $\mu(V)$.

Definition 2.2.11. Δ_2 is the *discriminant*, denoted $\Delta(V)$.

The formulas for these are

$$\Delta_0(F) = \log(\mathrm{ch}_0(F)),$$

$$\mu(F) = \frac{\mathrm{ch}_1(E)}{\mathrm{ch}_0(F)}, \text{ and}$$

$$\Delta(F) = \frac{1}{2}\mu(F)^2 - \frac{\mathrm{ch}_2(F)}{\mathrm{ch}_0(F)}.$$

Note that in the case of a locally free sheaf, $\mathrm{ch}_0(F)$ is just the rank of F and that these notions are easily extended to Chern characters in $K\left(\mathbb{P}^1 \times \mathbb{P}^1\right) \otimes \mathbb{R}$.

On a Picard rank one variety, the slope is a generalization of the degree of a line bundle to higher rank vector bundles. On higher Picard rank varieties, we can think of the slope as a generalization of a multi-degree that carries the information of the degree for every choice of embedding. Sometimes we would like an analog to the degree with respect to a specific embedding so we give the following definition.

Definition 2.2.12. The *slope with respect to an ample divisor* H is

$$\mu_H(E) = \mu(E) \cdot H.$$

We also call this the H-*Slope* of E. The two ample divisors that we will use in this paper are $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,1)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(1,2)$ for which the slopes will be denoted $\mu_{1,1}$ and $\mu_{1,2}$, respectively.

2.2.4 The Todd Class

So far we have given three formulations of the data of Chern classes. The Chern classes are natural from the perspective of vector bundles and multiply on short exact sequences. The Chern character has the nice property of additivity on short exact sequences and multiplicativity over tensor products. The logarithmic Chern character has the nice property of being additive on tensor products. Now, we will give one last reformulation that is natural from the perspective of tangent bundles. We want to "normalize" the highest degree part of the data with respect to the tangent bundle.

Before we formalize this notion, recall that both the Chern character and the logarithmic Chern character were polynomials in the Chern classes so we would like our formulation to again be a polynomial in the Chern classes. Call this polynomial $\mathrm{Td}(X) = \sum_{i=0}^{\dim(X)} \mathrm{Td}_i(X)$ where $\mathrm{Td}_i(X)$ is a polynomial in terms of the codimension i Chern class products: c_i, $c_{i-1} \cdot c_1$, etc. The property that we would like this formulation to have is that

$$\chi(\mathcal{O}_X) = \mathrm{Td}_{\dim(X)}(\mathcal{T}_X)$$

and that they are functorial. The polynomial with this property is the *Todd character*, and the pieces, Td_i, are the *Todd classes*. Again, requiring this property on good schemes is enough to force a unique set of Todd classes on all schemes. In fact, requiring this statement to be true just for products of projective spaces is sufficient to determine the polynomial we want. This definition of the Todd character is intuitive, but unhelpful. We can explicitly give an alternative definition, which can be derived from the first definition.

Definition 2.2.13. Given a Chern polynomial

$$c(V) = \prod_{i=0}^{\dim(X)} (1 + \alpha_i)$$

the *Todd character* is given by

$$\mathrm{Td}(V) = \prod_{i=0}^{\dim(X)} \frac{\alpha_i}{1 - e^{-\alpha_i}}$$

and the i-th Todd classes are the codimension i parts of the formula.

Note that all four of these formulations (the Chern polynomial, the Chern character, the logarithmic Chern character, and the Todd character) carry the exact same information, just in different ways that can be more convenient. We will switch back and forth interchangeably, but mostly use the logarithmic Chern character.

2.2.5 The Hirzebruch-Riemann-Roch Theorem

The Todd class is perhaps most famous not for its motivational definition but for its use in a generalization of the classical Riemann-Roch theorem. Recall that the Riemann-Roch theorem computes the Euler characteristic of a divisor on a curve. The Hirzebruch-Riemann-Roch theorem generalizes this theorem and computes the Euler characteristic of a sheaf on a variety. We will state the theorem without proof.

Theorem 2.2.14. *If X is a smooth projective variety of dimension n and $\mathcal{F}$ a coherent sheaf on X, then*

$$\chi(\mathcal{F}) = \left(\mathrm{ch}(\mathcal{F})\mathrm{Td}(\mathcal{T}_X)\right)_n$$

where the subscript n indicates the codimension n part of the equation.

This theorem was further generalized to the Groethendieck-Riemann-Roch theorem, but that level of generality will not be needed in this paper.

Example 2.2.1. Before we move beyond characteristic classes, we want to work out the various classes of the tangent bundle of $\mathbb{P}^1 \times \mathbb{P}^1$. For brevity, we denote $\mathcal{T}_{\mathbb{P}^1 \times \mathbb{P}^1}$ by V. First, recall that for $\mathbb{P}^1$, the Euler sequence is

$$0 \to \mathcal{O}_{\mathbb{P}^1} \to \mathcal{O}_{\mathbb{P}^1}(1)^2 \to \mathcal{T}_{\mathbb{P}^1} \to 0.$$

As all vector bundles split on $\mathbb{P}^1$, this immediately tells us that $\mathcal{T}_{\mathbb{P}^1} = \mathcal{O}_{\mathbb{P}^1}(2)$. Given a product of schemes $A \times B$, we know that $\mathcal{T}_{A \times B} = \pi_1^*(\mathcal{T}_A) \bigoplus \pi_2^*(\mathcal{T}_B)$ where π_1 and π_2 are the projection maps to A and B, respectively, (Exercise 2.8.3 (13)). Thus, we have

$$V = \pi_1^*(\mathcal{O}_{\mathbb{P}^1}(2)) \bigoplus \pi_2^*(\mathcal{O}_{\mathbb{P}^1}(2)) = \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,0) \bigoplus \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,2).$$

To find the Chern classes of V, first consider what the sections of V are. A section of V is a section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,0)$ plus a section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,2)$. A section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,0)$ is just a bidegree $(2,0)$ polynomial. Similarly, section of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,2)$ is just a bidegree $(0,2)$ polynomial. Let $\sigma_i = f_i + g_i$ be sections of V for $i = 0,1,2$ where f_i and g_i are sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2,0)$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0,2)$, respectively.

As always, we have that

$$c_0(V) = \left[\mathbb{P}^1 \times \mathbb{P}^1\right]$$

because three sections of a rank two bundle are linearly dependent everywhere. You can see this pointwise because sections are maps into a dimension two vector space so any three such maps are dependent.

Now, for the first Chern class, we are interested in the linear dependence of σ_0 and σ_1. Writing this out explicitly, we have

$$\sigma_0 \wedge \sigma_1 = (f_0 + g_0) \wedge (f_1 + g_1) = f_0 \wedge f_1 + f_0 \wedge g_1 + g_0 \wedge f_1 + g_0 \wedge g_1.$$

As both of the f_i are sections of a line bundle, $f_0 \wedge f_1$ vanishes everywhere. Similarly, $g_0 \wedge g_1$ vanishes everywhere. Thus,

$$\sigma_0 \wedge \sigma_1 = f_0 \wedge g_1 + g_0 \wedge f_1.$$

Then g_i and f_i are sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 0)$ and $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(0, 2)$, respectively. We then see that $g_i \wedge f_j$ as the vanishing locus of a polynomial of bidegree $(2, 2)$. Therefore, the first Chern is the locus where the sum of two bidegree $(2, 2)$ polynomials is equal to zero. Summing two bidegree $(2, 2)$ polynomials just gives a different polynomial with the same bidegree. Thus,

$$c_1(V) = [\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)] .$$

Finally for the second Chern class, we are interested in locus where a single section σ_0 is linearly dependent. For a single section, linear dependence is equivalent to vanishing. Then we are looking for the locus where σ_0 vanishes. In order for σ_0 to vanish, f_0 and g_0 must both vanish. Then f_0 vanishes on a curve of type $(2, 0)$ and g_0 vanishes on a curve of type $(0, 2)$ so they both vanish exactly on the intersection of the those two curves. By previous example, these two curves intersect in $(2 \cdot 2 + 0 \cdot 0)\,[\text{point}] \cdot 0 = 4\,[\text{point}]$. Thus, we have

$$c_2(V) = 4\,[\text{point}] .$$

We can summarize these Chern classes as

$$c(V) = \left[\mathbb{P}^1 \times \mathbb{P}^1\right] + [\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 2)] + 4\,[\text{point}] = (1, (2, 2), 4).$$

We will now omit the descriptions of the classes where they are clear.

Now, we would like to transform the Chern classes into the Chern character. Alternatively, we could calculate them directly using that V is a direct sum. Preparing to use the splitting principle, we think of the Chern polynomial of V as

$$c(V) = (1+\alpha)(1+\beta) = 1 + (\alpha+\beta) + (\alpha\cdot\beta).$$

Then the Chern character of V is

$$\mathrm{ch}(V) = e^{\alpha} + e^{\beta} = 1 + \alpha + \frac{1}{2}\alpha^2 + 1 + \beta + \frac{1}{2}\beta^2.$$

All powers of α or β above two are omitted as they vanish. Now, we manipulate the equation to match up similar parts with the Chern polynomial:

$$\mathrm{ch}(V) = e^{\alpha} + e^{\beta} = 2 + (\alpha+\beta) + \frac{1}{2}(\alpha^2+\beta^2).$$

First, notice that we have

$$\mathrm{ch}_0(V) = 2 \text{ and}$$

$$\mathrm{ch}_1(V) = c_1(V) = (2,2).$$

Calculating ch_2 requires manipulating symmetric functions. We see that

$$\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\cdot\beta.$$

Using the Chern polynomial, we see that

$$\mathrm{ch}_2(V) = \frac{1}{2}c_1(V)^2 - c_2(V) = \frac{1}{2}(2,2)^2 - 4 = \frac{1}{2}8 - 4 = 0.$$

Thus, the Chern character of V is

$$\mathrm{ch}(V) = (2,(2,2),0).$$

Next, we would like to transform the Chern character/polynomial into the logarithmic Chern character. By the given formulas, we know that

$$\Delta_0(V) = \log(\mathrm{rank}(V)) = \log(2),$$

$$\Delta_1(V) = \mu(V) = \frac{\mathrm{ch}_1(V)}{\mathrm{ch}_0(V)} = \frac{(2,2)}{2} = (1,1), \text{ and}$$

$$\Delta_2(V) = \Delta(V) = \frac{1}{2}\mu(V)^2 - \frac{\mathrm{ch}_2(V)}{\mathrm{ch}_0(V)} = \frac{1}{2}2 - \frac{0}{2} = 1.$$

Thus, the logarithmic Chern character is

$$\Delta_*(V) = (\log(2),(1,1),1).$$

Lastly, we would like to transform this data into the Todd character. Using the splitting principle, we have that

$$\begin{aligned}
\mathrm{Td}(V) &= \frac{\alpha}{1-e^{-\alpha}} \cdot \frac{\beta}{1-e^{-\beta}} \\
&= \frac{\alpha}{1-(1-\alpha+\frac{1}{2}\alpha^2-\frac{1}{6}\alpha^3+\cdots)} \cdot \frac{\beta}{1-(1-\beta+\frac{1}{2}\beta^2-\frac{1}{6}\beta^3+\cdots} \\
&= \frac{\alpha}{\alpha-\frac{1}{2}\alpha^2+\frac{1}{6}\alpha^3-\cdots} \cdot \frac{\beta}{\beta-\frac{1}{2}\beta^2+\frac{1}{6}\beta^3-\cdots} \\
&= \frac{1}{1-\frac{1}{2}\alpha+\frac{1}{6}\alpha^2-\cdots} \cdot \frac{1}{1-\frac{1}{2}\beta+\frac{1}{6}\beta^2-\cdots} \\
&= \left(1+\frac{1}{2}\alpha+\frac{1}{12}\alpha^2+\cdots\right)\left(1+\frac{1}{2}\beta+\frac{1}{12}\beta^2+\cdots\right) \\
&= 1+\frac{1}{2}(\alpha+\beta)+\frac{1}{12}\left(\alpha^2+\beta^2\right)+\frac{1}{4}\alpha\beta \\
&= 1+\frac{1}{2}(\alpha+\beta)+\frac{1}{12}\left((\alpha+\beta)^2+\alpha\beta\right),
\end{aligned}$$

where all the higher order terms vanish. From this expression, we see that

$$\mathrm{Td}_0(V) = 1,$$

$$\mathrm{Td}_1(V) = \frac{1}{2}c_1(V) = (1,1), \text{ and}$$

$$\mathrm{Td}_2(V) = \frac{1}{12}\left(c_1(V)^2+c_2(V)\right) = \frac{1}{12}(8+4) = 1.$$

Thus, we have

$$\mathrm{Td}(V) = (1,(1,1),1).$$

2.3 The Minimal Model Program

Chern classes and cycles give us the necessary background to study varieties using the Minimal Model Program. In particular, the divisorial cones that we defined are part of a larger framework called the *stable base locus decomposition*. See Lazarsfeld's book (1) for a more complete discussion of this framework.

2.3.1 The Stable Base Locus Decomposition

We need a few technical definitions to set up the decomposition.

Definition 2.3.1. The *base locus* of a line bundle L, $\mathrm{Bs}(\mathsf{L})$, is the set of points where all of its global sections vanish.

Alternatively, it is the common intersection of every element of $|\mathsf{L}|$.

Definition 2.3.2. The *stable base locus* of a line bundle L, $\mathbf{B}(\mathsf{L})$, is defined as the intersection of the base loci of all positive multiples of L:

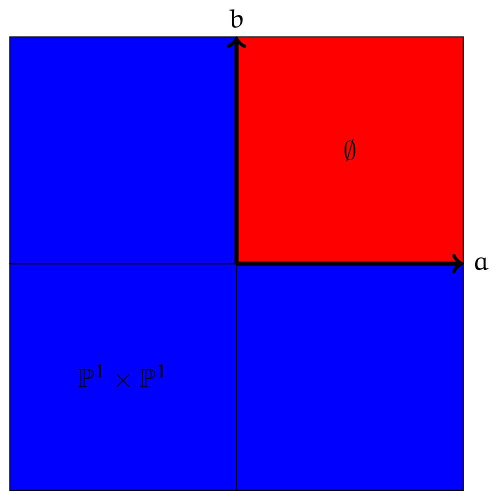
$$\mathbf{B}(\mathsf{L}) = \cap_{k \geq 1}(\mathsf{Bs})(k\mathsf{L}).$$

Definition 2.3.3. The *stable base locus decomposition*, or *Mori decomposition* of the Néron-Severi space breaks the space up into chambers, called *Mori chambers*, where the divisors in a fixed chamber all have the same stable base locus.

We should note that the stable base locus fails to be invariant under numerical equivalence on the boundaries between different Mori chambers. We can correct for this by using the augmented stable base locus, but that will be unnecessary in this paper.

We can rephrase the definitions of the previous cones in terms of this decomposition. The big cone is the union of all chambers whose stable base locus is not the whole space. The movable cone is the union of all chambers whose stable base locus is codimension at least 2. The ample cone is the chamber whose stable base locus is empty.

Example 2.3.1. By previous examples, the ample cone is the interior of the effective cone on $\mathbb{P}^1 \times \mathbb{P}^1$. This fact gives the two chambers of the stable base locus decomposition.



2.3.2 Mori Dream Spaces

For a large class of "nice" objects, the stable base locus decomposition controls all of the birational geometry of X. The results of this chapter do not require our smoothness assumption. These spaces are called Mori dream spaces, but we need a couple of other definitions before we can define them. The following subsection summarizes a discussion from the seminal paper by Hu and Keel (2). A line bundle, L, is *semiample* if there exists an integer m such that $|mL|$ is base point free. A *small* $\mathbb{Q}$-*factorial modification*, or *SQM*, of a projective variety X is a birational contraction $f : X \dashrightarrow X'$ that is an isomorphism in codimension one with X' projective and $\mathbb{Q}$-factorial.

With these definitions in hand, we can define a Mori dream space.

Definition 2.3.4. A normal projective variety X is a *Mori Dream Space*, or *MDS*, provided the following hold:

(1) X is $\mathbb{Q}$-factorial, and $\mathrm{Pic}(X) \otimes \mathbb{Q} = \mathrm{NS}(X)$.

(2) $\mathrm{Nef}(X)$ is the affine hull of finitely many semi-ample line bundles.

(3) There is a finite collection of SQMs $f_i : X \dashrightarrow X_i$ such that each X_i satisfies (2) and $\mathrm{Mov}(X)$ is the union of the $f_i^*(\mathrm{Nef}(X_i))$.

As a consequence of being a MDS, the Mori decomposition is particularly enlightening.

Theorem 2.3.5 ((2))**.** *Let X be a MDS. Then*

(1) All of the Mori chambers are polyhedral cones with finitely many faces.

(2) Each Mori chamber of the effective cone is the pull back of the nef cone from a birational contraction, $g_i : X \dashrightarrow Y_i$, and these are the only birational contractions of X with $\mathbb{Q}$-factorial image.

(3) Each Mori chamber of the movable cone is the pull back of the nef cone of an SQM of X, $f : X \dashrightarrow X_i$, and these are the only SQMs of X.

Each of the Y_i's from the theorem is a *model* of X.

Example 2.3.2. Our running example of $\mathbb{P}^1 \times \mathbb{P}^1$ is a MDS. The model corresponding to the ample cone chamber is $\mathbb{P}^1 \times \mathbb{P}^1$ itself (this will be true for all MDS's). The two edges of this chamber correspond to the the two projection maps $\pi_1, \pi_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$.

The spaces that will be the focus of our study are, in fact, Mori dream spaces so we have a reasonable hope of understanding their birational geometry. Let's now move to describing those spaces.

2.4 Hilbert Schemes of Points

The spaces that we want to study are spaces that parametrize sheaves with a fixed Chern character on $\mathbb{P}^1 \times \mathbb{P}^1$. These spaces are examples of *moduli spaces*.

2.4.1 Moduli Spaces

Informally, a *moduli space* is a scheme whose points are in bijection with equivalence classes of a certain type of object.

Example 2.4.1. Projective space is "the moduli space" of affine lines in $\mathbb{A}^{n+1}$.

Example 2.4.2. $\mathbb{P}^{\frac{(d+2)(d+1)}{2}-1}$ is "the moduli space" of curves of degree d in $\mathbb{P}^2$.

Let's start to make this notion rigorous.

Definition 2.4.1. A *family* of subschemes of X with the property P is a subscheme $Y \subset X \times_k B$ with the property that $Y_b \subset X$ has the property P for all closed points $b \in B$. We call B the *base* of the family.

Intuitively, we want "the moduli space" to be a family with some additional properties guaranteeing that it has a closed point for every equivalence class and that every two closed points represent different equivalence classes. Say that we have a moduli space M parameterizing subschemes of X with property P and Y is a family of subschemes with the same property with base B. Then we get a map $f : B \to M$, at least on the level of topological spaces, defined by taking the closed point $b \in B$ to the point of M corresponding to class Y_b. In fact, we would want any family for which B is a base to give such a map. This motivates our next definition.

Definition 2.4.2. A family Y of subschemes of X with property P with base M is called *universal* if any other family Y' with base B is the pull back $f^*(Y)$ of a morphism $f : B \to M$.

Note that a universal family is unique up to isomorphism if it exists. Ideally, our moduli space would be the base of the universal family of subschemes with property P. Using categorical language, we reformulate this notion to give the formal definition of a *fine moduli space*.

Definition 2.4.3. A *fine moduli space*, M, of subschemes of X with property P is the unique space such that $\mathrm{Hom}(-, M) = \mathfrak{F}$ where $\mathfrak{F} : \mathcal{S}ch \to \mathcal{S}et$ is the functor from the category of schemes to the category of sets sending a scheme B to the set of families of subschemes with property P over the base B. This is called *(co)representing the functor*.

Being a fine moduli space is a very difficult condition to satisfy. This is one of the motivations for using stacks in algebraic geometry, but discussing that would take us too far afield. We tweak the formal definition to weaken this condition to a more satisfiable condition.

Definition 2.4.4. A *coarse moduli space*, M, of subschemes of X with property P is the unique space such that there is a natural transformation $\Phi : \mathfrak{F} \rightsquigarrow \mathrm{Hom}(-, M)$ where $\mathfrak{F} : \mathcal{S}ch \to \mathcal{S}et$ is the functor from the category of schemes to the category of sets sending a scheme B to the set of families of subschemes with property P with base B with the properties that

1) There is a bijection between $\mathrm{Hom}(\mathsf{Spec}(\mathbb{C}), M)$ and $\mathfrak{F}(\mathsf{Spec}(\mathbb{C}))$ and

2) Given a natural transformation $\Psi : \mathfrak{F} \rightsquigarrow \mathrm{Hom}(-, M')$ there exists a map $f : M \to M'$ such that $\Psi = \Phi \circ \Xi_f$ where Ξ_f is the natural transformation $\Xi_f : \mathrm{Hom}(-, M) \rightsquigarrow \mathrm{Hom}(-, M')$ induced by f.

2.4.2 Hilbert Schemes

Some of the most classical examples of moduli spaces are the spaces that parameter 0-dimensional cycles of a space. We have no hope to parametrize all 0-dimensional cycles at once because that would be an infinite dimensional space (as there are at least $n * m$ dimensions of 0-dimensional subschemes with m points counting multiplicity). Using numerical equivalence, we can tell apart two cycles which have different equivalence classes, so a way to get a nice parameter space is to study all of the cycles with a fixed numerical cycle class. For 0-cycles, the number of points (counting multiplicities) determines this class. The space parameterizing the set of 0-cycles of X with m points is just the standard symmetric product $X^{(m)} \cong X^m/\mathfrak{S}_m$.

For curves, the symmetric product is a great moduli space. It is smooth and has dimension n. However, the symmetric product does not have many of the properties we want a moduli space to have once we move to higher dimensional varieties. For example, it is singular. In particular, its singularity is along the locus of cycles where we have a point with multiplicity higher than one. On surfaces, we can resolve the singularities by keeping track of the scheme structure underlying the cycle rather than just the multiplicities, which leads to our definition of the *Hilbert scheme of points*.

2.4.3 Definition and Properties

Definition 2.4.5. The *Hilbert scheme of* m*-points* on X is the parameter space of 0-dimensional subschemes of length m on X, denoted $\mathrm{Hilb}^m(X)$ or $X^{[m]}$.

On curves, the Hilbert scheme of m points is exactly the symmetric product. The Hilbert schemes of points on surfaces have very rich geometries and have been studied extensively. On higher dimensional varieties, the Hilbert schemes of points start to have very messy geometry. They can have arbitrarily bad singularities, arbitrarily many components, and non-reduced components. If we restrict ourselves to the surface case, this space now has many of the properties we would want in a parameter space. In particular, in this paper we will primarily concern ourselves with the case $X = \mathbb{P}^1 \times \mathbb{P}^1$. In that case, the Hilbert scheme is smooth, irreducible, and dimension $2m$.

The Hilbert scheme resolves the singularities of the symmetric product via the *Hilbert-Chow* morphism,

$$hc : X^{[m]} \to X^{(m)},$$

which forgets the scheme structure and turns the length of the scheme supported at a point into the multiplicity of that point. It is worth noting that this is a *crepant* resolution, which means that $hc^*(K_{X^{(m)}}) = K_{X^{[m]}}$, and that there is an open set of $X^{[m]}$ which is isomorphic via hc to the open set of $X^{(m)}$ consisting of m-tuples where all m points are distinct. Fogarty proved even stronger results than these about the Hilbert schemes of points on surfaces.

Theorem 2.4.6 ((15))**.** *If* $n = 2$ *and* $h^1(X, \mathcal{O}_X) = 0$*, then* $X^{[m]}$ *is a smooth irreducible projective variety of dimension* $2m$ *whose Picard rank is exactly one more than the Picard rank of* X*.*

This theorem tells us that all but one of the generators of the Picard group of $X^{[m]}$ are the pull backs of generators of the Picard group of X. In order to see this connection explicitly, start with a line bundle L on X that is a generator of the Picard group. Letting $\pi_i : X^n \to X$ be the ith projection of the Cartesian product, pull L back to a line bundle, $\pi_i^* L$, on X^n. Define the line bundle $L \boxtimes L \cdots \boxtimes L$ to be the the tensor product of all of these pullbacks, $\pi_1^*(L) \otimes \cdots \otimes \pi_m^*(L)$ (In general, the bundle $L \boxtimes L'$ on the product $X \times Y$ is the bundle $\pi_X^*(L) \otimes \pi_Y^*(L')$). This line bundle descends to a line bundle L_0 on the symmetric product as it is invariant under the action of the symmetric group. Then define a line bundle $L[m]$ on $X^{[m]}$ by pulling L_0 back along hc (i.e. $L[m] = hc^*(L_0)$). We will sometimes abuse notation and refer to $L[m]$ as L when the context makes our meaning clear. These line bundles correspond to divisors on X which have very geometric interpretations. If L is effective on X, the divisor corresponding to $L[m]$ consists of those collections of m points whose support intersect a fixed divisor of type L on X. Fogarty proved that once you have the generators of the Picard group corresponding in this fashion to those generating X's Picard group, the only remaining generator that we need to span the Néron-Severi space corresponds to the locus of non-reduced schemes, denoted as B.

2.5 Moduli Spaces of Sheaves

2.5.1 Alternate Descriptions of $X^{[m]}$

Although we described the objects that the Hilbert scheme of points parametrizes using the geometric description of the length of a 0-dimensional subscheme, we could have alternatively described them in two different ways that will generalize more easily.

The first description requires the notation of the (reduced) Hilbert polynomial.

Definition 2.5.1. The *Hilbert polynomial of a sheaf $\mathcal{F}$ with respect to an ample line bundle* H is

$$P_{\mathcal{F}}(k) = \chi\left(\mathcal{F}(k)\right) = \frac{\alpha_d}{d!}k^d + \cdots + \alpha_1 k + \alpha_0$$

where $\mathcal{F}(k) = \mathcal{F} \otimes H^{\otimes k}$, where the dimension of $\mathcal{F}$ is d, and where we think of this as a polynomial in the variable k.

Definition 2.5.2. The *reduced Hilbert polynomial* of a sheaf $\mathcal{F}$ with respect to an ample line bundle H is defined to be

$$p_{\mathcal{F}}(k) = \frac{P_{\mathcal{F}}(k)}{\alpha_d}$$

where d is the dimension of $\mathcal{F}$.

The Hilbert polynomial, unlike the individual cohomology groups it sums over, is a numerical object that is entirely determined by the Chern character of the sheaf and the Chern character of the ample line bundle.

Using the Hilbert polynomial, the Hilbert scheme is the space that parametrizes sheaves $\mathcal{F}$ on X with Hilbert polynomial equal to the constant equation m. This definition gives the Hilbert schemes their name, but it is not the description that we will focus on in this paper.

Another alternative description of $X^{[m]}$ is in terms of the Chern character and stability of these sheaves. Black boxing the words stability and stable for the moment, the Hilbert scheme of points on X is the moduli space of stable sheaves $\mathcal{F}$ on X with logarithmic Chern character equal to $(1, \mathbf{0}, m)$. We have written the middle coordinate as a vector because there is one coordinate for each generator of the Néron-Severi space. By defining stability, we hope to generalize the construction of the Hilbert schemes of m points to the construction of moduli spaces of sheaves with any fixed Chern character.

2.5.2 Classical Stability Conditions

Let us define stability so that we can define our spaces. There are multiple meanings of stable. We will start with the most classical version, *slope stability.*

Definition 2.5.3. A sheaf $\mathcal{F}$ is *slope (semi-)stable* with respect to an ample line bundle H if for all proper subsheaves $\mathcal{F}' \subset \mathcal{F}$,

$$\mu_{\mathsf{H}}(\mathcal{F}')\,(\leq) < \mu_{\mathsf{H}}(\mathcal{F}).$$

A stronger notation of stability is the notion of *Gieseker stability*, which is also known as γ stability.

Definition 2.5.4. A sheaf $\mathcal{F}$ is *Gieseker (semi-)stable* with respect to an ample divisor H if for all proper subsheaves $\mathcal{F}' \subset \mathcal{F}$, $p_{\mathcal{F}'}\,(\leq) < p_{\mathcal{F}}$ where the polynomials are compared for all sufficiently large input values. We will also call this γ (semi-)stability.

The ordering on the polynomials could also have been phrased as the lexiographic ordering on their coefficients starting from the highest degree term's coefficient and working down. The condition on the Hilbert polynomial is equivalent on surfaces to requiring $\mu_{\mathsf{H}}(\mathcal{F}') \leq \mu_{\mathsf{H}}(\mathcal{F})$ and, in the case of equality, $\Delta(\mathcal{F}')\,(\geq) > \Delta(\mathcal{F})$.

These two notations of stability and semi-stability are related by a string of implications that seems slightly odd at first, but becomes clear using this last equivalence:

$$\text{slope stable} \to \text{Gieseker stable} \to \text{Gieseker semi-stable} \to \text{slope semi-stable}.$$

As we will be focused on $\mathbb{P}^1 \times \mathbb{P}^1$, one additional notion of stability will be relevant.

Definition 2.5.5. A sheaf $\mathcal{F}$ is $\overline{\gamma}$ *(semi-)stable* if it is Gieseker semi-stable with respect to $\mathcal{O}(1,1)$ and for all proper subsheaves $\mathcal{F}' \subset \mathcal{F}$, if $\mu_{1,1}(\mathcal{F}') = \mu_{1,1}(\mathcal{F})$ and $\Delta(\mathcal{F}') = \Delta(\mathcal{F})$, then $\mu_{1,2}(\mathcal{F}')\,(\leq) < \mu_{1,2}(\mathcal{F})$.

Again we have implications:

$$\text{slope stable} \rightarrow \text{Gieseker stable} \rightarrow \overline{\gamma} \text{ stable}$$

$$\rightarrow \overline{\gamma} \text{ semi-stable} \rightarrow \text{Gieseker semi-stable} \rightarrow \text{slope semi-stable}.$$

Because Gieseker stability generalizes to all varieties, it might seem odd to add a third, very variety specific, condition. However, adding this third condition allows $\overline{\gamma}$ stability to order all Chern characters in a way that was not possible for slope or Gieseker stability. This order is possible with $\overline{\gamma}$ stability because it has three conditions that can distinguish the three variables for a Chern character on $\mathbb{P}^1 \times \mathbb{P}^1$: the two coordinates of c_1 and the coordinate of c_2.

2.5.3 $M(\xi)$

Keeping these notions of stability in mind, we return to the problem of generalizing the Hilbert scheme of points to moduli spaces of sheaves with other fixed Chern characters.

Theorem 2.5.6 ((16))**.** *For all choices of ample divisor H and Chern character ξ on X, there exists a (possibly empty) moduli space $M_{\mu_H}(\xi)$ ($M(\xi)$) parametrizing slope(Gieseker) semi-stable sheaves on X with Chern character ξ.*

On $\mathbb{P}^1 \times \mathbb{P}^1$, our third type of stability also yields moduli spaces.

Theorem 2.5.7 ((7))**.** *For all choices of Chern character ξ on $\mathbb{P}^1 \times \mathbb{P}^1$, there exists a (possibly empty) moduli space $M_{\overline{\gamma}}(\xi)$ parametrizing $\overline{\gamma}$ semi-stable sheaves on X with Chern character ξ.*

These moduli spaces of stable sheaves are the primary spaces of interest for this paper. In particular, we will work on computing their effective cones, which are unknown in general. However, these spaces have been extensively studied, and much is known about them.

Proposition 2.5.8. *If $M(\xi)$ is zero dimensional, it is a single reduced point (10).*

If $M(\xi)$ is positive dimensional, it is an irreducible (8), normal (8), projective variety (9) of dimension $r^2(2\Delta - 1) + 1$ (9). In this case, they are smooth along their stable locus (9).

We will assume that the strictly semistable locus is codimension at least two so we can work with the stable locus $M^s(\xi)$ when convenient. In Chapter 3, we will explain why this assumption is justified. For now, merely note that this holds for the Hilbert schemes of points as they have empty strictly semistable locus.

One simple property of these spaces that we cannot prove with the machinery we have developed so far is when they are nonempty. We defer answering this question until later in this chapter where we will explain Rudakov's answer to it. For now, we will study the Picard groups of these spaces.

2.6 The Picard Group of $M(\xi)$

We have a good description of the Picard group of $M(\xi)$ if we assume that the Picard rank is three so we add this as a standing assumption. Again, we will give a justification for this assumption in Chapter 3. For now, just note that it holds for Hilbert schemes of points by Fogarty's theorem. Linear and numerical equivalence coincide on $M(\xi)$, so we have

$$\mathrm{NS}(M(\xi)) = \mathrm{Pic}(M(\xi)) \otimes \mathbb{R}.$$

As mentioned above, we will work with $M^s(\xi)$ when it is convenient. $M^s(\xi)$ is a coarse moduli space for the stable sheaves. In contrast, $M(\xi)$ is not a coarse moduli space, unless ξ is a primitive character, as it identifies S-equivalence classes of (strictly semistable) sheaves.

In order to understand the classes of the divisors that will span the effective cone, we have to understand the isomorphism

$$\xi^{\perp} \cong \mathrm{NS}(M(\xi)).$$

We construct this isomorphism by uniquely defining a line bundle on families of sheaves for each element of $\xi^{\perp}$ as done in (17) and (6) for the case of $\mathbb{P}^2$.

Let $\mathcal{U}/S$ be a flat family of semistable sheaves with Chern character ξ where S is a smooth variety. Define the two projections $p : S \times \mathbb{P}^1 \times \mathbb{P}^1 \to S$ and $q : S \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. Then consider the composition of maps

$$\lambda_{\mathcal{U}} : K(\mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{q^*} K(S \times \mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{\cdot[\mathcal{U}]} K(S \times \mathbb{P}^1 \times \mathbb{P}^1) \xrightarrow{-p!} K(S) \xrightarrow{\det} \mathrm{Pic}(S)$$

where $p! = \sum(-1)^i R^i P_*$. Colloquially, we are taking a sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$, pulling it back along the projection to $S \times \mathbb{P}^1 \times \mathbb{P}^1$, restricting it to our family, pushing it forward along the projection onto S, and then taking its determinant to get a line bundle on S.

Tensoring our family $\mathcal{U}$ by p^*L, where L is a line bundle on S, does not change the given moduli map $f : S \to M(\xi)$, but does change the λ map as follows:

$$\lambda_{\mathcal{U} \otimes p^* L}(\zeta) = \lambda_{\mathcal{U}}(\zeta) \otimes L^{\otimes -(\xi, \zeta)}.$$

Now given a class $\zeta \in \xi^{\perp}$, we want a λ_M map which commutes with the moduli map in the sense that we should have

$$\lambda_{\mathcal{U}}(\zeta) = f^* \lambda_M(\zeta)$$

for all $\mathcal{U}$. This equality determines a unique line bundle $\lambda_M(\zeta)$ on $M(\xi)$ and gives a linear map $\lambda_M : \xi^{\perp} \to \mathrm{NS}(M(\xi))$ which is an isomorphism under our assumptions.

Caveat: We have normalized λ_M using $-p!$ as in (6) rather than $p!$ as in (18) and (17) so that the positive rank Chern characters form the "primary half space" that we define later this chapter.

2.7 A Basis for the Picard Group

Using this isomorphism, we want to construct a basis for the Picard group. Following Huybrechts and Lehn (18) and modifying it for $\mathbb{P}^1 \times \mathbb{P}^1$, we define three Chern characters ζ_0, ζ_a, and ζ_b. Bundles with these Chern characters will be a basis for the Picard space. These Chern characters depend on the character ξ of the moduli space. Let a be the Chern character of a line of the first ruling and b be the Chern character of a line of the second ruling on $\mathbb{P}^1 \times \mathbb{P}^1$. We define our Chern characters by the equations

$$\zeta_0 = r(\xi)\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1} - \chi(\xi^*, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1})\mathcal{O}_p,$$

$$\zeta_a = r(\xi)a_1 - \chi(\xi^*, a)\mathcal{O}_p, \text{ and}$$

$$\zeta_b = r(\xi)b_1 - \chi(\xi^*, b)\mathcal{O}_p$$

where $\mathcal{O}_p$ is the structure sheaf of a point on $\mathbb{P}^1 \times \mathbb{P}^1$. (Note the difference from the analogous definition in (18) by a sign due to our convention for λ_M.)

Given that $(\xi, \mathcal{O}_p) = r(\xi)$, it should be clear that they are all in $\xi^\perp$. They can also be shown to be in $\xi^\perp$ by using Riemann-Roch given that the Chern characters $(\mathrm{ch}_0, \mathrm{ch}_1, \mathrm{ch}_2)$ of $\zeta_0, \zeta_a, \zeta_b$ are

$$\zeta_0 = \left(r(\xi), (0,0), -\chi(\xi)\right),$$

$$\zeta_a = \left(0, \left(r(\xi), 0\right), -r(\xi) - c_1(\xi)\cdot(0,1)\right), \text{ and}$$

$$\zeta_b = \left(0, \left(0, r(\xi)\right), -r(\xi) - c_1(\xi)\cdot(1,0)\right).$$

Using the map from the previous section, define $\mathcal{L}_0$, $\mathcal{L}_a$, $\mathcal{L}_b$ by the formulas $\mathcal{L}_0 = \lambda_M(\zeta_0)$, $\mathcal{L}_a = \lambda_M(\zeta_a)$, and $\mathcal{L}_b = \lambda_M(\zeta_b)$. They are a basis for the Picard space. We know that for $n >> 0$, $\mathcal{L}_0 \otimes \mathcal{L}_a^n$ and $\mathcal{L}_0 \otimes \mathcal{L}_b^n$ are ample.

Now, ζ_a and ζ_b span the plane of rank zero sheaves in $\xi^\perp$. Define the *primary half space* of $\xi^\perp$ to be the open half space of positive rank Chern characters and the *secondary half space* to be the closed half space of negative rank and rank zero Chern characters in $\xi^\perp$. Similarly, define the *primary and secondary halves* of NS($M(\xi)$) as the images of these under the isomorphism λ_M. Every extremal ray of the effective and nef cones sits in one of our halves. Call an extremal ray of the effective or nef cone *primary* or *secondary* according to which half it lies in.

We know that the ample cone of $M(\xi)$ lies in the primary half space because $\mathcal{L}_0$ lies in that half space.

2.8 Brill-Noether Divisors

Now that we have constructed a basis for the Picard space, we discuss the divisors that we will construct to span the effective cone. These divisors will be examples of *Brill-Noether divisors*. A Brill-Noether locus in general is the place where the rank of some cohomology group jumps.

Definition 2.8.1. A *Brill-Noether cycle* on $M(\xi)$ is a cycle of the form

$$Z_{V,i,j} = \{U \in M(\xi) : h^i(U \otimes V) > j \text{ for a fixed sheaf } V, i \in \mathbb{Z}, \text{ and } j \in \mathbb{Z}\}.$$

In particular, in this paper we will be concerned with Brill-Noether divisors of the form

$$D_V = \{U \in M(\xi) : h^1(U \otimes V) > 0 \text{ for a fixed sheaf } V \text{ with } \chi(U \otimes V) = 0\}.$$

In order for D_V to be a divisor, we must know that $h^1(U \otimes V) = 0$ for the general $U \in M(\xi)$. If this equality does not hold for general U, then $D_V = M(\xi)$. As we are computing the effective cone, we hope that these loci are effective divisors and that we can compute their class. One way to guarantee that h^1 is zero is make sure all of the h^i are zero.

Definition 2.8.2. Bundles V such that $h^i(U \otimes V) = 0$ for all i are *cohomologically orthogonal* to U.

Proposition 2.8.3. *Suppose* $V \in M(\zeta)$ *is a stable vector bundle and is cohomologically orthogonal to the general* $U \in M(\xi)$*. Put the natural determinantal scheme structure on the locus*

$$D_V = \overline{\{U \in M^s(\xi) : h^1(U \otimes V) \neq 0\}}$$

.

(1) D_V *is an effective divisor.*

(2) If $\mu_{1,1}(U \otimes V) > -4$*, then* $\mathcal{O}_{M(\xi)}(D_V) \cong \lambda_M(\zeta)$*.*

(3) If $\mu_{1,1}(U \otimes V) < 0$*, then* $\mathcal{O}_{M(\xi)}(D_V) \cong \lambda_M(\zeta)^* \cong \lambda_M(-\zeta)$*.*

Proof. After replacing slope with $(1,1)$-slope and -3 with -4 (the first is the slope of $K_{\mathbb{P}^2}$ while the second is the $(1,1)$-slope of $K_{\mathbb{P}^1 \times \mathbb{P}^1}$), the proof is identical to that of Prop 5.4 in (6). □

Given a Brill-Noether divisor, a natural question to ask is whether it lies in the primary or the secondary half of NS($M(\xi)$). The answer is immediate from the computation of the class of the Brill-Noether divisors.

Corollary 2.8.4. *Keep the notation and hypotheses from Prop. 2.8.3.*

(1) If $\mu_{1,1}(U \otimes V) > -4$*, then [*$D_V$*] lies in the primary half of* NS($M(\xi)$).

(2) If $\mu_{1,1}(U \otimes V) < 0$*, then [*$D_V$*] lies in the secondary half of* NS($M(\xi)$).

Note that since $\mu_{1,1}(U \otimes V)$ cannot be between -4 and 0 there is no contradiction in these results.

2.9 The Interpolation Problem

As we previously mentioned, in order to construct a Brill-Noether divisor, it is necessary to find a sheaf V for which $h^1(U \otimes V) = 0$ for the general $U \in M(\xi)$ and the easiest way to do so is to find a sheaf that is cohomologically orthogonal to U. Cohomological orthogonality implies that $\chi(U \otimes V) = 0$. The vanishing of that Euler characteristic is a strictly weaker condition. For example, it might be the case that $h^0 = h^1 = 1$ and $h^2 = 0$. We would like an added condition which would make them equivalent. A bundle V is *non-special with respect to* U if $\chi(U \otimes V)$ determines the cohomology groups (i.e. they are lowest ranks allowed by the Euler characteristic). We can rephrase cohomological orthogonality as V having $\chi(U \otimes V) = 0$ and being non-special with respect to U. In general, there will be many such sheaves, but those that interest us will be those of "minimum slope." Finding a sheaf like this is a form of the interpolation problem.

The Interpolation Problem. *Given invariants* ξ *of a vector bundle and a polarization* H *of* $\mathbb{P}^1 \times \mathbb{P}^1$*, find a stable vector bundle* V *with minimum* μ_H *that is cohomologically orthogonal to the general element of* $M(\xi)$*.*

Note, if we restrict our interest to line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$ and $M(\xi) = \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[m]}$, this is the classical interpolation for points lying on $\mathbb{P}^1 \times \mathbb{P}^1$: find the "lowest" bidegree (a, b) such that m points lie on a curve of type (a, b).

Example 2.9.1. Let's work out an example of a divisor on the Hilbert scheme of two points on $\mathbb{P}^1 \times \mathbb{P}^1$. We want to consider the locus

$$Z = \{z \in M(\xi) : z \text{ lies on a } (1, 0) \text{ curve.}\}$$

Intuitively, we can see that Z is a divisor by counting dimensions. Z is three dimensional because you can pick any point on $\mathbb{P}^1 \times \mathbb{P}^1$ for your first point, which adds two dimensions, but then you have to pick your second point (or scheme structure) on the unique $(1,0)$ curve that the first point was on, which adds one more dimension. Its effectiveness is clear by construction. We want to see that this divisor from the perspective of interpolation. By Hirzebruch-Riemann-Roch, we see that

$$\chi(\mathcal{I}_z \otimes \mathcal{O}(1,0)) = \chi(I_Z(1,0)) = 1((1+1)(1+0)-2) = 0.$$

We can use a little geometry to show that $\mathcal{O}(1,0)$ is also non-special. First, $h^0(I_z \otimes \mathcal{O}(1,0)) = 0$ because two general points do not lie on a curve of type $(1,0)$. Next, we have the short exact sequence

$$0 \to \mathcal{I}_z \to \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1} \to \mathcal{O}_z \to 0.$$

Tensoring by $\mathcal{O}(1,0)$, we see that $h^2(\mathcal{I}_z \otimes \mathcal{O}(1,0)) = 0$. Thus, $Z = D_{\mathcal{O}(1,0)}$.

$\mathcal{O}(1,0)$ has minimal $\mathcal{O}(2,1)$ slope of all bundles with $\chi(V \otimes \mathcal{I}_z) = 0$. This minimal slope will follow from later theorems, but, for now, note that it has the minimal $\mathcal{O}(2,1)$ slope among all line bundles corresponding to effective divisors that have this property. To see this, first compute

$$\chi(\mathcal{I}_z \otimes \mathcal{O}(a,b)) = \chi(\mathcal{I}_Z(a,b)) = 1((1+a)(1+b)-2) = 0.$$

Solving for a gives

$$a = \frac{2}{b+1} - 1,$$

which implies that b is -3, -2, 0, or 1. These, in turn, imply that $\mathcal{O}(a,b)$ would have to be $\mathcal{O}(-2,-3)$, $\mathcal{O}(-3,-2)$, $\mathcal{O}(1,0)$, or $\mathcal{O}(0,1)$. The first two are not effective, and the fourth has higher $\mathcal{O}(2,1)$ slope.

By construction of the Brill-Noether divisors, any solution of the interpolation problem will give an effective Brill-Noether divisor. Our goal is to give a method to construct Brill-Noether divisors which are sufficient to span the effective cone in many examples. We show that they span in examples by providing an alternate construction of the divisors. This description provides dual moving curves which prove that they are extremal divisors in the effective cone.

The alternate construction starts by using the cohomological vanishing guaranteed by these Brill-Noether divisors to resolve the general objects of $M(\xi)$. In general, we then use these resolutions to construct maps from $M(\xi)$ to Picard rank one varieties that have positive dimensional fibers. The Brill-Noether divisors will match up with the pull backs of an ample divisor on these Picard rank one varieties. We will prove the extremality of each divisor using the dual moving curves that cover the fibers of the morphism.

2.10 Kronecker Moduli Spaces

The simpler varieties that we will map to are moduli spaces of *Kronecker modules*. Kronecker modules are one of the simplest examples of *quiver representations*. To introduce these, let's back up and begin with quivers themselves. Good references for an introduction to quivers are the papers by Derksen and Weyman (19), Reineke (20), and Brion (21).

2.10.1 Quivers

A *quiver* is a directed graph. The data of a quiver can be given as the set of points, Q_0, and the set of arrows between them, Q_1. An arrow points from its *tail vertex* to its *head vertex*. There are set maps $h, t : Q_1 \to Q_0$ which map an arrow to its head and tail vertex, respectively.

Example 2.10.1. An example of a quiver, Q, is the set of points

$$Q_0 = \{p_0, p_1, p_2\}$$

and the set of arrows

$$Q_1 = \{A_0, A_1, A_2\}.$$

We can draw this as follows.

$$\begin{tikzcd}
p_0 \arrow[rr, "A_0"] & & p_1 \arrow[dl, "A_1"] \\
& p_2 \arrow[ul, "A_2"] &
\end{tikzcd}$$

Then the tail morphism is

$$t(A_i) = p_i$$

and the head morphism is

$$h(A_i) = p_{i+1(\mathrm{mod})3}.$$

A *representation,* R*, of a quiver,* Q, is an assignment of a vector space to each vertex and a linear map between the vector spaces for each arrow. Equivalently, R is given by the data of maps $rk : Q_0 \to \mathbb{Z}$ and $map : Q_1 \to \bigoplus_{A \in Q_1} \mathrm{Hom}(\mathbb{C}^{rk(t(A))}, \mathbb{C}^{rk(h(A))})$ which give the rank of the (complex) vector space at each vertex and assign to each arrow $(A : i \to j) \in Q_1$ a linear map in the vector space $\mathrm{Hom}(\mathbb{C}^{rk(i)}, \mathbb{C}^{rk(j)})$. The *dimension vector,* $\underline{\dim}\ R$*,* of a representation R is the list of each vertex with its rank which is equivalent to the graph of rk.

Example 2.10.2. A representation R of our previous example Q is the assignment of $\mathbb{C}^{i+1}$ at each vertex and the assignment of a map M_i at each arrow A_i where

$$M_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \text{ and } M_2 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix},$$

Its dimension vector would then be $(1, 2, 3)$.

We will be interested in moduli spaces of quiver representations. Like moduli spaces of sheaves, we have to restrict to *stable* representations. Defining this notion of stability will not be necessary for our work so we defer it to the references. What is important for our work is that the existence of semistable Kronecker modules can be detected by the components of a version of the relative Euler characteristic.

The *relative Euler characteristic* of two representations of the same quiver with dimension vectors e and f is given by the formula

$$\chi(e, f) = \sum_{p \in Q_0} rk_e(p) rk_f(p) - \sum_{A \in Q_1} rk_e(t(A)) rk_f(h(A)).$$

Example 2.10.3. Returning to our previous example, we have

$$\chi(r, r) = 1 * 1 + 2 * 2 + 3 * 3 - 1 * 2 - 2 * 3 - 3 * 1 = 3.$$

The components of the relative Euler characteristic detect the semistability of a dimension vector through the following proposition.

Proposition 2.10.1 ((22)). *The general quiver representation S of a quiver Q with dimension vector f is semistable iff there exists a quiver representation R of Q with nonzero dimension vector e such that*

$$\mathrm{Hom}(R, S) = \mathrm{Ext}^1(R, S) = 0.$$

2.10.2 Kronecker Modules

The quivers that we will be interested in have two points and N arrows in one direction between the points. If there are multiple, say N, arrows with the same head and tail, we will only draw one arrow in our quivers but label that arrow with N.

$$p_0 \xrightarrow{\quad N \quad} p_1$$

Definition 2.10.2. This type of quiver, with two vertices and arrows in only direction between them, is a *Kronecker quiver*.

Definition 2.10.3. A *Kronecker* V*-module* is a representation of this quiver and is equivalent to a linear map $A \otimes V \to B$ where V is a vector space of dimension N and A and B are arbitrary vector spaces.

Definition 2.10.4. The moduli space of semistable Kronecker V-modules with dimension vector $r = (a, b)$ is $\mathrm{Kr}_N(a, b)$.

The *expected dimension* of this space is

$$(\mathrm{edim})\ \mathrm{Kr}_N(a, b) = 1 - \chi(r, r) = Nab - a^2 - b^2 + 1.$$

For Kronecker moduli spaces, we also know that they are nonempty and of the expected dimension if their expected dimension is nonnegative (20). Another fact about Kronecker moduli spaces is that they are Picard rank one (23). We use this fact later when we create effective divisors on our moduli spaces $M(\xi)$ by constructing fibrations from them to these Kronecker moduli spaces and pulling back a generator of the ample cone of $Kr_N(a, b)$. In order to do this, we need to get Kronecker modules from maps between *exceptional bundles* which we will introduce in the next section.

2.11 Exceptional Bundles

Exceptional bundles also give us the perfect place to start our study of derived categories. For a more complete discussion of exceptional bundles in this setting, see the paper by Gorodentsev (10). A culmination of our study of $M(\xi)$ so far was generating divisors by solving an interpolation problem. The interpolation problem was defined in terms of the Euler characteristic of a sheaf. In this section, we start with a relative version of Euler characteristic. We will eventually see that sheaves E with relative Euler characteristic $\chi(E, E) = 1$ will control the geometry of our moduli spaces.

2.11.1 The Relative Euler Characteristic

The *relative Euler characteristic* of two sheaves on a variety X of dimension n is

$$\chi(E, F) = \sum_{i=0}^{n} (-1)^i \mathrm{ext}^i(E, F)$$

where $\mathrm{ext}^i(E, F) = \dim(\mathrm{Ext}^i(E, F))$. For locally free sheaves, we can equivalently define it by the formula

$$\chi(E, F) = \chi(E^* \otimes F) = \sum_{i=0}^{n} (-1)^i h^i(E^* \otimes F).$$

Restricting to the case where $n = 2$, we again write out Hirzebruch-Riemann-Roch in order to explicitly compute this as

$$\chi(E, F) = r(E)r(F)\left(P\left(\mu(E) - \mu(F)\right) - \Delta(E) - \Delta(F)\right)$$

where $P(x, y) = (x + 1)(y + 1)$ is the Euler characteristic of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(x, y)$.

Example 2.11.1. Let's compute the relative Euler characteristic of two general line bundles on $\mathbb{P}^1 \times \mathbb{P}^1$.

$$\chi(\mathcal{O}(a, b), \mathcal{O}(c, d)) = 1 * 1((c - a + 1)(d - b + 1) - 0 - 0) = (c - a + 1)(d - b + 1)$$

If $c \geq a$ and $d \geq b$, we see that this number matches our naive expectation that

$$\chi(\mathcal{O}(a, b), \mathcal{O}(c, d)) = \hom(\mathcal{O}(a, b), \mathcal{O}(c, d)) = \hom(\mathcal{O}, \mathcal{O}(c - a, d - b)) = \chi(\mathcal{O}(c - a, d - b)).$$

Using the alternate definition of relative Euler characteristic, we define a bilinear pairing on $K\left(\mathbb{P}^1 \times \mathbb{P}^1\right)$ by

$$(E, F) = \chi(E^*, F) = \chi(F^*, E) = \chi(E \otimes F).$$

Then we define the *orthogonal complement* of E, denoted $\mathrm{ch}(E)^{\perp}$, to be all bundles F such that $(E, F) = 0$. Note that the pairing is symmetric so the orthogonal complement does not depend on whether E is the first or second element in the pairing.

The Ext groups we used to define the relative Euler characteristic also allow us to describe a set of vector bundles that control the geometry of $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.11.1. A sheaf E is *exceptional* if $\mathrm{Hom}(E, E) = \mathbb{C}$ and $\mathrm{Ext}^i(E, E) = 0$ for all $i > 0$.

We say a Chern character (slope) is *exceptional* if there is an exceptional sheaf with that character (slope). The prototypical exceptional sheaves are the line bundles, $\mathcal{O}(a, b)$. Many of the properties of line bundles pass to exceptional sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. We list some of these properties without repeating the proofs. Their proofs are in the previously mentioned paper by Gorodentsev (10).

Proposition 2.11.2. *On $\mathbb{P}^1 \times \mathbb{P}^1$, we have the following results.*

- *Every exceptional sheaf is a vector bundle.*

- *There exists a unique exceptional bundle for each exceptional slope.*
- *The two coordinates of the first Chern class are coprime with the rank of* E.
- *The denominator of* $\mu_{1,1}(E)$ *is the rank of* E.
- *Exceptional bundles are stable with respect to* $\mathcal{O}(1,1)$.

Given these properties, it makes sense to define the *rank* of an exceptional slope ν to be the smallest $r \in \mathbb{Z}$ such that $r\mu_{1,1}(\nu) \in \mathbb{Z}$. Also, set the notation $E_{\frac{a}{r},\frac{b}{r}}$ for the unique exceptional bundle with slope $\left(\frac{a}{r}, \frac{b}{r}\right)$. Given a Chern character ξ with slope $\alpha = \left(\frac{a}{r}, \frac{b}{r}\right)$, we will interchangeably write $E_{-\alpha} = E^*_\alpha = E_{\frac{-a}{r},\frac{-b}{r}}$. Similarly, we will abuse notation and write the slope and the whole Chern character interchangeably.

We can characterize exceptional bundles among all stable bundles by the Euler characteristic $\chi(E, E)$. For stable E, Serre duality implies $\mathrm{Ext}^2(E, E) = \mathrm{Hom}(E, E(K)) = 0$. Similarly, the stability of E implies that $\mathrm{Hom}(E, E) = \mathbb{C}$. Then for stable E we have

$$\chi(E, E) = 1 - \mathrm{ext}^1(E, E) \leq 1$$

with equality precisely when E is exceptional. Conversely, we use Hirzebruch-Riemann-Roch to explicitly compute

$$\chi(E, E) = r(E)^2 (1 - 2\Delta).$$

Putting these together, we see that for a stable bundle

$$\Delta(E) = \frac{1}{2}\left(1 - \frac{\chi(E, E)}{r(E)^2}\right).$$

For exceptional bundles, this reduces to

$$\Delta(E) = \frac{1}{2}\left(1 - \frac{1}{r(E)^2}\right).$$

Since $\chi(F, F) \leq 0$ for all other stable bundles, we see that exceptional bundles are the only stable bundles with $\Delta < \frac{1}{2}$. As there is a unique exceptional bundle for an exceptional slope and there can be no other stable bundles with that discriminant, the moduli space of stable bundles with an exceptional Chern character is a single reduced point (10). Giving an explicit description of the exceptional bundles analogous to the description of the exceptional bundles on $\mathbb{P}^2$ given in (6) and (17) is an open question.

2.12 Exceptional Collections

Exceptional bundles naturally sit inside of collections.

Definition 2.12.1. A collection of exceptional bundles $(E_0, \cdots, E_n)$ is an *exceptional collection* if for all $i < j$, $\text{Ext}^k(E_j, E_i) = 0$ for all k and there is at most one k such that $\text{Ext}^k(E_i, E_j) \neq 0$.

An exceptional collection is *strong* if $k = 0$ for all pairs (i, j). We say the *length* an exceptional collection $(E_0, \cdots, E_n)$ is $n + 1$. An *exceptional pair* is an exceptional collection of length two. A *coil* is a maximal exceptional collection, which in the case of $\mathbb{P}^1 \times \mathbb{P}^1$ is length four. Our stereotypical (strong) coil is $(\mathcal{O}, \mathcal{O}(1, 0), \mathcal{O}(0, 1), \mathcal{O}(1, 1))$.

Every exceptional bundle sits inside of an exceptional collection, and every exceptional collection can be completed to a coil (7). Given an exceptional collection of length three (E_0, E_1, E_2) there are exactly four ways to complete it to a coil:

$$(A, E_0, E_1, E_2), (E_0, B, E_1, E_2), (E_0, E_1, C, E_2), \text{ and } (E_0, E_1, E_2, D).$$

In other words, once you pick where you would like the fourth bundle to be, there is a unique way to complete the exceptional collection to a coil. This uniqueness follows as an easy consequence of the requirement that the fourth bundles forms an exceptional pair in the correct way with the other three bundles. First, each bundle we require it to be in an exceptional pair with imposes an independent condition on its rank and first Chern classes so they are determined. Then the rank and first Chern class determine its discriminant, and the bundle is uniquely determined by its Chern classes.

Before we can state how to extend an exceptional collection of length two to a coil, we first must explain a process that turns an exceptional collection into different exceptional collections called *mutation* or *reconstruction*. We first define mutation for exceptional pairs and then bootstrap this definition into a definition for all exceptional collections. The definitions of mutation that we use are equivalent on $\mathbb{P}^1 \times \mathbb{P}^1$ to the general definitions (10).

Definition 2.12.2. The *left mutation of an exceptional pair* $(\mathsf{E}_0, \mathsf{E}_1)$ is the exceptional pair $\mathsf{L}(\mathsf{E}_0, \mathsf{E}_1) = (\mathsf{L}_{\mathsf{E}_0}\mathsf{E}_1, \mathsf{E}_0)$ where $\mathsf{L}_{\mathsf{E}_0}\mathsf{E}_1$ is determined by one of the following short exact sequences:

$$\text{(regular) } 0 \to \mathsf{L}_{\mathsf{E}_0}\mathsf{E}_1 \to \mathsf{E}_0 \otimes \mathrm{Hom}(\mathsf{E}_0, \mathsf{E}_1) \to \mathsf{E}_1 \to 0,$$

$$\text{(rebound) } 0 \to \mathsf{E}_0 \otimes \mathrm{Hom}(\mathsf{E}_0, \mathsf{E}_1) \to \mathsf{E}_1 \to \mathsf{L}_{\mathsf{E}_0}\mathsf{E}_1 \to 0, \text{ or}$$

$$\text{(extension) } 0 \to \mathsf{E}_1 \to \mathsf{L}_{\mathsf{E}_0}\mathsf{E}_1 \to \mathsf{E}_0 \otimes \mathrm{Ext}^1(\mathsf{E}_0, \mathsf{E}_1) \to 0.$$

One of these sequences exists by Gorodentsev (10). By rank considerations only one of the previous short exact sequences is possible so the left mutation is unique. Rebound and extension mutations are called *non-regular*. *Right mutation of an exceptional pair*, denoted $\mathsf{R}(\mathsf{E}_0, \mathsf{E}_1) = (\mathsf{E}_1, \mathsf{R}_{\mathsf{E}_1}\mathsf{E}_0)$, is defined similarly by tensoring the Hom or Ext with E_1 rather than with E_0. Note that left and right mutation are inverse operations in the sense that

$$L(R(E_0, E_1)) = R(L(E_0, E_1)) = (E_0, E_1).$$

Example 2.12.1. Start with the exceptional pair $\{\mathcal{O}, \mathcal{O}(1,1)\}$. Then by Hirzebruch-Riemann-Roch, we have

$$\chi(\mathcal{O}, \mathcal{O}(1,1)) = 1 * 1((1+1-0)(1+1-0) - 0 - 0) = 4.$$

By rank considerations, the left mutation of this pair is regular and given by

$$0 \to E \to \mathcal{O}^4 \to \mathcal{O}(1,1) \to 0$$

where $E = L_{\mathcal{O}}\mathcal{O}(1,0)$.

Let's compute E's logarithmic Chern character. Counting ranks, it is clear that $\mathrm{rk}(E) = 3$. By our earlier comment, this gives E's discriminant

$$\Delta(E) = \frac{1}{2}\left(1 - \frac{1}{3^2}\right) = \frac{4}{9}.$$

All that remains is to find E's slope. Since we know E's rank, this reduces to computing its first Chern class. Recall that the first Chern class is equal to the first Chern character class. Then the Chern character classes add on short exact sequences so we have that

$$c_1(E) = \mathrm{ch}_1(E) = \mathrm{ch}_1(\mathcal{O}^4) - \mathrm{ch}_1(\mathcal{O}(1,1)) = (0,0) - (1,1) = (-1,-1).$$

Thus,

$$\mu(E) = \frac{(-1,-1)}{3}$$

so the logarithmic Chern character of E is

$$\left(\log(3), \left(\frac{-1}{3}, \frac{-1}{3}\right), \frac{4}{9}\right).$$

Similarly, the right mutation of them is an exceptional bundle with logarithmic Chern character

$$\left(\log(3), \left(\frac{4}{3}, \frac{4}{3}\right), \frac{4}{9}\right).$$

We can also mutate any part of an exceptional collection. In particular, replacing any adjacent exceptional pair in an exceptional collection with any of its left or right mutations gives another exceptional collection. For example, given an exceptional collection (E_0, E_1, E_2, E_3),

$$(L(E_0, E_1), E_2, E_3), (R(E_0, E_1), E_2, E_3), (E_0, L(E_1, E_2), E_3)$$

$$(E_0, R(E_1, E_2), E_3), (E_0, E_1, L(E_2, E_3)), \text{ and } (E_0, E_1, R(E_2, E_3))$$

are all exceptional collections. Rudakov proved that all possible exceptional collections can be gotten from our stereotypical collection via these pairwise reconstructions (24).

Mutating an exceptional collection is then just mutating all of the bundles in a systematic way. We define the *left(right) mutation of an exceptional collection* $(E_0, E_1, E_2, , \cdots, E_n)$ as

$$(L_{E_0} \cdots L_{E_{n-1}} E_n, \cdots, L_{E_0} L_{E_1} E_2, L_{E_0} E_1, E_0)$$

$$((E_n, R_{E_n}E_{n-1}, R_{E_n}R_{E_{n-1}}E_{n-2}, \cdots, R_{E_n}\cdots R_{E_1}E_0)).$$

For a coil (E_0, E_1, F_0, F_1) on $\mathbb{P}^1 \times \mathbb{P}^1$, its left mutation is $(F_1(K), F_2(K), E_{-1}, E_0)$ and its right mutation $(F_1, F_2, E_{-1}(-K), E_0(-K))$.

Example 2.12.2. The left and right mutations of our standard coil $\{\mathcal{O}, \mathcal{O}(1,0), \mathcal{O}(0,1), \mathcal{O}(1,1)\}$ are respectively

$$\{\mathcal{O}(-1,-1), \mathcal{O}(0,-1), \mathcal{O}(-1,0), \mathcal{O}\} \text{ and}$$

$$\{\mathcal{O}(1,1), \mathcal{O}(2,1), \mathcal{O}(1,2), \mathcal{O}(2,2)\}.$$

Now, let's return to the problem of completing an exceptional pair to a coil. Say that we could extend (E_0, E_1) to the coil (E_0, E_1, F_0, F_1), this extension is not unique, even up to placement because $(E_0, E_1, L(F_0, F_1))$ and $(E_0, E_1, R(F_0, F_1))$ are also coils. To make a unique notation of extension, we need the notion of a *system*.

Definition 2.12.3. Using mutation, each exceptional pair (E_0, E_1) generates a *system of exceptional bundles* $\{E_i\}_{i\in\mathbb{Z}}$ where we inductively define $E_{i+1} = R_{E_i}E_{i-1}$ and $E_{i-1} = L_{E_i}E_{i+1}$.

Given an exceptional pair (E_0, E_1), we define the *right completion system* of it to be the unique system $\{F_i\}$ such that (E_0, E_1, F_i, F_{i+1}) is a coil. *Left* and *center* completion systems are defined analogously. A *left (right,center) completion pair* is any pair (F_i, F_{i+1}) coming from the left (right,center) completion system. By Prop. 4.5 & 4.8 of Rudakov's paper (24), the completion system is either a system of line bundles or has a *minimally ranked completion pair* where the minimally ranked completion pair is the pair in the system with the lowest sum of the two ranks of the bundles in the system.

Completing an exceptional collection of length one to a coil can be reduced to the two steps of completing it to an exceptional pair (which is highly not unique) and then completing the pair to a coil. In this paper, we will start with pairs and provide a unique way to extend them to a coil.

Given a complex $W : A^a \to B^b$ of powers of an exceptional pair (A, B), we extend the idea of mutation to the complex. Define LW to be the complex

$$LW : (L_A B)^a \to A^b$$

and similarly define RW to be the complex

$$RW : B^a \to (R_B A)^b.$$

Example 2.12.3. Let's start with the complex

$$W : \mathcal{O} \to \mathcal{O}(1,1)^4.$$

Then the left mutation of $\{\mathcal{O}, \mathcal{O}(1,1)\}$ is $\{E_{\frac{-1}{3},\frac{-1}{3}}, \mathcal{O}\}$. Thus, the left mutated complex is

$$LW : E_{\frac{-1}{3},\frac{-1}{3}} \to \mathcal{O}^4.$$

We also define the mutations relative to an exceptional bundle C where $\{C, A, B\}$ ($\{A, B, C\}$) is an exceptional collection as follows: If $\{C, A, B\}$ is an exceptional collection, define $L_C W$ to be the complex

$$L_C W : (L_C A)^a \to (L_C B)^b,$$

and similarly, if $\{A, B, C\}$ is an exceptional collection, define $R_C W$ to be the complex

$$R_C W : (R_C A)^a \to (R_C B)^b.$$

Example 2.12.4. Let's again start with the complex

$$W : \mathcal{O} \to \mathcal{O}(1,1)^4.$$

Then the left mutation of $\{\mathcal{O}, \mathcal{O}(1,1)\}$ by $\mathcal{O}(-1,0)$ is $\{\mathcal{O}(-2,0), E_{\frac{-9}{5},\frac{-1}{5}}\}$. Thus, the left mutated complex is

$$L_{\mathcal{O}(-1,0)}W : \mathcal{O}(-2,0) \to E^4_{\frac{-9}{5},\frac{-1}{5}}.$$

2.12.1 Kronecker Modules from Complexes

Given a complex $W : A^a \to B^b$ where $\{A, B\}$ is an exceptional pair, we get a Kronecker Hom(A, B)-module R with dimension vector $r = (a, b)$. The properties of exceptional bundles tell us that homomorphisms of the Kronecker module are exactly the homomorphisms of W and that $\chi(r, r') = \chi(W, W')$ where R' is the Kronecker module corresponding to a complex $W' : A^{a'} \to B^{b'}$. We will get these complexes between exceptional bundles from resolutions of the general objects of our moduli spaces $M(\xi)$.

2.13 Spectral Sequences

We will find these resolutions using a generalized Beilinson spectral sequence. Before we do so, let's back up and review spectral sequences. A good reference for spectral sequences is the introductory book by McCleary (25). We will only give a simplified definition of a spectral sequence in this section followed by generalities about their use. The best way to understand this definition and its usefulness is to work out the examples given in Chapter 8. As a technical note, we will only describe spectral sequences of *cohomological type* as all spectral sequences we use are of that type.

Definition. A *spectral sequence* in a category $\mathcal{C}$ consists of a triple array of objects of $\mathcal{C}$, $E_r^{p,q}$, and a set of *differentials*, $\{d_r : E_r^{*,*} \to E_r^{*,*}\}$, where $d_r = \{d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q+1-r}\}$, subject to the condition that $E_{r+1}^{p,q} \cong H^{p,q}(E_r^{*,*}, d_r)$ for $r \geq 0$. $E_r^{*,*}$ is called the *r-th page* of the spectral sequence.

Generally, the objects on an early page of a spectral sequence depend upon some input object. For example, one might have that the objects on the 0-th page are the Hodge cohomology groups: $E_0^{p,q} = H^{p,q}(X)$. When the spectral sequence depends upon an input object, the spectral sequence sometimes provides a generalization of the notion of a resolution of that object. While we say that a resolution resolves an object, we will say that a spectral sequence *converges to* an object.

Definition. A spectral sequence *converges to* H *along the diagonal* if H has a filtration F such that

$$E_\infty^{p,p} = \lim_{r\to\infty} E_r^{p,p} = F^s/F^{s-1}.$$

The first question one might ask after seeing this definition is how can you define a limit of objects in a general category. This seems problematic, but in all spectral sequences in this paper (and more generally in most useful spectral sequences), for a choice of p and q there exists M such that $E_N^{p,q} = E_{N+1}^{p,q}$ for all $N > M$. This degeneration makes the "limit" well-defined.

We can now state a theorem of Gorodentsev which establishes the spectral sequence that we need on $\mathbb{P}^1 \times \mathbb{P}^1$.

Theorem 2.13.1 ((10))**.** *Let* U *be a coherent sheaf on* $\mathbb{P}^1 \times \mathbb{P}^1$*, and let* (E_0, E_1, F_0, F_1) *be a coil. Write* $\mathcal{A} = (A_0, A_1, A_2, A_3) = (E_0, E_1, F_0, F_1)$ *and* $\mathcal{B} = (B_{-3}, B_{-2}, B_{-1}, B_0) = (F_1(K), F_2(K), E_{-1}, E_0)$ *There is a spectral sequence with* $E_1^{p,q}$*-page*

$$E_1^{p,q} = B_p \otimes \operatorname{Ext}^{q-\Delta_p}(A_{-p}, U)$$

that converges to U *in degree 0 and to 0 in all other degrees where* Δ_p *is the number of non-regular mutations in the string* $L_0 ... L_{p-1} A_p$ *which mutates* A_p *into* B_{-p}.

It should be clear that $\Delta_0 = 0$. Considering the spectral sequence converging to different bundles of the coil allows us to deduce that $\Delta_3 = 1$ and that the other two are either 0 or 1 [Rmk. 1.5.2, (26)]. Also, notice that $\mathcal{B}$ is the left mutation of $\mathcal{A}$.

2.14 Derived Categories

One way to interpret the Beilinson spectral sequence is that it gives a "resolution" of any sheaf on $\mathbb{P}^1 \times \mathbb{P}^1$ in terms of the elements of the coil. Another way to interpret it is to say that any coil on $\mathbb{P}^1 \times \mathbb{P}^1$ generates the "derived category of bounded coherent sheaves" on $\mathbb{P}^1 \times \mathbb{P}^1$. The derived category is a more modern way of studying varieties and our third major theme.

To construct this category, start with the category of coherent sheaves on X, $\mathsf{Coh}(X)$. As we are studying moduli spaces of sheaves, this is a natural category to consider; the coherent sheaf associated to the point of one of our moduli spaces is an object of $\mathsf{Coh}(X)$. Formally, we have

$$\mathsf{Obj}(\mathsf{Coh}(X)) = \{\text{Coherent sheaves on } X\}$$

and

$$\mathrm{Hom}_{\mathsf{Coh}(X)}(F, G) = \mathrm{Hom}_{\mathcal{O}_X}(F, G) \text{ for all } F, G \in \mathsf{Obj}(\mathsf{Coh}(X)).$$

Given the sheer amount of data that this category contains, it is perhaps unsurprising that it determines the variety.

Theorem 2.14.1 ((27)). *Let* X *and* Y *be varieties. Then* $X \cong Y$ *iff* $\mathsf{Coh}(X)$ *is equivalent to* $\mathsf{Coh}(Y)$.

We should also mention that $\mathsf{Coh}(X)$ is an Abelian category with enough injective objects to resolve every object. Because we use complexes to resolve sheaves, it is beneficial to "enlarge" $\mathsf{Coh}(X)$ to a category $\mathsf{Kom}^b(\mathsf{Coh}(X))$ whose objects are bounded complexes of coherent sheaves,

$$\mathsf{Obj}(\mathsf{Kom}^b(\mathsf{Coh}(X))) =$$

$$\{F^{\cdot} = \{0 \to F^i \to F^{i+1} \to \cdot \to F^{j-1} \to F^j \to 0\} : F^k \in \mathsf{Obj}(\mathsf{Coh}(X)), i \leq k \leq j, i \in \mathbb{Z}, j \in \mathbb{Z}\},$$

and whose morphisms are morphism of complexes as you would expect,

$$\mathrm{Hom}_{\mathsf{Kom}^b(\mathsf{Coh}(X))}(F^{\cdot}, G^{\cdot}) = \{\phi^{\cdot} = \{\phi^i\} : \phi^i \in \mathrm{Hom}_{\mathsf{Coh}(X)}(F^i, G^i)\}.$$

Kom^b clearly also carries enough information to reconstruct the variety from it so it still carries one of the main benefits of Coh, but both Coh and Kom^b carry so much information that they are unwieldy to work with. One way in which they carry too much information is that they let us tell apart some objects which are quasi-isomorphic but not isomorphic. In many ways, we do not care to tell such things apart. We modify Kom^b to rectify this problem by first associating maps up to chain homotopy to construct the *homotopy category*, $K(\mathsf{Coh}(X))$, and then by inverting all the quasi-isomorphisms in that category to construct the *derived category* (of bounded complexes of coherent sheaves), $D^b(\mathsf{Coh}(X))$.

Definition 2.14.2. The *derived category of bouunded complexes of coherent sheaves* on a variety X is the category, $D(X) = D^b(X)$, which we define as the category of bounded complexes of coherent sheaves on X where maps are associated up to chain homotopy and all quasi-isomorphisms are formally inverted.

The homotopy category has the advantage of being more understandable than $D^b(\mathsf{Coh}(X))$ because the only quasi-isomorphisms that are inverted are those which have an inverse up to chain homotopy. However, the derived category has the advantage of being a *triangulated category*; none of the other categories we have constructed have this property in general. Defining a triangulated category would take us too far afield as we will not make use of this fact. It is worth mentioning that the derived category is in some sense a localization of Kom^b.

The derived category though does lose information relative to $\mathsf{Coh}(X)$. We are no longer able to reconstruct a variety given its derived category in general, but there is a partial reconstruction theory due to Bondal and Orlov.

Theorem 2.14.3 ((28))**.** *Let X and Y be varieties with ample canonical or anti-canonical bundles. Then $X \cong Y$ iff $D^b(\mathsf{Coh}(X))$ is equivalent to $D^b(\mathsf{Coh}(Y))$.*

There are many known examples of non-isomorphic (even non-birational) varieties with equivalent derived categories when the Kodaira dimension is intermediate so this theorem is the strongest possible theorem.

As $\mathbb{P}^1 \times \mathbb{P}^1$ has ample anti-canonical bundle, $\mathcal{O}(2,2)$, we see that the derived category on it carries a lot of information about its geometry. Because exceptional bundles generate the derived category, they carry this information too. One specific part of the geometry that they explicitly control is the existence of stable bundles.

Let E be exceptional with Chern character e. Then, E imposes a numerical condition on stable bundles with "nearby" Chern characters. To see this condition, start with an exceptional bundle E and another stable bundle F, we know that there are no maps from E to F if the $(1,1)$-slope of E is bigger than F's by F's stability so

$$\hom(E, F) = 0.$$

Similarly, there are no maps from F to E twisted by the canonical if the $(1,1)$-slope of F is greater than the $(1,1)$-slope of $E(K)$ so

$$\hom(F, E(K)) = 0.$$

By Serre duality,

$$\hom(F, E(K)) = \operatorname{ext}^2(E, F) = 0.$$

Thus, if we have both of these conditions, we know that

$$\chi(E, F) = \operatorname{ext}^1(E, F) \leq 0.$$

This is the numeric condition that E imposes on nearby stable bundles. Given a fixed exceptional E, we can encode this data by saying that the Chern character of F must lie on or above a certain surface, δ_E, in the (μ, Δ) space. We define $\delta_E(\mu)$

$$\delta_E(\mu) = \left\{ \begin{array}{ll} \chi(E, \mu) = 0 & \text{if } \mu_{1,1}(E) - 4 < \mu_{1,1}(\mu) \text{ and } \overline{\gamma}(\mu) < \overline{\gamma}(E) \\ \chi(\mu, E) = 0 & \text{if } \mu_{1,1}(\mu) < \mu_{1,1}(E) + 4 \text{ and } \overline{\gamma}(E) < \overline{\gamma}(\mu) \\ 0 & \text{otherwise} \end{array} \right\}.$$

Then F's Chern character lying on or above δ_E means that

$$\Delta(F) \geq \delta_E(\mu(F)).$$

Using these conditions, each exceptional bundle gives an inequality that a Chern character must satisfy in order to be stable. We combine all of these conditions into one by looking at the maximum over all of the inequalities. Formally, let $\mathcal{E}$ be the set of exceptional bundles and define the δ surface by

Definition 2.14.4.

$$\delta(\mu) = \sup_{\{E \in \mathcal{E}\}} \delta_E(\mu).$$

Then saying that a stable Chern character, ζ, must satisfy all of the inequalities from exceptional bundles is equivalent to

$$\Delta(F) \geq \delta(\mu(F)).$$

Alternatively, we say that a stable Chern character must lie on or above the δ-surface.

Rudakov proved that lying above the δ surface was not only necessary but also sufficient for a Chern character to be stable.

Theorem 2.14.5. *[Main theorem, (7)] Let $\xi = (r, \mu, \Delta)$ be a Chern character of postive integer rank. There exists a positive dimensional moduli space of $\overline{\gamma}$-semistable sheaves $M_{\overline{\gamma}}(\xi)$ with Chern character ξ if and only if $c_1(\mu) \cdot (1,1) = r\mu_{1,1} \in \mathbb{Z}$, $\chi = r\,(P(\mu) - \Delta) \in \mathbb{Z}$, and $\Delta \geq \delta(\mu)$(7). The same conditions are necessary and sufficient for γ-semistability as long as $\Delta > \frac{1}{2}$ and $\mu \notin \mathcal{E}\mathbb{Z}$(7).*

2.15 Fourier-Mukai Transforms

Before we tie derived categories back into moduli spaces, we take a slight detour to define *Fourier-Mukai transforms.* A Fourier-Mukai transform will be a functor between the derived categories of two varieties X and Y constructed in a specific way.

Definition 2.15.1. Given a sheaf M on the product $X \times Y$, the *Fourier-Mukai transform with kernel* M is the map

$$\Phi_M : D^b(\mathrm{Coh}(X)) \to D^b(\mathrm{Coh}(Y))$$

given by

$$\Phi_M(\mathcal{F}) = \pi_{Y*}\left(\pi_X^*(\mathcal{F}) \otimes M\right).$$

Colloquially, we are pulling $\mathcal{F}$ back to the product, tensoring it with our kernel, and then pushing forward onto Y. This is a very similar operation to the map we constructed earlier from $K(\mathbb{P}^1 \times \mathbb{P}^1)$ to Pic(S) where S was the base of a family of sheaves on X.

2.16 Bridgeland Stability

The machinery of the derived category gives us Rudakov's important theorem as well as the Beilinson spectral sequence. We now want to tie this machinery together with the goals of the minimal model program for our moduli spaces. Bridgeland provided a way to do this by generalizing the notion of stability to the setting of derived categories (29). Bridgeland's two original papers on the subject are good references for this subject ((29), (30)). A paper of Bertram and Coskun gives a gentle introduction to Bridgeland stability in the context of $\mathbb{P}^1 \times \mathbb{P}^1$ (31).

Conjecturally, *Bridgeland stability* will provide the models of our moduli spaces corresponding to the chambers of the effective cones of our moduli spaces. Roughly speaking, the models of our moduli spaces will be the moduli spaces parameterizing the stable objects for different notions of stability. In order to give the formal definitions of these new stability conditions, we have to define the *Groethendieck group*, $\mathsf{K}(\mathsf{D})$, of a category, D, which is the category modded out by short exact sequences. In the case of the derived category, the Groethendieck group is just the group of Chern characters. We will abuse notation and write $\mathsf{K}(\mathsf{X})$ for the Groethendieck group of the derived category of X.

Definition 2.16.1 ((29))**.** A stability condition (Z, P) on a triangulated category D consists of a group homomorphism $Z : K(D) \to \mathbb{C}$, called the central charge, and full additive subcategories, $P(\phi) \subset D$ for each $\phi \in \mathbb{R}$, satisfying the following axioms:

(a) if $E \in P(\phi)$, then $Z(E) = m(E)\exp(i\pi\phi)$ for some $m(E) \in \mathbb{R}_{>0}$,

(b) for all $\phi \in \mathbb{R}$, $P(\phi + 1) = P(\phi)[1]$,

(c) if $\phi_1 > \phi_2$ and $A_j \in P(\phi_j)$, then $\mathrm{Hom}_D(A_1, A_2) = 0$,

(d) if every nonzero object $E \in D$ has a finite filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$ whose factors $F_j = E_j/E_{j-1}$ are semistable objects of D with $\phi(F_1) > \phi(F_2) > \cdots > \phi(F_n)$.

We can apply this definition to the derived categories of varieties. On every curve, there is a Bridgeland stability condition on corresponding to Gieseker stability with respect to H where

$$Z_H(E) = -\text{rank}(E)\mu(E) \cdot H + i\text{rank}(E).$$

Similarly on every surface, there is a Bridgeland stability condition given by

$$Z_{H,D}(E) = -\text{ch}_2 + D \cdot \text{ch}_1 - \frac{D^2 - H^2}{2}\text{ch}_0 + i(H \cdot \text{ch}_1 - D \cdot H \cdot \text{ch}_0)$$

where D is a divisor on X and H is again an ample divisor. In both cases, we have to "tilt" the derived category to a slightly different category in order for these conditions to be stability conditions. We will not take the detour necessary to define tilting. Showing the existence of stability conditions on other varieties is an open area of research though some progress has been made on threefolds. For examples of this work see papers by Macrì (32), Schmidt (33), Bayer, Macrì and Toda (34), and Bayer, Macrì, and Stellari (35).

Bridgeland found these conditions particularly well behaved (after he imposed a technical condition called the support property on them) because he was able to show that when they exist, there is a manifold, $\text{Stab}(D)$, parameterizing them.

Theorem 2.16.2 ((29)). *Let D be a triangulated category and $\text{Stab}(D)$ be the space of stability conditions on it. For each connected component $\Sigma \subset \text{Stab}(D)$ there is a linear subspace $V(\Sigma) \subset \text{Hom}_Z(K(D), C)$, with a well-defined linear topology, and a local homeomorphism $Z : \Sigma \to V(\Sigma)$ which maps a stability condition (Z, P) to its central charge Z. In particular, $\text{Stab}(D)$ is a manifold.*

Moreover, Bridgeland showed the existence of a chamber structure on this manifold analogous to the stable base locus decomposition of the effective cone. In this case, the objects of D which are stable with respect to each condition are fixed within each chamber like the base locus of a divisor was fixed within a chamber before. There is a conjectured relationship between these two sets of walls. This conjectured relationship will motivate our approach to finding divisors spanning the effective cone, but will not be directly used in the construction so we will not elaborate on it.

CHAPTER 3

NEW BASIC PROPERTIES

With this background, we can begin our original work on $M(\xi)$. In the last chapter, we listed many properties of the nonempty moduli spaces. In this chapter, we prove that the moduli spaces are $\mathbb{Q}$-factorial and are Mori dream spaces. In fact, we prove these results for all moduli spaces of Gieseker semistable sheaves on any del Pezzo surface, not just $\mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 3.0.3. *Let M be the moduli space of semistable sheaves with a fixed Chern character on a del Pezzo surface. Then M is $\mathbb{Q}$-factorial.*

Proof. By (36), M is a geometric quotient of a smooth variety. Applying Thm. 4 of (37) immediately allows us to conclude that M is $\mathbb{Q}$-factorial. $\square$

The proof that these spaces are Mori dream spaces is slightly more involved, but similar in flavor.

Theorem 3.0.4. *Let M be the moduli space of semistable sheaves with a fixed Chern character on a del Pezzo surface. Then M is a Mori dream space.*

Proof. The proof follows the same basic outline as the proof for moduli spaces of sheaves on $\mathbb{P}^2$ (6). By (38), a log Fano variety is a Mori dream space. This result reduces the theorem to showing that M is a log Fano variety. Since the anticanonical bundle of M is nef [(18),Thm. 8.2.8 & 8.3.3] and there exists effective divisors E such that $-K_M - \epsilon E$ is ample for all sufficiently small $\epsilon > 0$, showing M is log Fano reduces to showing that $(M, \epsilon E)$ is a klt-pair for all effective divisors E. Showing that $(M, \epsilon E)$ is a klt-pair for all effective divisors E further reduces to showing that M has canonical singularities.

We now show that M has canonical singularities. By (36), M is also a geometric quotient of a smooth variety. By (39), a geometric quotient of a variety with rational singularities has rational singularities so M has rational singularities. As a M is 1-Gorenstein [Thm. 8.3.3, (18)], it has canonical singularities [Thm. 11.1, (40)]. $\square$

3.1 Additional Assumptions

There are two previously mentioned properties that we would also like our moduli spaces on $\mathbb{P}^1 \times \mathbb{P}^1$ to have. We want the complement of the stable locus to be codimension at least two, and we want them to have Picard rank 3. As mentioned before, the first assumption allows us to ignore the strictly semistable locus when we are working with divisors, and the second assumption lets us use properties of the Picard group that we need. These assumptions are justified for a few reasons. First, they hold for the Hilbert schemes of points, which are the primary examples of such moduli spaces. Second, Yoshioka proved that the second assumption holds for $\mathsf{M}(\xi)$ where one of the slope components of ξ is an integer and ξ is above the δ surface (41). Lastly, there are no examples of $\mathsf{M}(\xi)$ where ξ is above the δ surface for which either assumption is known to fail. In fact, both assumptions are believed to be true above the δ surface. Proving that they hold in this region is the focus of current research.

We also assume that ξ is above the δ surface and is not a multiple of an exceptional Chern character.

CHAPTER 4

CORRESPONDING EXCEPTIONAL PAIRS

In this chapter, we use exceptional pairs to identify the Brill-Noether divisors that we expect to span the effective cone by identifying possible solutions to the interpolation problem.

Let $U \in M(\xi)$ be a general element. Recall that to solve the interpolation problem we wanted to find bundles that were cohomologically orthogonal to U and that being cohomologicaly orthogonal is equivalent to V having $\chi(U \otimes V) = 0$ and V being cohomologically non-special with respect to ξ. Now, we are able find all cohomologically orthogonal bundles by imposing each of these two conditions separately.

4.1 Bundles with Vanishing Euler Characteristic

First, we find bundles V with $\chi(U \otimes V) = 0$. As we are trying to compute the effective cone, scaling Chern characters is relatively unimportant. In particular, we can scale a Chern character so that $\mathrm{ch}_0 = 1$ (unless it was 0 to start).

Definition 4.1.1. When a Chern character ξ has positive rank, the *orthogonal surface to* ξ is

$$Q_\xi = \{(\mu, \Delta) : (1, \mu, \Delta) \text{ lies in } \xi^\perp\} \subset \mathbb{R}^3.$$

Using the orthogonal surface rather than the full $\xi^\perp$ has the advantage of working in the three dimensional (μ, Δ)-space instead of in the full four dimensional $K(\mathbb{P}^1 \times \mathbb{P}^1)$. We define the *reference surface*, Q_{ξ_0}, to be the orthogonal surface to $\xi_0 = \mathrm{ch}(\mathcal{O}) = (1, (0, 0), 0)$.

Using Hirzebruch-Riemann-Roch to compute the equation of the orthogonal surface to ξ gives the formula

$$Q_\xi : P(\mu(\xi) + \mu) - \Delta(\xi) = \Delta$$

where $P(x, y) = (x+1)(y+1)$. This equation defines a saddle surface that is a shift of the reference surface. It has unique saddle point at the point $(-1-x_0, -1-y_0)$ where $(x_0, y_0) = \mu(\xi)$. Consequently, any two such surfaces intersect in a parabola that lies over a line in the slope plane.

Using this language, the condition that $\chi(U \otimes V) = 0$ can be rephrased as saying that V must lie on Q_ξ. In other words, we can restrict our search for solutions to the interpolation problem to bundles that lie on the orthogonal surface (to ξ).

As a side note, we can now give an alternative description of each fractal part of the δ surface using orthogonal surfaces. Given an exceptional bundle E_α, δ_E can equivalently be written as

$$\delta_E(\mu) = \begin{cases} Q_{E^*}(\mu) & \text{if } \mu_{1,1}(E) - 4 < \mu_{1,1}(\mu) \text{ and } \overline{\gamma}(\mu) < \overline{\gamma}(E) \\ Q_{E^*(K)}(\mu) & \text{if } \mu_{1,1}(\mu) < \mu_{1,1}(E) + 4 \text{ and } \overline{\gamma}(E) < \overline{\gamma}(\mu) \end{cases}$$

Thus, every saddle subsurface of the surface $\delta(\mu) = \Delta$ can be seen to be a portion of some surface Q_{E_α}.

4.2 Cohomologically Non-special Bundles

Now that we have found the bundles with $\chi(U \otimes V) = 0$, we would like to impose the second condition of cohomological orthogonality, being *non-special*. In some cases, we can find numerical conditions defining being non-specialty as well. Those conditions will be in terms of certain exceptional bundles that we have to pick out. Picking them out begins with studying Q_ξ again.

Note that Q_ξ intersects the plane $\Delta = \frac{1}{2}$. The exceptional bundles that we want to pick out are those exceptional bundles that control this intersection.

Definition 4.2.1. A *controlling exceptional bundle* of ξ is an exceptional bundle, E_α, for which there exists a slope ν for which $\delta(\nu) = \delta_{E_\alpha}(\nu)$ and $Q_\xi(\nu) = \frac{1}{2}$.

As promised above, each controlling exceptional bundle will provide a necessary condition for a stable Chern character to be non-special with respect to the general element of $M(\xi)$. The condition that a controlling exceptional bundles imposes on a Chern character is that the Chern character must be on or above a surface corresponding to that exceptional bundle.

Definition 4.2.2. The *corresponding surface* to an exceptional bundle E_α for ξ is defined as

$$Q_{\alpha,\xi}(\nu) = \begin{cases} Q_{E_\alpha^*}(\nu) & : \text{if } \chi(E_\alpha^*, U) > 0 \\ Q_{E_\alpha^*(K)}(\nu) & : \text{if } \chi(E_\alpha^*, U) < 0. \end{cases}$$

Then in some, if not all, cases, the Chern characters ν such that $Q_{\alpha,\xi}(\nu) > 0$ for all controlling bundles E_α of ξ are precisely the non-special Chern characters with respect to ξ.

4.3 Potential Extremal Rays

The solutions to interpolation are the intersection of the orthogonal surface and the maximum of the corresponding surfaces. Each part of this intersection is where the orthogonal surface intersects a corresponding surface, i.e. where $Q_\xi = Q_{\alpha,\xi}$ for a controlling exceptional E_α of ξ. The corners of the intersection are where the orthogonal surface intersects two different corresponding surfaces, i.e. where $Q_\xi = Q_{\alpha,\xi} = Q_{\beta,\xi}$ for controlling exceptionals E_α and E_β of ξ.

As α, β, and ξ are three linearly independent rational Chern characters, $Q_\xi \cap Q_{\alpha,\xi} \cap Q_{\beta,\xi}$ is a single point that corresponds to the intersection of the 3-planes $\alpha^\perp \cap \beta^\perp \cap \gamma^\perp$, $(\alpha+K)^\perp \cap \beta^\perp \cap \gamma^\perp$, $\alpha^\perp \cap (\beta+K)^\perp \cap \gamma^\perp$, or $(\alpha+K)^\perp \cap (\beta+K)^\perp \cap \gamma^\perp$. We determine which intersection it is by which cases of $Q_{\alpha,\xi}$ and $Q_{\beta,\xi}$ we are using. We want to find the corners made in this way in order to get effective divisors. To find those divisors, we first define all of the possible triple intersections that might work.

Definition 4.3.1. The *corresponding orthogonal point* of a pair of controlling exceptional bundles E_α and E_β is one of the following

(1) the unique point $(\mu^+, \Delta^+) \in Q_\xi \cap Q_{\alpha,\xi} \cap Q_{\beta,\xi}$,

(2) β if $\beta \in Q_\xi \cap Q_{\alpha,\xi}$, or

(3) α if $\alpha \in Q_\xi \cap Q_{\beta,\xi}$.

(2) and (3) can occur simultaneously, but you can treat them individually so we will say "the orthogonal point".

Some of the corresponding orthogonal points will not actually be what we want, as they can both fail to satisfy interpolation (by being below one of the other corresponding surfaces) or by not being extremal (as they have a slope that is not linearly independent of other solutions). We want to pick out the only the solutions that satisfy interpolation and are extremal among those solutions.

Definition 4.3.2. A *controlling pair* of ξ is an exceptional pair of controlling exceptional bundles E_α and E_β of ξ with a corresponding orthogonal point (μ^+, Δ^+) that is stable.

We now want to turn these points back into Chern characters for convenience.

Definition 4.3.3. A *potential (primary) orthogonal Chern character* ξ^+ to ξ is defined by any character $\xi^+ = (r^+, \mu^+, \Delta^+)$ where r^+ is sufficiently large and divisible and (μ^+, Δ^+) is the corresponding orthogonal point of a controlling pair of ξ.

We call them *potential* because we will see in the next chapter that we need a few additional conditions to make sure that they actually span an extremal ray of the effective cone. We will give an approach to showing that some of the potential primary orthogonal Chern characters span the solutions of the interpolation problem and the effective cone of $M(\xi)$ is spanned by the Brill-Noether divisors D_V for many examples of $M(\xi)$ where V are bundles whose Chern character is an orthogonal Chern character ξ^+.

The behavior of these extremal rays depends on the sign of $\chi(E_\alpha^*, U)$ and $\chi(E_\beta^*, U)$. Keeping the identification of $\mathrm{NS}(M(\xi)) \cong \xi^\perp$ in mind, recall that the primary half of the space corresponds to characters of positive rank.

(1) If $\chi(\xi_{-\alpha},\xi) > 0$ and $\chi(\xi_{-\beta},\xi) > 0$, the ray is spanned by a positive rank Chern character in $\xi^\perp \cap \xi_{-\alpha}^\perp \cap \xi_{-\beta}^\perp$.

(2) If $\chi(\xi_{-\alpha},\xi) < 0$ and $\chi(\xi_{-\beta},\xi) > 0$, the ray is spanned by a positive rank Chern character in $\xi^\perp \cap (\xi_{-\alpha+K})^\perp \cap \xi_{-\beta}^\perp$.

(3) If $\chi(\xi_{-\alpha},\xi) < 0$ and $\chi(\xi_{-\beta},\xi) < 0$, the ray is spanned by a positive rank Chern character in $\xi^\perp \cap (\xi_{-\alpha+K})^\perp \cap (\xi_{-\beta+K})^\perp$.

(4) If $\chi(\xi_{-\alpha},\xi) = 0$ or $\chi(\xi_{-\beta},\xi) = 0$, the ray is spanned by α or β, respectively.

CHAPTER 5

THE BEILINSON SPECTRAL SEQUENCE

In this chapter, we find a resolution of the general object $U \in M(\xi)$ for each orthogonal Chern character via the generalized Beilinson spectral sequence. In the next chapter, we use these resolutions to construct fibrations $M(\xi) \dashrightarrow Kr_V(m, n)$. In the second last chapter, these fibrations will give us effective divisors on $M(\xi)$.

An orthogonal Chern character already has exceptional pairs associated to it. In order to use the spectral sequence, we have to complete each of those exceptional pairs to a coil.

The coil we use to resolve U depends on the behavior of the extremal ray that we exhibited in the last chapter.

(1) If $\chi(E_{-\alpha}, U) \geq 0$ and $\chi(E_{-\beta}, U) \geq 0$, we will decompose U according to the coil

$\left(F_0^*, F_{-1}^*, E_{-\beta}, E_{-\alpha}\right)$ where (F_0, F_1) is a minimally ranked right completion pair of (E_α, E_β).

(2) If $\chi(E_{-\alpha}, U) \leq 0$ and $\chi(E_{-\beta}, U) \geq 0$, we will decompose U according to the coil

$\left(E_{-\alpha}(K), F_0^*, F_{-1}^*, E_{-\beta}\right)$ where (F_0, F_1) is a minimally ranked right completion pair of (E_α, E_β).

(3) If $\chi(E_{-\alpha}, U) \leq 0$ and $\chi(E_{-\beta}, U) \leq 0$, we will decompose U according to the coil

$\left(E_{-\beta}(K), E_{-\alpha}(K), F_0^*, F_{-1}^*\right)$ where (F_0, F_1) is a minimally ranked right completion pair of (E_α, E_β).

The spectral sequence will only give a resolution under some assumptions on the controlling pairs. These are the additional conditions needed to make a potential orthogonal Chern character span an extremal ray of the effective cone. We call those controlling pairs that satisfy the needed conditions *extremal pairs*; there will be a different definition for if a controlling pair is extremal based on the signs of some Euler characteristics so we defer the definition to the following three sections.

Extremal pairs pick out the exact Chern characters that correspond to extremal effective divisors using our approach as promised in the previous chapter.

Definition 5.0.4. A *primary orthogonal Chern character* is the potential primary orthogonal Chern character associated to an extremal pair

It is a current area of research to show the following conjecture for extremal pairs (including those as defined analogously in the next three sections).

Conjecture 5.0.5. *Every ξ above the Rudakov δ-surface has an extremal pair and every controlling exceptional bundle of it that is in an extremal pair is in two extremal pairs.*

Proving the conjecture would show that the process laid out in this paper computes the effective cone of $M(\xi)$ for all ξ above the δ surface.

5.1 The "Mixed Type" Spectral Sequence.

Assume $\chi(E_{-\alpha}, U) \leq 0$ and $\chi(E_{-\beta}, U) \geq 0$. Let the right mutated coil of $\left(E_{-\alpha}(K), F_0^*, F_{-1}^*, E_{-\beta}\right)$ be $\left(E_{-\beta}, E_{-1}^*, E_{-2}^*, E_{-\alpha}\right)$. Let Δ_i be as in the spectral sequence with input these two coils.

Definition 5.1.1. A (mixed type) controlling pair of ξ, (E_α, E_β), with corresponding orthogonal slope and discriminant (μ^+, Δ^+) is called *extremal* if it satisfies the following conditions:

(1) They are within a unit in both slope coordinates.

(2) $\mu_{1,1}(E_\alpha)$, $\mu_{1,1}(E_{-2})$, $\mu_{1,1}(E_{-1})$, and $\mu_{1,1}(E_\beta)$ are all greater than $\mu_{1,1}(U) - 4$.

(3) ($\Delta_2 = 1$ and $\chi\left(E_{-2}^*, U\right) \geq 0$) or ($\Delta_2 = 0$ and $\chi\left(E_{-2}^*, U\right) \leq 0$).

(4) ($\Delta_1 = 1$ and $\chi\left(E_{-1}^*, U\right) \geq 0$) or ($\Delta_1 = 0$ and $\chi\left(E_{-1}^*, U\right) \geq 0$).

(5)$\mathscr{H}om(E_{-\alpha}(K), F_{-1}^*)$, $\mathscr{H}om(E_{-\alpha}(K), E_{-\beta})$, $\mathscr{H}om(F_0^*, F_{-1}^*)$, and $\mathscr{H}om(F_0^*, E_{-\beta})$ are all globally generated.

(6) Any bundle sitting in a triangle $\left(F_{-1}^*\right)^{m_1} \bigoplus E_{-\beta}^{m_0} \to U \to \left(E_{-\alpha}(K)^{m_3} \bigoplus (F_0^*)^{m_2}\right)[1]$ is prioritary.

Given an extremal pair, we can resolve the general object of $M(\xi)$.

Theorem 5.1.2. *The general* $U \in M(\xi)$ *admits a resolution of the following form*

$$0 \to E_{-\alpha}(K)^{m_3} \bigoplus (F_0^*)^{m_2} \xrightarrow{\phi} \left(F_{-1}^*\right)^{m_1} \bigoplus E_{-\beta}^{m_0} \to U \to 0.$$

Proof. Consider a bundle U defined by the sequence

$$0 \to E_{-\alpha}(K)^{m_3} \bigoplus (F_0^*)^{m_2} \xrightarrow{\phi} \left(F_{-1}^*\right)^{m_1} \bigoplus E_{-\beta}^{m_0} \to U \to 0$$

where the map $\phi \in \text{Hom}\left(E_{-\alpha}(K)^{m_3} \bigoplus (F_0^*)^{m_2}, \left(F_{-1}^*\right)^{m_1} \bigoplus E_{-\beta}^{m_0}\right)$ is general.

The proof proceeds in 4 steps: calculate that $\text{ch}(U) = \xi$, show ϕ is injective, confirm the expected vanishings in the spectral sequence, and prove that U is stable.

Step 1: Calculate $\text{ch}(U) = \xi$. We do not know if ϕ is injective yet, but we can compute the Chern character of the mapping cone of ϕ in the derived category. Assuming ϕ is injective, this computes the Chern character of U.

This computation follows from the generalized Beilinson spectral sequence's convergence to U. Specifically, we have a spectral sequence with E_1-page

$$\begin{array}{ccccccc}
E_{-\alpha}(K) \otimes \mathbb{C}^{m_{32}} & \to & F_0^* \otimes \mathbb{C}^{m_{22}} & \to & F_{-1}^* \otimes \mathbb{C}^{m_{12}} & \to & 0 \\
E_{-\alpha}(K) \otimes \mathbb{C}^{m_{31}} & \to & F_0^* \otimes \mathbb{C}^{m_{21}} & \to & F_{-1}^* \otimes \mathbb{C}^{m_{11}} & \to & E_{-\beta} \otimes \mathbb{C}^{m_{01}} \\
0 & \to & F_0^* \otimes \mathbb{C}^{m_{20}} & \to & F_{-1}^* \otimes \mathbb{C}^{m_{10}} & \to & E_{-\beta} \otimes \mathbb{C}^{m_{00}}
\end{array}$$

that converges in degree 0 to the sheaf U and to 0 in all other degrees. We omitted all nonzero rows and columns. In particular, row 3 is zero by the vanishing of h^2 for all the sheaves (follows from Def. 5.1.1). Also note that either the top or bottom element of each of the middle two rows vanishes (depending on the value of Δ_1 and Δ_2). An easy computation shows that

$$\begin{aligned} \mathrm{ch}(U) = &-(m_{32}-m_{31})\mathrm{ch}(E_{-\alpha}(K)) + (m_{22}-m_{21}+m_{20})\mathrm{ch}(F_0^*) \\ &-(m_{12}-m_{11}+m_{10})\mathrm{ch}(F_{-1}^*) + (m_{00}-m_{01})\mathrm{ch}(E_{-\beta}). \end{aligned}$$

In our situation, we see that this gives

$$\xi = -m_{-3}\mathrm{ch}(E_{-\alpha}(K)) + m_{-2}\mathrm{ch}(F_0^*) - m_{-1}\mathrm{ch}(F_{-1}^*) + m_0\mathrm{ch}(E_{-\beta})$$

where the m_i are defined in the obvious way.

Step 2: Show ϕ is injective. The sheaves

$$\mathscr{H}om(E_{-\alpha}(K), F_{-1}^*),\ \mathscr{H}om(E_{-\alpha}(K), E_{-\beta}),\ \mathscr{H}om(F_0^*, F_{-1}^*),\ \text{and}\ \mathscr{H}om(F_0^*, E_{-\beta})$$

are all globally generated by the controlling pair being an extremal pair. Those bundles being globally generated immediately implies that

$$\mathscr{H}om(E_{-\alpha}(K)^{m_3} \bigoplus (F_0^*)^{m_2}, (F_{-1}^*)^{m_1} \bigoplus E_{-\beta}^{m_0})$$

is globally generated as well. Using a Bertini-type theorem [Prop 2.6, (42)] and the fact that the virtually computed rank of U is positive, we see that ϕ is injective.

Step 3: Verify U's spectral sequence has the correct vanishings. We know that $\chi(E_{-\alpha}, U) \leq 0$ and

$\chi(E_{-\beta}, U) \geq 0$. We also know that $(\chi(E^*_{-2}, U) \leq 0$ and $\Delta_2 = 0)$ or $(\chi(E^*_{-2}, U) \geq 0$ and $\Delta_2 = 1)$. Analogously, we know that $(\chi(E^*_{-1}, U) \leq 0$ and $\Delta_1 = 0)$ or $(\chi(E^*_{-1}, U) \geq 0$ and $\Delta_1 = 1)$.

Since we know that all of the groups other than possibly ext^1 and hom vanish, it is enough to check that

$$\hom(E_{-\alpha}, U) = \mathrm{ext}^{\Delta_1}(E^*_{-1}, U) = \mathrm{ext}^{\Delta_2}(E^*_{-2}, U) = \mathrm{ext}^1(E_{-\beta}, U) = 0.$$

These vanishings for the specific U we have resolved will follow from the orthogonality properties of exceptional bundles and the relevant long exact sequences. Once they vanish for a specific U, they will vanish for a general U as needed. We show these four vanishings in order.

First, $\hom(E_{-\alpha}, F^*_{-1})$, $\hom(E_{-\alpha}, E_{-\beta})$, and $\mathrm{ext}^1(E_{-\alpha}, F^*_0)$ all vanish since $\left(F^*_0, F^*_{-1}, E_{-\beta}, E_{-\alpha}\right)$ is a coil and $\mathrm{ext}^1(E_{-\alpha}, E_{-\alpha}(K)) = \mathrm{ext}^1(E_{-\alpha}(K), E_{-\alpha}(K))$ which vanishes since exceptional bundles are rigid. This gives $\hom(E_{-\alpha}, U) = 0$ as desired.

For the next vanishing, we have two cases: $\Delta_1 = 0$ and $\Delta_1 = 1$.

Assume $\Delta_1 = 0$, then we need to show that $\hom(E^*_{-1}, U) = 0$. Next, $\mathrm{ext}^1(E^*_{-1}, E_{-\alpha}(K))$, $\hom(E^*_{-1}, E_{-\beta})$, and $\mathrm{ext}^1(E^*_{-1}, F^*_0)$ all vanish since $\left(E_{-\alpha}(K), F^*_0, E_{-\beta}, E^*_{-1}\right)$ is a coil. Then the only remaining vanishing we need to prove is $\hom(E^*_{-1}, F^*_{-1})$. By assumption on Δ_1, we can write $0 \to F^*_{-1} \to E^a_{-\beta} \to E^*_{-1} \to 0$ where $a = \chi(E_{-\beta}, E_{-1})$. From this resolution, we see $\hom(E^*_{-1}, F^*_{-1}) \hookrightarrow \hom(E^*_{-1}, E_{-\beta}) = 0$, and the vanishing follows.

Assume $\Delta_1 = 1$, then we need to show that $\mathrm{ext}^1(E^*_{-1}, U) = 0$. Next, $\mathrm{ext}^2(E^*_{-1}, E_{-\alpha}(K))$, $\mathrm{ext}^1(E^*_{-1}, E_{-\beta})$, and $\mathrm{ext}^2(E^*_{-1}, F^*_0)$ all vanish since $\left(E_{-\alpha}(K), F^*_0, E_{-\beta}, E^*_{-1}\right)$ is a coil. Then the only remaining vanishing we need to prove is $\mathrm{ext}^1(E^*_{-1}, F^*_{-1})$. By assumption on Δ_1, we either have the short exact sequence

$$0 \to E^a_{-\beta} \to E^*_{-1} \to F^*_{-1} \to 0$$

or the short exact sequence

$$0 \to E^*_{-1} \to F^*_{-1} \to E^a_{-\beta} \to 0$$

where $a = |\chi(E_{-\beta}, E_{-1})|$. If we have the first sequence, $\mathrm{ext}^1(E^*_{-1}, F^*_{-1})$ vanishes since both $\mathrm{ext}^1(E^*_{-1}, E^*_{-1})$ $= 0$ by the rigidity of exceptional bundles and $\mathrm{ext}^2(E^*_{-1}, E_{-\beta})$ vanishes by a properties of them being an exceptional pair. If we have the second sequence, $\mathrm{ext}^1(E^*_{-1}, F^*_{-1})$ vanishes since both $\mathrm{ext}^1(E^*_{-1}, E^*_{-1}) = 0$ by the rigidity of exceptional bundles and $\mathrm{ext}^1(E^*_{-1}, E_{-\beta})$ vanishes by the properties of them being an exceptional pair. So we have shown the vanishing for both sequences, and so $\mathrm{ext}^1(E^*_{-1}, U) = 0$.

Finally, we have shown $\mathrm{ext}^{\Delta_1}(E^*_{-1}, U) = 0$ as desired.

Then for the next vanishing, we again have two cases: $\Delta_2 = 0$ and $\Delta_2 = 1$.

Assume $\Delta_2 = 0$, then we need to show that $\hom(E^*_{-2}, U) = 0$. Then, $\mathrm{ext}^1(E^*_{-2}, E_{-\alpha}(K))$, $\hom(E^*_{-2}, E_{-\beta})$, and $\hom(E^*_{-2}, F^*_{-1})$ all vanish since $\left(E_{-\alpha}(K), F^*_{-1}, E_{-\beta}, E^*_{-2}\right)$ is a coil. Then the only remaining vanishing we need to prove is $\mathrm{ext}^1(E^*_{-2}, F^*_0)$. By assumption on Δ_2, we can write $0 \to F^*_0 \to E^a_{-\beta} \to L_{E^*_{-1}} E^*_{-2} \to 0$ where $a = \chi(E_{-\beta}, L_{E^*_{-1}} E^*_{-2})$. From this resolution we see that $\mathrm{ext}^1(E^*_{-2}, F^*_0) \cong \hom(E^*_{-2}, L_{E^*_{-1}} E^*_{-2})$ since $\hom(E^*_{-2}, E_{-\beta}) = \mathrm{ext}^1(E^*_{-2}, E_{-\beta}) = 0$. Thus, we reduce to showing that $\hom(E^*_{-2}, L_{E^*_{-1}} E^*_{-2}) = 0$. Again, by assumption on Δ_2, we know that we can write $0 \to L_{E^*_{-1}} E^*_{-2} \to (E^*_{-1})^a \to E^*_{-2} \to 0$ from which we can easily see $\hom(E^*_{-2}, L_{E^*_{-1}} E^*_{-2}) \hookrightarrow \hom(E^*_{-2}, E^*_{-1}) = 0$, and the vanishing follows.

Assume $\Delta_2 = 1$, then we need to show that $\mathrm{ext}^1(E^*_{-2}, U) = 0$. Then, $\mathrm{ext}^2(E^*_{-2}, E_{-\alpha}(K))$, $\mathrm{ext}^1(E^*_{-2}, E_{-\beta})$, and $\mathrm{ext}^1(E^*_{-2}, F^*_{-1})$ all vanish since $\left(E_{-\alpha}(K), F^*_{-1}, E_{-\beta}, E^*_{-2}\right)$ is a coil. Now, the only remaining vanishing we need to prove is $\mathrm{ext}^2(E^*_{-2}, F^*_0)$. Then, we have one of the following short exact sequences where $a = \left|\chi(E^*_{-1}, E^*_{-2})\right|$:

$$\text{(A) } 0 \to E^*_0 \to E^{*a}_{-1} \to E^*_{-2} \to 0,$$

$$\text{(B) } 0 \to E^{*a}_{-1} \to E^*_{-2} \to E^*_0 \to 0, \text{ or}$$

$$\text{(C) } 0 \to E^*_{-2} \to E^*_0 \to E^{*a}_{-1} \to 0.$$

If we are in case (A), then, since $\Delta_2 = 1$, we have two possibilities where $b = |\chi(E_{-\beta}, E^*_0)|$:

$$\text{(A.A) } 0 \to E^{*b}_{-\beta} \to E^*_0 \to F^*_0 \to 0 \text{ and}$$

$$\text{(A.B) } 0 \to E^*_0 \to F^*_0 \to E^{*b}_{-\beta} \to 0.$$

If we are in case (B) or (C), since $\Delta_2 = 1$, we have the short exact sequence

$$\text{(B.A)\&(C.A) } 0 \to F^*_0 \to E^{*b}_{-\beta} \to E^*_0 \to 0$$

where $b = \chi(E_{-\beta}, E^*_0)$. We have to treat each of these four cases (A.A), (A.B), (B.A), and (C.A) separately.

In case (A.A), using the short exact sequence (A.A), we see that $\mathrm{ext}^2(E^*_{-2}, E^*_0) \twoheadrightarrow \mathrm{ext}^2(E^*_{-2}, F^*_0)$, so it suffices to show that $\mathrm{ext}^2(E^*_{-2}, E^*_0) = 0$. Using the short exact sequence (A), we see that $\mathrm{ext}^2(E^*_{-2}, E^*_0)$ vanishes since both $\mathrm{ext}^2(E^*_{-2}, E^*_{-1}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E^*_{-2}, E^*_{-2}) = 0$ by the rigidity of exceptional bundles. Thus, in case (A.A), we have that $\mathrm{ext}^2(E^*_{-2}, F^*_0)$ vanishes.

In case (A.B), using the short exact sequence (A.B), we see that $\mathrm{ext}^2(E^*_{-2}, E^*_0) \twoheadrightarrow \mathrm{ext}^2(E^*_{-2}, F^*_0)$ since $\mathrm{ext}^2(E^*_{-2}, E_{-\beta}) = 0$ as they are an exceptional pair, so it suffices to show that $\mathrm{ext}^2(E^*_{-2}, E^*_0) = 0$. Using the short exact sequence (A), we see that $\mathrm{ext}^2(E^*_{-2}, E^*_0)$ vanishes since both $\mathrm{ext}^2(E^*_{-2}, E^*_{-1}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E^*_{-2}, E^*_{-2}) = 0$ by the rigidity of exceptional bundles. Thus, in case (A.B), we have that $\mathrm{ext}^2(E^*_{-2}, F^*_0)$ vanishes.

In case (B.A), using the short exact sequence (B.A), we see that $\mathrm{ext}^1(E^*_{-2}, E^*_0) \twoheadrightarrow \mathrm{ext}^2(E^*_{-2}, F^*_0)$ since $\mathrm{ext}^2(E^*_{-2}, E_{-\beta}) = 0$ as they are an exceptional pair, so it suffices to show that $\mathrm{ext}^1(E^*_{-2}, E^*_0) = 0$. Using the short exact sequence (B), we see that $\mathrm{ext}^1(E^*_{-2}, E^*_0)$ vanishes since both $\mathrm{ext}^2(E^*_{-2}, E^*_{-1}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E^*_{-2}, E^*_{-2}) = 0$ by the rigidity of exceptional bundles. Thus, in case (B.A), we have that $\mathrm{ext}^2(E^*_{-2}, F^*_0)$ vanishes.

In case (C.A), using the short exact sequence (C.A), we see that $\mathrm{ext}^1(E^*_{-2}, E^*_0) \twoheadrightarrow \mathrm{ext}^2(E^*_{-2}, F^*_0)$ since $\mathrm{ext}^2(E^*_{-2}, E_{-\beta}) = 0$ as they are an exceptional pair, so it suffices to show that $\mathrm{ext}^1(E^*_{-2}, E^*_0) = 0$. Using the short exact sequence (C), we see that $\mathrm{ext}^1(E^*_{-2}, E^*_0)$ vanishes since both $\mathrm{ext}^1(E^*_{-2}, E^*_{-1}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E^*_{-2}, E^*_{-2}) = 0$ by the rigidity of exceptional bundles. Thus, in case (C.A), we have that $\mathrm{ext}^2(E^*_{-2}, F^*_0)$ vanishes. Finally, we have shown the desired vanishing in the case that $\Delta_2 = 1$.

Then, we have shown $\mathrm{ext}^{\Delta_2}(E^*_{-2}, U) = 0$ as desired.

Finally, we will show $\mathrm{ext}^1(E_{-\beta}, U) = 0$. Then $\mathrm{ext}^1(E_{-\beta}, F^*_{-1})$, $\mathrm{ext}^2(E_{-\beta}, E_{-\alpha}(K))$, and $\mathrm{ext}^2(E_{-\beta}, F^*_0)$ all vanish since $\left(E_{-\alpha}(K), F^*_0, F^*_{-1}, E_{-\beta}\right)$ is a coil while $\mathrm{ext}^1(E_{-\beta}, E_{-\beta}) = 0$ since exceptional bundles are rigid. This gives the vanishing of $\mathrm{ext}^1(E_{-\beta}, U)$.

Step 4: Prove that U is stable. Let

$$S \subset \mathrm{Hom}\left(E_{-\alpha}(K)^{m_3} \bigoplus (F^*_0)^{m_2}, \left(F^*_{-1}\right)^{m_1} \bigoplus E^{m_0}_{-\beta}\right)$$

be the open subset of sheaf maps that are injective and have torsion-free cokernels.

By the argument of 5.3 of (6), it follows that S is non-empty.

Consider the family ${}^{\mathcal{U}}/_{S}$ of quotients parametrized by S. We need to show that $\mathcal{U}$ is a complete family of prioritary sheaves. Recall that a prioritary sheaf is a torsion free sheaf U such that $\mathrm{Ext}^2(U, U(0,-1)) = 0$ or $\mathrm{Ext}^2(U, U(-1,0)) = 0$ or, equivalently in our case, that $\mathrm{Hom}(U, U(-1,-2)) = 0$ or $\mathrm{Hom}(U, U(-2,-1)) = 0$. By Def. 5.1.1, the elements of $\mathcal{U}$ are prioritary. Again by the general argument of 5.3 of (6), the family is a complete family. By Thm. 1 of (8), the Artin stack of prioritary sheaves with Chern character ξ is an irreducible stack that contains the stack of semistable sheaves with Chern character ξ as a dense open subset. It is then clear that S parametrizes the general sheaves in $M(\xi)$. □

5.2 The "Negative Type" Spectral Sequence.

Assume $\chi(E_{-\alpha}, U) \leq 0$ and $\chi(E_{-\beta}, U) \leq 0$. Let the right mutated coil of $\left(E_{-\beta}(K), E_{-\alpha}(K), F_0^*, F_{-1}^*\right)$ be $\left(F_{-1}^*, F_{-2}^*, E_{-\gamma}, E_{-\beta}\right)$. Let Δ_i be as in the spectral sequence with input these two coils.

Definition 5.2.1. A (negative type) controlling pair of ξ, (E_α, E_β), with corresponding orthogonal slope and discriminant (μ^+, Δ^+) is called *extremal* if it satisfies the following conditions:

(1) They are within a unit in both slope coordinates.

(2) $\mu_{1,1}(E_\alpha)$, $\mu_{1,1}(F_{-1})$, $\mu_{1,1}(F_{-2})$, and $\mu_{1,1}(E_\beta)$ are all greater than $\mu_{1,1}(U) - 4$.

(3) ($\Delta_2 = 0$ and $\chi(E_{-\gamma}, U) \leq 0$) or ($\Delta_2 = 1$ and $\chi(E_{-\gamma}, U) \geq 0$).

(4) $\Delta_1 = 0$, $\chi\left(F_{-1}^*, U\right) \geq 0$, and $\chi\left(F_{-2}^*, U\right) \geq 0$.

(5) $\mathcal{H}om(E_{-\beta}(K), F_{-1}^*)$, $\mathcal{H}om(E_{-\alpha}(K), F_{-1}^*)$, and $\mathcal{H}om(F_0^*, F_{-1}^*)$ are all globally generated.

(6) Any bundle sitting in a triangle $F_{-1}^{m_0} \to U \to \left((F_0^*)^{m_1} \bigoplus E_{-\alpha}(K)^{m_2} \bigoplus E_{-\beta}(K)^{m_3}\right)[1]$ is prioritary.

Given an extremal pair, we can resolve the general object of $M(\xi)$.

Theorem 5.2.2. *The general* $U \in M(\xi)$ *admits a resolution of the following forms*

$$0 \to E_{-\beta}(K)^{m_3} \bigoplus E_{-\alpha}(K)^{m_2} \bigoplus (F_0^*)^{m_1} \to \left(F_{-1}^*\right)^{m_0} \to U \to 0.$$

Proof. The only part of the proof which changes significantly is which vanishing we need to show and how to show them so we omit the rest of the proof. We know that $\chi(E_{-\alpha}, U) \le 0$ and $\chi(E_{-\beta}, U) \le 0$. We also know that $\chi(F^*_{-2}, U) \ge 0$ and $\chi(F^*_{-1}, U) \ge 0$. Finally, we know that ($\Delta_2 = 0$ and $\chi\left(E_{-\gamma}, U\right) \le 0$) or ($\Delta_2 = 1$ and $\chi\left(E_{-\gamma}, U\right) \ge 0$).

Since we know that all of the groups other than possibly ext^1 and hom vanish, it is enough to check that

$$\text{hom}(E_{-\beta}, U) = \text{ext}^{\Delta_2}(E_{-\gamma}, U) = \text{ext}^1(F^*_{-2}, U) = \text{ext}^1(F^*_{-1}, U) = 0.$$

These vanishings for the specific U we have resolved will follow from the orthogonality properties of exceptional bundles and the relevant long exact sequences. Once they vanish for a specific U, they will vanish for a general U as needed. We show these four vanishings in order.

First, $\text{ext}^1(E_{-\beta}, E_{-\alpha}(K))$, $\text{ext}^1(E_{-\beta}, F^*_0)$, and $\text{hom}(E_{-\beta}, F^*_{-1})$ all vanish since $\left(E_{-\alpha}(K), F^*_0, F^*_{-1}, E_{-\beta}\right)$ is a coil and $\text{ext}^1(E_{-\beta}, E_{-\beta}(K)) = \text{ext}^1(E_{-\beta}(K), E_{-\beta}(K))$ which vanishes since exceptional bundles are rigid. This gives $\text{hom}(E_{-\beta}, U) = 0$ as desired.

For the next vanishing, we have two cases: $\Delta_2 = 0$ and $\Delta_2 = 1$.

Assume $\Delta_2 = 0$, then we need to show that $\text{hom}(E_{-\gamma}, U) = 0$. Next, $\text{ext}^1(E_{-\gamma}, E_{-\beta}(K))$, $\text{ext}^1(E_{-\gamma}, F^*_0)$, and $\text{hom}(E_{-\gamma}, F^*_{-1})$ vanish since $\left(E_{-\beta}(K), F^*_0, F^*_{-1}, E_{-\gamma}\right)$ is a coil. Thus, the only remaining vanishing we need to prove is $\text{ext}^1(E_{-\gamma}, E_{-\alpha}(K))$. Then, $\text{ext}^1(E_{-\gamma}, E_{-\alpha}(K)) = \text{ext}^1(E_{-\alpha}, E_{-\gamma})$ by Serre duality. Since $\Delta_2 = 0$, we have the short exact sequence

$$0 \to L_{F^*_{-2}} E_{-\gamma} \to F^{*a}_{-2} \to E_{-\gamma} \to 0$$

where $a = \chi(F^*_{-2}, E_{-\gamma})$. Looking at the long exact sequence, $\operatorname{ext}^1(E_{-\alpha}, E_{-\gamma}) \cong \operatorname{ext}^2(E_{-\alpha}, L_{F^*_{-2}} E_{-\gamma})$ since $\operatorname{ext}^i(E_{-\alpha}, F^*_{-2}) = 0$ for $i = 1$ and $i = 2$ as they are an exceptional pair. So we have reduced to showing that $\operatorname{ext}^2(E_{-\alpha}, L_{F^*_{-2}} E_{-\gamma})$ vanishes. Again, since $\Delta_2 = 0$, we have the short exact sequence

$$0 \to E_{-\alpha}(K) \to F^{*b}_{-1} \to L_{F^*_{-2}} E_{-\gamma} \to 0$$

where $b = \chi(F^*_{-1}, L_{F^*_{-2}} E_{-\gamma})$. Again looking at the long exact sequence, $\operatorname{ext}^2(E_{-\alpha}, F^{*b}_{-1}) \twoheadrightarrow \operatorname{ext}^2(E_{-\alpha}, L_{F^*_{-2}} E_{-\gamma})$. Then $\operatorname{ext}^2(E_{-\alpha}, F^{*b}_{-1}) = 0$ since they are an exceptional pair, and we have shown $\hom(E_{-\gamma}, U) = 0$.

Assume $\Delta_2 = 1$, then we need to show that $\operatorname{ext}^1(E_{-\gamma}, U) = 0$. Next, $\operatorname{ext}^2(E_{-\gamma}, E_{-\beta}(K))$, $\operatorname{ext}^2(E_{-\gamma}, F^*_0)$, and $\operatorname{ext}^1(E_{-\gamma}, F^*_{-1})$ all vanish since $\left(E_{-\beta}(K), F^*_0, F^*_{-1}, E_{-\gamma}\right)$ is a coil. Then the only remaining vanishing we need to prove is $\operatorname{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$. Since $\Delta_2 = 1$, we have one of the following short exact sequences where $a = \left|\chi(F^*_{-2}, E_{-\gamma})\right|$ and $E^* = L_{F^*_{-2}} E_{-\gamma}$:

(A) $0 \to E^* \to F^{*a}_{-2} \to E_{-\gamma} \to 0$,

(B) $0 \to F^{*a}_{-2} \to E_{-\gamma} \to E^* \to 0$, or

(C) $0 \to E_{-\gamma} \to E^* \to F^{*a}_{-2} \to 0$.

If we are in case (A), then since $\Delta_2 = 1$ we have two possibilities where $b = \left|\chi(F^*_{-1}, E^*)\right|$:

(A.A) $0 \to F^{*b}_{-1} \to E^* \to E_{-\alpha}(K) \to 0$, or

(A.B) $0 \to E^* \to E_{-\alpha}(K) \to F^{*b}_{-1} \to 0$.

If we are in case (B) or (C), since $\Delta_2 = 1$, we have the short exact sequence

(B.A)&(C.A) $0 \to E_{-\alpha}(K) \to F^{*b}_{-1} \to E^* \to 0$

again where $b = \left|\chi(F^*_{-1}, E^*)\right|$. We have to treat each of these four cases (A.A), (A.B), (B.A), and (C.A) separately.

In case (A.A), using the short exact sequence (A.A), we see that $\mathrm{ext}^2(E_{-\gamma}, E^*) \twoheadrightarrow \mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$, so it suffices to show that $\mathrm{ext}^2(E_{-\gamma}, E^*) = 0$. Using the short exact sequence (A), we see that $\mathrm{ext}^2(E_{-\gamma}, E^*)$ vanishes since both $\mathrm{ext}^2(E_{-\gamma}, F^*_{-2}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E_{-\gamma}, E_{-\gamma})$ $= 0$ by the rigidity of exceptional bundles. Thus, in case (A.A), we have that $\mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$ vanishes.

In case (A.B), using the short exact sequence (A.B), we see that $\mathrm{ext}^2(E_{-\gamma}, E^*) \twoheadrightarrow \mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$ since $\mathrm{ext}^2(E_{-\gamma}, F^*_{-1}) = 0$ as they are an exceptional pair, so it suffices to show that $\mathrm{ext}^2(E_{-\gamma}, E^*) = 0$. Using the short exact sequence (A), we see that $\mathrm{ext}^2(E_{-\gamma}, E^*)$ vanishes since both $\mathrm{ext}^2(E_{-\gamma}, F^*_{-2}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E_{-\gamma}, E_{-\gamma}) = 0$ by the rigidity of exceptional bundles. Thus, in case (A.B), we have that $\mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$ vanishes.

In case (B.A), using the short exact sequence (B.A), we see that $\mathrm{ext}^1(E_{-\gamma}, E^*) \twoheadrightarrow \mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$ since $\mathrm{ext}^2(E_{-\gamma}, F_{-1}(K)) = 0$ as they are an exceptional pair, so it suffices to show that $\mathrm{ext}^1(E_{-\gamma}, E^*) = 0$. Using the short exact sequence (B), we see that $\mathrm{ext}^2(E_{-\gamma}, E^*)$ vanishes since both $\mathrm{ext}^2(E_{-\gamma}, F^*_{-2}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E_{-\gamma}, E_{-\gamma}) = 0$ by the rigidity of exceptional bundles. Thus, in case (B.A), we have that $\mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$ vanishes.

In case (C.A), using the short exact sequence (C.A), we see that $\mathrm{ext}^1(E_{-\gamma}, E^*) \twoheadrightarrow \mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$ since $\mathrm{ext}^2(E_{-\gamma}, F_{-1}(K)) = 0$ as they are an exceptional pair, so it suffices to show that $\mathrm{ext}^1(E_{-\gamma}, E^*) = 0$. Using the short exact sequence (C), we see that $\mathrm{ext}^2(E_{-\gamma}, E^*)$ vanishes since both $\mathrm{ext}^1(E_{-\gamma}, F^*_{-2}) = 0$ by them being an exceptional pair and $\mathrm{ext}^1(E_{-\gamma}, E_{-\gamma}) = 0$ by the rigidity of exceptional bundles. Thus, in case (C.A), we have that $\mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K))$ vanishes.

Then, we have shown $\mathrm{ext}^2(E_{-\gamma}, E_{-\alpha}(K)) = 0$ as desired in each of these cases.

Thus, $\mathrm{ext}^{\Delta_2}(E_{-\gamma}, U)$ vanishes for both possible values of Δ_2.

Next, we show $\mathrm{ext}^1(F^*_{-2}, U) = 0$. Then, $\mathrm{ext}^2(F^*_{-2}, E_{-\alpha}(K))$, $\mathrm{ext}^2(F^*_{-2}, E_{-\beta}(K))$, and $\mathrm{ext}^1(F^*_{-2}, F^*_{-1})$ all vanish since $\left(E_{-\beta}(K), E_{-\alpha}(K), F^*_{-1}, F^*_{-2}\right)$ is a coil. Then the only remaining vanishing we need to prove is $\mathrm{ext}^2(F^*_{-2}, F^*_0)$. By assumption on Δ_1, we can write $0 \to F^*_0 \to F^a_{-1} \to F^*_{-2} \to 0$ where $a = \chi(F^*_{-1}, F^*_{-2})$. From this resolution we see that $\mathrm{ext}^2(F^*_{-2}, F^*_0)$ vanishes since $\mathrm{ext}^1(F^*_{-2}, F^*_{-2}) = 0$ by the rigidity of F_{-2} and $\mathrm{ext}^2(F^*_{-2}, F^*_{-1}) = 0$ as they are an exceptional pair. From this, the vanishing follows.

Finally, we show $\mathrm{ext}^1(F^*_{-1}, U) = 0$. Then, $\mathrm{ext}^2(F^*_{-1}, E_{-\beta}(K))$, $\mathrm{ext}^2(F^*_{-1}, E_{-\alpha}(K))$, and $\mathrm{ext}^2(F^*_{-1}, F^*_0)$ all vanish since $\left(E_{-\beta}(K), E_{-\alpha}(K), F^*_0, F^*_{-1}\right)$ is a coil while $\mathrm{ext}^1(F^*_{-1}, F^*_{-1}) = 0$ since exceptional bundles are rigid. This gives the vanishing of $\mathrm{ext}^1(F^*_{-1}, U)$. □

5.3 The "Positive Type" Spectral Sequence.

Assume $\chi(E_{-\alpha}, U) \geq 0$ and $\chi(E_{-\beta}, U) \geq 0$. Let the right mutated coil of $\left(F^*_0, F^*_{-1}, E_{-\beta}, E_{-\alpha}\right)$ be $(E_{-\alpha}, E_{-\gamma}, F^*_1(-K), F^*_0(-K))$. Let Δ_i be as in the spectral sequence with input these two coils.

Definition 5.3.1. A (positive type) controlling pair of ξ, (E_α, E_β), with corresponding orthogonal slope and discriminant (μ^+, Δ^+) is called *extremal* if it satisfies the following conditions:

(1) They are within a unit in both slope coordinates.

(2) $\mu_{1,1}(E_\alpha)$, $\mu_{1,1}(F_1(K))$, $\mu_{1,1}(F_0(K))$, and $\mu_{1,1}(E_\beta)$ are all greater than $\mu_{1,1}(U) - 4$.

(3) ($\Delta_1 = 1$ and $\chi(E_{-\gamma}, U) \geq 0$) or ($\Delta_1 = 0$ and $\chi(E_{-\gamma}, U) \leq 0$).

(4) $\chi\left(F^*_{-1}(-K), U\right) \leq 0$, $\chi(F^*_0(-K), U) \leq 0$, and $\Delta_2 = 1$.

(5) $\mathscr{H}om(F^*_{-0}, F^*_{-1})$, $\mathscr{H}om(F^*_{-0}, E_{-\beta})$, and $\mathscr{H}om(F^*_0, E_{-\alpha})$ are all globally generated.

(6) Any bundle sitting in a triangle $F^{m_2}_{-1} \bigoplus E^{m_1}_{-\beta} \bigoplus E^{m_0}_{-\alpha} \to U \to (F^*_0)^{m_3}[1]$ is prioritary.

Given an extremal pair, we can resolve the general object of $M(\xi)$.

Theorem 5.3.2. *The general* $U \in M(\xi)$ *admits a resolution of the following forms*

$$0 \to (F_0^*)^{m_3} \to \left(F_{-1}^*\right)^{m_2} \bigoplus E_{-\beta}^{m_1} \bigoplus E_{-\alpha}^{m_0} \to U \to 0.$$

Proof. Again, the only part of the proof which changes significantly is which vanishing we need to show and how to show them so we omit the rest of the proof. We know that $\chi(E_{-\alpha}, U) \geq 0$ and $\chi(E_{-\beta}, U) \geq 0$. We also know that $\chi(F_1^*(-K), U) \leq 0$ and $\chi(F_0^*(-K), U) \leq 0$. Finally, we know that ($\Delta_1 = 1$ and $\chi(E_{-\gamma}, U) \geq 0$) or ($\Delta_1 = 0$ and $\chi(E_{-\gamma}, U) \leq 0$).

Since we know that all of the groups other than possibly ext^1 and hom vanish, it is enough to check that

$$\mathrm{ext}^1(E_{-\alpha}, U) = \hom(F_1^*(-K), U) = \hom(F_0^*(-K), U) = \mathrm{ext}^{\Delta_1}(E_{-\gamma}, U) = 0.$$

These vanishings for the specific U we have resolved will follow from the orthogonality properties of exceptional bundles and the relevant long exact sequences. Once they vanish for a specific U, they will vanish for a general U as needed. We show these four vanishings in order.

First, $\mathrm{ext}^1(E_{-\alpha}, E_{-\beta})$, $\mathrm{ext}^1(E_{-\alpha}, F_{-1}^*)$, and $\mathrm{ext}^2(E_{-\alpha}, F_0^*)$ all vanish since $\left(F_0^*, F_{-1}^*, E_{-\beta}, E_{-\alpha}\right)$ is a coil and $\mathrm{ext}^1(E_{-\alpha}, E_{-\alpha}) = 0$ since exceptional bundles are rigid. This gives $\mathrm{ext}^1(E_{-\alpha}, U) = 0$.

Next, $\hom(F_1^*(-K), E_{-\alpha})$, $\hom(F_1^*(-K), E_{-\beta})$, and $\mathrm{ext}^1(F_1^*(-K), F_0^*)$ all vanish since $(F_0^*, E_{-\beta}, E_{-\alpha}, F_1^*(-K))$ is a coil. Then the only remaining vanishing we need to prove is $\hom(F_1^*(-K), F_{-1}^*)$. By Serre duality, $\hom(F_1^*(-K), F_{-1}^*) = \mathrm{ext}^2(F_{-1}^*, F_1^*)$. Since F_1^* sits in a short exact sequence with F_0^* and F_{-1}^* and $\mathrm{ext}^i(F_{-1}^*, F_0^*) = \mathrm{ext}^i(F_{-1}^*, F_{-1}^*)$ for $i = 1$ and $i = 2$, $\mathrm{ext}^2(F_{-1}^*, F_1^*)$ vanishes. The vanishing we wanted follows immediately.

Then, $\hom(F_0^*(-K), E_{-\alpha})$, $\hom(F_0^*(-K), E_{-\beta})$, and $\hom(F_0^*(-K), F_{-1}^*)$ all vanish since $\left(F_{-1}^*, E_\beta, E_{-\alpha}, F_0^*(-K)\right)$ is a coil while $\mathrm{ext}^1(F_0^*(-K), F_0^*) = \mathrm{ext}^1(F_0^*, F_0^*)$ vanishes since exceptional bundles are rigid and Serre duality. This gives the vanishing of $\hom(F_0^*(-K), U)$.

For the final vanishing, we have two cases: $\Delta_1 = 0$ and $\Delta_1 = 1$.

Assume $\Delta_1 = 0$, then we need to show $\hom(E_{-\gamma}, U)$. Then $\hom(E_{-\gamma}, E_{-\alpha})$, $\hom(E_{-\gamma}, F^*_{-1})$, and $\mathrm{ext}^1(E_{-\gamma}, F^*_0)$ all vanish since $\{F^*_0, F^*_{-1}, E_{-\alpha}, E_{-\gamma}\}$ is a coil. Then the only remaining vanishing we need to prove is $\hom(E_{-\gamma}, E_{-\beta})$. Since $\Delta_1 = 0$, we have the short exact sequence

$$0 \to E_{-\beta} \to E^a_{-\alpha} \to E_{-\gamma} \to 0$$

where $a = \chi(E_{-\alpha}, E_{-\gamma})$. Then $\hom(E_{-\gamma}, E_{-\beta})$ vanishes since $\hom(E_{-\gamma}, E_{-\alpha}) = 0$ as they are an exceptional pair and $\hom(E_{-\gamma}, E_{-\beta}) \hookrightarrow \hom(E_{-\gamma}, E_{-\alpha})$. This gives the desired vanishing in this case.

Now assume $\Delta_1 = 1$, then we need to show $\mathrm{ext}^1(E_{-\gamma}, U)$. Then $\mathrm{ext}^1(E_{-\gamma}, E_{-\alpha})$, $\mathrm{ext}^1(E_{-\gamma}, F^*_{-1})$, and $\mathrm{ext}^2(E_{-\gamma}, F^*_0)$ all vanish since $\{F^*_0, F^*_{-1}, E_{-\alpha}, E_{-\gamma}\}$ is a coil. Then the only remaining vanishing we need to prove is $\mathrm{ext}^1(E_{-\gamma}, E_{-\beta})$. Since $\Delta_1 = 1$, we either have the short exact sequence

$$0 \to E^a_{-\alpha} \to E_{-\gamma} \to E_{-\beta} \to 0$$

or we have the short exact sequence

$$0 \to E_{-\gamma} \to E_{-\beta} \to E^a_{-\alpha} \to 0$$

where $a = |\chi(E_{-\alpha}, E_{-\gamma})|$. If we have the first one, $\mathrm{ext}^1(E_{-\gamma}, E_{-\beta})$ vanishes since both $\mathrm{ext}^1(E_{-\gamma}, E_{-\gamma})$ vanishes by the rigidity of exceptional bundles and $\mathrm{ext}^2(E_{-\gamma}, E_{-\alpha})$ vanishes by them being an exceptional pair. If we have the second one, $\mathrm{ext}^1(E_{-\gamma}, E_{-\beta})$ vanishes since both $\mathrm{ext}^1(E_{-\gamma}, E_{-\gamma})$ vanishes by the rigidity of exceptional bundles and $\mathrm{ext}^1(E_{-\gamma}, E_{-\alpha})$ vanishes by them being an exceptional pair. Thus, we have the vanishing we wanted for both short exact sequences.

We now have $\mathrm{ext}^{\Delta_1}(E_{-\gamma}, U) = 0$ as desired. $\square$

CHAPTER 6

THE KRONECKER FIBRATION

In this section, we use the resolutions constructed in the last chapter to construct a map to a moduli space of Kronecker modules. In the next chapter, these maps produce effective divisors on $M(\xi)$.

One thing that we need to know in order for this to work is that all the homomorphisms in the derived category are morphisms of complexes.

Lemma 6.0.3. *Consider a pair of two term complexes*

$$W = E^* \otimes \mathbb{C}^{m_3} \bigoplus F_0^* \otimes \mathbb{C}^{m_2} \to F_{-1}^* \otimes \mathbb{C}^{m_1}$$

and

$$W' = E^* \otimes \mathbb{C}^{m_3'} \bigoplus F_0^* \otimes \mathbb{C}^{m_2'} \to F_{-1}^* \otimes \mathbb{C}^{m_1'}$$

each sitting in degrees 0 and -1. Every homomorphism $W \to W'$ in the derived category $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ is realized by a homomorphism of the complexes, so

$$\mathrm{Hom}_{D^b(\mathbb{P}^1\times\mathbb{P}^1)}(W, W') = \mathrm{Hom}_{\mathsf{Kom}(\mathbb{P}^1\times\mathbb{P}^1)}(W, W').$$

Similarly, consider a pair of two term complexes

$$W = F_0^* \otimes \mathbb{C}^{m_2} \to F_{-1}^* \otimes \mathbb{C}^{m_1} \bigoplus E^* \otimes \mathbb{C}^{m_0}$$

and

$$W' = F_0^* \otimes \mathbb{C}^{m_2'} \to F_{-1}^* \otimes \mathbb{C}^{m_1'} \bigoplus E^* \otimes \mathbb{C}^{m_0'}$$

each sitting in degrees 0 and -1. Every homomorphism $W \to W'$ *in the derived category* $D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ *is realized by a homomorphism of the complexes, so*

$$\mathrm{Hom}_{D^b(\mathbb{P}^1\times\mathbb{P}^1)}(W, W') = \mathrm{Hom}_{Kom(\mathbb{P}^1\times\mathbb{P}^1)}(W, W').$$

Proof. The proof of each statement are nearly identical to that of Lemma 5.5 in (6) after switching $\mathbb{P}^1 \times \mathbb{P}^1$ in place of $\mathbb{P}^2$ and noting that $\mathrm{Hom}\left(A^a \bigoplus B^b, A^a \bigoplus B^b\right) \cong \mathrm{GL}(a) \times \mathrm{GL}(b) \times M_{b,a}(\mathrm{Hom}(A, B))$ where $\{A, B\}$ is an exceptional pair and where $M_{b,a}(\mathrm{Hom}(A, B))$ is the group of b by a matrices with entries in $\mathrm{Hom}(A, B)$. ☐

Let $\{E_\alpha, E_\beta\}$ be an extremal controlling pair to ξ, $\{F_{-1}, F_0\}$ be the left mutation of the minimally ranked right completion pair of $\{E_\alpha, E_\beta\}$, ξ^+ the primary orthogonal Chern character associated to that exceptional pair $\{E_\alpha, E_\beta\}$, and $U \in M(\xi)$ be a general element. For simplicity, for the rest of the paper, we assume that $\chi(E_\beta^*, U) \geq 0$ and $\chi(E_\alpha^*, U) \leq 0$; proving the other cases is similar. In Chapter 5, we saw that U has a resolution of the form

$$0 \to E_\alpha^*(K)^{m_3} \bigoplus F_0^{*m_2} \to F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0.$$

Using the resolution, we construct dominant rational maps from $M(\xi)$ to different Kronecker moduli spaces.

Which Kronecker moduli space we use, as well as the behavior of the map, depends on which, if any, of the m_i are zero. At most two of the m_i are zero because ξ is not the Chern class of E^f for any exceptional E. We break up the cases by the number of m_i which are zero.

If no m_i is zero, we construct a dominant rational map from $M(\xi)$ to $Kr_N(m_2, m_1)$ where $N = \hom(F_0^*, F_{-1}^*)$.

If exactly one m_i is zero, then we could construct a dominant rational map to a certain Kronecker moduli space, but that space would always be a single point as one part of the dimension vector would be 0. The constant map tells us nothing about our space, so we will not construct it here.

If m_{i_0} and m_{i_1} are zero, we construct a dominant rational map from $M(\xi)$ to $Kr_N(m_{j_1}, m_{j_0})$ where N is the dimension of the appropriate group of homomorphisms and $\{i_0, i_1, j_0, j_1\}$ is some permutation of the set $\{0, 1, 2, 3, \}$ and $j_1 < j_0$.

6.1 The Case When Two Powers Vanish

First note that the cases $m_3 = m_2 = 0$ and $m_1 = m_0 = 0$ cannot occur due to the form of the spectral sequence. This leaves four cases where two exponents vanish to deal with. The proof of the proposition is identical in each case after you replace the two bundles with nonzero exponents so we will only explicitly prove the proposition in the first case.

6.1.1 When the Second and Third Powers Vanish

For this subsection, assume that $m_2 = 0$ and $m_1 = 0$ in the resolution

$$0 \to E_\alpha^*(K)^{m_3} \bigoplus F_0^{*m_2} \xrightarrow{\phi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0.$$

We see that U determines a complex of the form

$$W : E_\alpha^*(K)^{m_3} \to E_\beta^{*m_0}$$

which in turn determines a Kronecker $\text{Hom}(E_\alpha^*(K), E_\beta^*)$-module.

Conversely, given a general such module and its determined complex, W', there exists an element $U' \in M(\xi)$ such that W' is the associated complex of U' by Thm. 5.1.2. Assuming that the Kronecker module associated to a general W is semistable, this constructs a rational map

$$\pi : M(\xi) \dashrightarrow Kr_N(m_3, m_0)$$

where $N = \hom(E^*_\alpha(K), E^*_\beta)$. In order to show that the Kronecker module associated to a general W is semistable, it suffices by Section 2.10 to show that $Kr_N(m_3, m_0)$ is positive dimensional.

Proposition 6.1.1. *With the above notation,* $Kr_N(m_3, m_0)$ *is positive dimensional, and the dominant rational map*

$$\pi : M(\xi) \dashrightarrow Kr_N(m_3, m_0)$$

is a birational map.

Proof. By construction, the primary orthogonal Chern characters to ξ are all semistable so ξ^+ is semistable. By assumption, the general $U \in M(\xi)$ has the resolution

$$0 \to E^*_\alpha(K)^{m_3} \to E_\beta^{*m_0} \to U \to 0.$$

A general map in a complex of the form

$$W : E^*_\alpha(K)^{m_3} \to E_\beta^{*m_0}$$

is injective and has a semistable cokernel with Chern character ξ by Thm. 5.1.2. We also know that any isomorphism of two general elements of $M(\xi)$ is induced by an isomorphism of their resolutions.

Recall from earlier in the chapter that W corresponds to a Kronecker hom$(E^*_\alpha(K), E^*_\beta)$-module e with dimension vector (m_3, m_0). Then we compute that

$$\dim(M(\xi)) = 1 - \chi(U, U) = 1 - \chi(e, e) = (\text{edim})\mathrm{Kr}_N(m_3, m_0).$$

As $\dim(M(\xi)) > 0$, we have that $(\text{edim})\mathrm{Kr}_N(m_3, m_0) > 0$. By the properties of Kronecker moduli spaces, $\mathrm{Kr}_N(m_3, m_0)$ is positive dimensional. Thus, the general such module is stable. As isomorphism of complexes corresponds exactly with isomorphism of Kronecker modules, we obtain a birational map

$$\pi : M(\xi) \dashrightarrow \mathrm{Kr}_N(m_3, m_0).$$

$\square$

6.1.2 When the Second and Fourth Powers Vanish

For this subsection, assume that $m_2 = 0$ and $m_0 = 0$ in the resolution

$$0 \to E^*_\alpha(K)^{m_3} \bigoplus F_0^{*m_2} \xrightarrow{\phi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0.$$

We see that U determines a complex of the form

$$W : E_{-\alpha}(K)^{m_3} \to F_{-1}^{*m_1}$$

which in turn determines a Kronecker Hom$(E_{-\alpha}(K), F^*_{-1})$-module.

Conversely, given a general such module and its determined complex, W', there exists an element $U' \in M(\xi)$ such that W' is the associated complex of U' by Thm. 5.1.2. Assuming that the Kronecker module associated to a general W is semistable, this constructs a rational map

$$\pi : M(\xi) \dashrightarrow Kr_N(m_3, m_1)$$

where $N = \hom(E_{-\alpha}(K), F^*_{-1})$. In order to show that the Kronecker module associated to a general W is semistable, it suffices by Section 2.10 to show that $Kr_N(m_3, m_1)$ is positive dimensional.

Proposition 6.1.2. *With the above notation,* $Kr_N(m_3, m_1)$ *is positive dimensional, and the dominant rational map*

$$\pi : M(\xi) \dashrightarrow Kr_N(m_3, m_1)$$

is a birational map.

6.1.3 When the First and Third Powers Vanish

For this subsection, assume that $m_3 = 0$ and $m_1 = 0$ in the resolution

$$0 \to E^*_\alpha(K)^{m_3} \bigoplus F_0^{*m_2} \xrightarrow{\phi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0.$$

We see that U determines a complex of the form

$$W : F_0^{*m_2} \to E_\beta^{*m_0}$$

which in turn determines a Kronecker $\mathrm{Hom}(F_0^*, E_\beta^*)$-module.

Conversely, given a general such module and its determined complex, W', there exists an element $U' \in M(\xi)$ such that W' is the associated complex of U' by Thm. 5.1.2. Assuming that the Kronecker module associated to a general W is semistable, this constructs a rational map

$$\pi : M(\xi) \dashrightarrow Kr_N(m_2, m_0)$$

where $N = \hom(F_0^*, E_\beta^*)$. In order to show that the Kronecker module associated to a general W is semistable, it suffices by Section 2.10 to show that $Kr_N(m_2, m_0)$ is positive dimensional.

Proposition 6.1.3. *With the above notation,* $Kr_N(m_2, m_0)$ *is positive dimensional, and the dominant rational map*

$$\pi : M(\xi) \dashrightarrow Kr_N(m_2, m_0)$$

is a birational map.

6.1.4 When the First and Fourth Powers Vanish

For this subsection, assume that $m_3 = 0$ and $m_0 = 0$ in the resolution

$$0 \to E_\alpha^*(K)^{m_3} \bigoplus F_0^{*m_2} \xrightarrow{\phi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0.$$

We see that U determines a complex of the form

$$W : F_0^{*m_2} \to F_{-1}^{*m_1}$$

which in turn determines a Kronecker $\mathrm{Hom}(F_0^*, F_{-1}^*)$-module.

Conversely, given a general such module and its determined complex, W', there exists an element $U' \in M(\xi)$ such that W' is the associated complex of U' by Thm. 5.1.2. Assuming that the Kronecker module associated to a general W is semistable, this constructs a rational map

$$\pi : M(\xi) \dashrightarrow Kr_N(m_2, m_1)$$

where $N = \hom(F_0^*, F_{-1}^*)$. In order to show that the Kronecker module associated to a general W is semistable, it suffices by Section 2.10 to show that $Kr_N(m_2, m_1)$ is positive dimensional.

Proposition 6.1.4. *With the above notation,* $Kr_N(m_2, m_1)$ *is positive dimensional, and the dominant rational map*

$$\pi : M(\xi) \dashrightarrow Kr_N(m_2, m_1)$$

is a birational map.

6.2 When All of the Powers Are Nonzero

For this section, assume that $m_i \neq 0$ for all i in the resolution

$$0 \to E_\alpha^*(K)^{m_3} \bigoplus F_0^{*m_2} \xrightarrow{\phi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0.$$

Forgetting most of the information of the resolution, U determines a complex of the form

$$W : F_0^{*m_2} \to F_{-1}^{*m_1}$$

which in turn determines a Kronecker $\mathrm{Hom}(F_0^*, F_{-1}^*)$-module.

Conversely, given a general such module and its determined complex, W', there exists an element $U' \in M(\xi)$ such that W' is the associated complex of U' by Thm. 5.1.2. Assuming that there is a semistable Kronecker module associated to a general W, this constructs a rational map

$$\pi : M(\xi) \dashrightarrow Kr_N(m_2, m_1)$$

where $N = \hom(F_0^*, F_{-1}^*)$. In order to verify that we get Kronecker modules, we have to prove a result about the map $E_\beta^{*m_0} \to U$ and $U \to E_\alpha^{*m_3}(K)[1]$ being canonical. To show that the general associated Kronecker module is semistable, we will again have to show that the Kronecker moduli space is nonempty. We now show that the map $E_\beta^{*m_0} \to U$ is canonical.

Proposition 6.2.1. *With the notation of this section, let $U \in M(\xi)$ be general. Let $W' \in D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ be the mapping cone of the canonical evaluation map*

$$E_\beta^* \otimes \mathrm{Hom}(E_\beta^*, U) \to U,$$

so that there is a distinguished triangle

$$E_\beta^* \otimes \mathrm{Hom}(E_\beta^*, U) \to U \to W' \to \cdot.$$

Then W' is isomorphic to a complex of the form

$$(E_\alpha^*(K) \otimes \mathbb{C}^{m_3}) \bigoplus (F_0^* \otimes \mathbb{C}^{m_2}) \to \left(F_{-1}^* \otimes \mathbb{C}^{m_1}\right)$$

sitting in degrees -1 and 0.

Furthermore, W' is also isomorphic to the complex $(E_\alpha^*(K)\otimes\mathbb{C}^{m_3})\bigoplus(F_0^*\otimes\mathbb{C}^{m_2})\to\left(F_{-1}^*\otimes\mathbb{C}^{m_1}\right)$ *appearing in the Beilinson spectral sequence for* U.

Proof. It is easy to show that if

$$0\to A\to B\oplus C\to D\to 0$$

is an exact sequence of sheaves, then the mapping cone of $C\to D$ is isomorphic to the complex $A\to B$ sitting in degrees -1 and 0.

Suppose we have a resolution

$$0\to E_\alpha^*(K)^{m_3}\bigoplus F_0^{*m_2}\xrightarrow{\phi}F_{-1}^{*m_1}\bigoplus E_\beta^{*m_0}\to U\to 0$$

of a general U as in Thm. 5.1.2. We have $m_0=\hom(E_\beta^*,U)$. Then since U is semistable, the map $E_\beta^{*m_0}\to U$ can be identified with the canonical evaluation $E_\beta^*\otimes\mathrm{Hom}(E_\beta^*,U)\to U$. Thus, the mapping cone of this evaluation is the complex given by the first component of ϕ. By Lemma 6.0.3, any two complexes of the form $E_\alpha^*(K)^{m_3}\bigoplus F_0^{*m_2}\to F_{-1}^{m_1}$ which are isomorphic in the derived category are in the same orbit of the $\mathrm{GL}(m_3)\times\mathrm{GL}(m_2)\times M_{m_2,m_3}(\mathrm{Hom}(E_\alpha^*(K),F_0^*))\times\mathrm{GL}(m_1)$ action.

Finally, we show that W' is isomorphic to the complex which appears in the Beilinson spectral sequence for U. For simplicity, we assume we are in the case $\Delta_1=\Delta_2=1$. We recall how to compute the $E_1^{p,q}$-page of the spectral sequence. Let $p_i:\left(\mathbb{P}^1\times\mathbb{P}^1\right)\times\left(\mathbb{P}^1\times\mathbb{P}^1\right)\to\mathbb{P}^1\times\mathbb{P}^1$ be the projections, and let $\Delta\subset\left(\mathbb{P}^1\times\mathbb{P}^1\right)\times\left(\mathbb{P}^1\times\mathbb{P}^1\right)$ be the diagonal. There is a resolution of the diagonal

$$0\to E_\alpha^*(K)\boxtimes E_\alpha\bigoplus F_0^*\boxtimes E_0\to F_{-1}^*\boxtimes E_1\to E_\beta^*\boxtimes E_\beta\to\mathcal{O}_\Delta\to 0.$$

We split the resolution of the diagonal into two short exact sequences

$$0 \to E_\alpha^*(K) \boxtimes E_\alpha \bigoplus F_0^* \boxtimes E_0 \to F_{-1}^* \boxtimes E_1 \to M \to 0 \text{ and}$$

$$0 \to M \to E_\beta^* \boxtimes E_\beta \to \mathcal{O}_\Delta \to 0.$$

Tensoring with $p_2^*(U)$ and applying Rp_{1*}, we get triangles

$$\Phi_{E_\alpha^*(K)\boxtimes E_\alpha}(U) \bigoplus \Phi_{F_0^*\boxtimes E_0}(U) \to \Phi_{F_{-1}^*\boxtimes E_1}(U) \to \Phi_M(U) \to \cdot \text{ and}$$

$$\Phi_M(U) \to \Phi_{E_\beta^*\boxtimes E_\beta}(U) \to \Phi_{\mathcal{O}_\Delta}(U) \to \cdot,$$

where $\Phi_F : D^b(\mathbb{P}^1 \times \mathbb{P}^1) \to D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ is the Fourier-Mukai transform with kernel F. Computing these transforms using Proposition 5.1.2, we obtain two different complexes

$$E_\alpha^*(K) \otimes H^1(E_\alpha \otimes U) \bigoplus F_0^* \otimes H^0(E_0 \otimes U) \to F_{-1}^* \otimes H^0(E_1 \otimes U) \to \Phi_M(U)[1] \to \cdot \text{ and}$$

$$\Phi_M(U) \to E_\beta^* \otimes \mathrm{Hom}(E_\beta^*, U) \to U \to \cdot$$

involving $\Phi_M(U)$; notice that the map $E_\beta^* \otimes \mathrm{Hom}(E_\beta^*, U) \to U$ is the canonical one since the map

$$E_\beta^* \boxtimes E_\beta \to \mathcal{O}_\Delta$$

is the trace map. Therefore $\Phi_M(U)[1]$ is isomorphic to W by the second triangle. On the other hand, $\Phi_M(U)[1]$ is also isomorphic to the complex in the Beilinson spectral sequence by the first triangle. $\square$

We now turn to showing that the map $U \to E_\alpha^*(K)^{m_0}[1]$ is canonical in order to show that the general W is associated to a Kronecker module.

Proposition 6.2.2. *With the notation of this section, let* $U \in M(\xi)$ *be general. Let* $W' \in D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ *be the mapping cone of the canonical evaluation map*

$$U \to E_\alpha^*(K)[1] \otimes \mathrm{Hom}(U, E_\alpha^*(K)[1]),$$

so that there is a distinguished triangle

$$U \to E_\alpha^*(K)[1] \otimes \mathrm{Hom}(U, E_\alpha^*(K)[1]) \to W' \to \cdot.$$

Then W' *is isomorphic to a complex of the form*

$$(F_0^* \otimes \mathbb{C}^{m_2}) \to \left(F_{-1}^* \otimes \mathbb{C}^{m_1}\right) \bigoplus \left(E_\beta^* \otimes \mathbb{C}^{m_0}\right)$$

sitting in degrees -2 *and* -1.

Furthermore, W' *is also isomorphic to the complex* $(F_0^* \otimes \mathbb{C}^{m_2}) \to \left(F_{-1}^* \otimes \mathbb{C}^{m_1}\right) \bigoplus \left(E_\beta^* \otimes \mathbb{C}^{m_0}\right)$ *appearing in the Beilinson spectral sequence for* U.

Proof. It is easy to show that if

$$0 \to A \oplus B \to C \to D \to 0$$

is an exact sequence of sheaves, then the mapping cone of $D \to A[1]$ is isomorphic to the complex $B \to C$ sitting in degrees -2 and -1. Suppose we have a resolution

$$0 \to E_\alpha^*(K)^{m_3} \bigoplus F_0^{*m_2} \xrightarrow{\phi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0$$

of a general U as in Thm. 5.1.2. We have $m_3 = \hom(U, E_\alpha^*(K)[1])$. Then since U is semistable, the map $U \to E_\alpha^*(K)^{m_3}[1]$ can be identified with the canonical co-evaluation $U \to E_\alpha^*(K)[1] \otimes \mathrm{Hom}(U, E_\alpha^*(K)[1])$. Thus, the mapping cone of this co-evaluation is the complex given by the second component of ϕ. By Lemma 6.0.3, any two complexes of the form $F_0^{*m_2} \to F_{-1}^{m_1} \bigoplus E_\beta^{*m_0}$ which are isomorphic in the derived category are in the same orbit of the $\mathrm{GL}(m_2) \times \mathrm{GL}(m_1) \times \mathrm{GL}(m_0) \times M_{m_0,m_1}(\mathrm{Hom}(F_{-1}^*, E_\beta^*))$ action.

Finally, we show that W' is isomorphic to the complex which appears in the Beilinson spectral sequence for U. For simplicity, we assume we are in the case $\Delta_1 = \Delta_2 = 0$. Recall how to compute the $E_1^{p,q}$-page of the spectral sequence. Let $p_i : \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \times \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \to \mathbb{P}^1 \times \mathbb{P}^1$ be the projections, and let $\Delta \subset \left(\mathbb{P}^1 \times \mathbb{P}^1\right) \times \left(\mathbb{P}^1 \times \mathbb{P}^1\right)$ be the diagonal. There is a resolution of the diagonal

$$0 \to E_\alpha^*(K) \boxtimes E_\alpha \to F_0^* \boxtimes E_0 \to F_{-1}^* \boxtimes E_1 \bigoplus E_\beta^* \boxtimes E_\beta \to \mathcal{O}_\Delta \to 0.$$

We split the resolution of the diagonal into two short exact sequences

$$0 \to F_0^* \boxtimes E_0 \to F_{-1}^* \boxtimes E_1 \bigoplus E_\beta^* \boxtimes E_\beta \to M \to 0 \text{ and}$$

$$0 \to M \to \mathcal{O}_\Delta \to \left(E_\alpha^*(K) \boxtimes E_\alpha\right)[1].$$

Tensoring with $p_2^*(U)$ and applying Rp_{1*}, we get triangles

$$\Phi_{F_0^* \boxtimes E_0}(U) \to \Phi_{F_{-1}^* \boxtimes E_1}(U) \bigoplus \Phi_{E_\beta^* \boxtimes E_\beta}(U) \to \Phi_M(U) \to \cdot \text{ and}$$

$$\Phi_M(U) \to \Phi_{\mathcal{O}_\Delta}(U) \to \Phi_{(E_\alpha^*(K) \boxtimes E_\alpha)[1]}(U) \to \cdot,$$

where $\Phi_F : D^b(\mathbb{P}^1 \times \mathbb{P}^1) \to D^b(\mathbb{P}^1 \times \mathbb{P}^1)$ is the Fourier-Mukai transform with kernel F. Computing these transforms using Thm. 5.1.2, we obtain two different complexes

$$F_0^* \otimes H^1(E_0 \otimes U) \to F_{-1}^* \otimes H^1(E_1 \otimes U) \bigoplus E_\beta^* \otimes \mathrm{Hom}(E_\beta^*, U) \to \Phi_M(U) \to \cdot \text{ and}$$

$$\Phi_M(U) \to U \to E_\alpha^*(K) \otimes \mathrm{Hom}(U, E_\alpha^*(K)[1]) \to \cdot$$

involving $\Phi_M(U)$; notice that the map $U \to E_\alpha^*(K)[1] \otimes \mathrm{Hom}(U, E_\alpha^*(K)[1])$ is the canonical one since the map

$$\mathcal{O}_\Delta \to (E_\alpha^*(K) \boxtimes E_\alpha)[1]$$

is the cotrace map. Therefore, $\Phi_M(U)$ is isomorphic to W' by the second triangle. On the other hand, $\Phi_M(U)$ is also isomorphic to the complex in the Beilinson spectral sequence by the first triangle. $\square$

We now use that these maps are canonical to establish that each resolution is associated to a Kronecker module.

Proposition 6.2.3. *Let $U \in M(\xi)$ be a general object with the resolution*

$$0 \to E_\alpha^*(K)^{m_3} \bigoplus F_0^{*m_2} \to F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to U \to 0$$

with subcomplex

$$W : F_0^{*m_2} \to F_{-1}^{*m_1}.$$

Then W is the complex appearing in the spectral sequence.

Proof. Let $W' : F_0^{*m_2} \to F_{-1}^{*m_1}$ be the complex appearing in U' spectral sequence. By a Bertini-like statement (42), W' is either surjective or injective.

Assume that it is injective. This means that the E_2 page of the spectral sequence is

$$\begin{array}{cccc} E_\alpha^*(K)^{m_3} & 0 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & E_\beta^{*m_0} \end{array}$$

where $K = \text{coker}(W')$. In turn this gives that, in the resolution coming from the spectral sequence, the map from $F_0^{*m_2}$ to $E_\beta^{*m_0}$ is zero. Then, by Prop. 6.2.2, we have two short exact sequences

$$0 \to F_0^{*m_2} \xrightarrow{\phi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to L \to 0 \text{ and}$$

$$0 \to F_0^{*m_2} \xrightarrow{\psi} F_{-1}^{*m_1} \bigoplus E_\beta^{*m_0} \to L \to 0$$

where W and W' are subcomplexes of the respective sequences and the second component of ψ is zero. The identity map on L induces an isomorphism of its resolutions, which implies that the the first component of ϕ is a scalar multiple of the first component of ψ. This then gives an isomorphism between W and W' given by dividing by that scalar multiple in the degree -1 and the identity in degree 0. Thus, W is the complex in the spectral sequence converging to U.

The case of surjectivity is similar but uses Prop. 6.2.1. □

We now finish the construction of the map $M(\xi) \dashrightarrow Kr_N(m_2, m_1)$.

Proposition 6.2.4. *With the above notation,* $Kr_N(m_2, m_1)$ *is nonempty and there is a dominant rational map*

$$\pi : M(\xi) \dashrightarrow Kr_N(m_2, m_1).$$

Proof. By construction, we know that the primary orthogonal Chern characters to ξ are all semistable so ξ^+ is semistable. Let $V \in M(\xi^+)$ be general. Then, by Thm. 5.1.2, V has the resolution

$$0 \to E_1^{n_1} \to V \to E_0^{n_2} \to 0$$

or one of the equivalent resolutions which have the same Kronecker module structure. Similarly, the general $U \in M(\xi)$ has the resolution

$$E_\beta^{*m_0} \to U \to \left(E_\alpha^*(K)^{m_3} \bigoplus W\right)[1]$$

where W is the complex

$$F_0^{*m_2} \to F_{-1}^{*m_1}.$$

As the point (μ^+, Δ^+) lies on the surface Q_ξ, we have that $\chi(V^*, U) = 0$. By design, the resolution of V immediately forces $\chi(V^*, E_\beta^*) = 0$ and $\chi(V^*, E_\alpha^*(K)) = 0$ because of the orthogonality properties of the coil $\{E_\alpha^*(K), E_\beta^*, E_1^*, E_0^*\}$. Then vanishings of $\chi(V^*, U)$, $\chi(V^*, E_\beta^*)$, and $\chi(V^*, E_\alpha^*(K))$ force $\chi(V^*, W) = 0$. Since $\chi(V^*, W) = 0$ and $\chi(E_\beta^*, W) = 0$, we have that $\chi(L_{E_\beta^*} V^*, W) = 0$. Shifting only shifts the indices so we have $\chi(L_{E_\beta^*} V^*[1], W) = 0$ as well. In the derived category $L_{E_\beta^*} V^*[1]$ is isomorphic to the complex

$$F_0^{*n_1} \to F_{-1}^{*n_2}$$

sitting in degrees -1 and 0. Thus, $L_{E_\beta^*} V^*[1]$ and W both correspond to Kronecker $\mathrm{Hom}(F_0^*, F_1^*)$-modules. Call them e and f respectively.

Then $\chi(L_{E_\beta^*} V^*[1], W) = 0$ tells us that $\chi(e, f) = 0$ which implies that $\underline{\dim}\ f$ if a right-orthogonal dimension vector to $\underline{\dim}\ e$. Since $M(V)$ is nonempty, Prop. 6.1.1 shows that $Kr_N(n_1, n_2)$ is nonempty.

If $\mathsf{Kr}_N(\mathfrak{n}_1, \mathfrak{n}_2)$ is positive(0) dimensional, the discussion at the end of Section 6.1 of (6) shows that $\mathsf{Kr}_N(\mathfrak{m}_2, \mathfrak{m}_1)$ is as well. Thus, $\mathsf{Kr}_N(\mathfrak{m}_2, \mathfrak{m}_1)$ is nonempty as promised.

$\square$

CHAPTER 7

PRIMARY EXTREMAL RAYS OF THE EFFECTIVE CONE

In this chapter, we use the maps from $M(\xi)$ to Kronecker moduli spaces that we constructed in the previous chapter to give an alternate description of effective Brill-Noether divisors and to show that they are extremal. Let ξ^+ a primary orthogonal Chern character to $M(\xi)$ with $V \in M(\xi^+)$ general. The way in which we can express the Brill-Noether divisor D_V depends greatly on the dimension of the Kronecker moduli space, K, that we map to (as dictated by the previous chapter).

If $\dim(K) = 0$ (or a single m_i is zero so we did not construct a fibration), then the Kronecker fibration is a map to a point so it does not give us any information so we do not use it at all. In this case, V is an exceptional bundle. This divisor consists exactly of those elements in $M(\xi)$ without the specified resolution, and the dual moving curve(s) are found by varying the maps in the resolution.

If $\dim(K) > 0$, then the Kronecker fibration is far more interesting. In this case, ξ^+ may or may not be exceptional and the Brill-Noether divisor D_V is the indeterminancy or exceptional locus of the map from $M(\xi)$ to the Kronecker moduli space. Either this map is birational, in which case the moving curve is gotten by varying the Kronecker module, or the map has positive dimensional fibers, in which case the moving curve(s) are gotten by varying the other maps in the resolution to cover the fibers of the map. If certain numeric inequalities hold, there are two dual moving curves covering the (positive dimensional) fibers of the map which implies that D_V is the pullback of a generator of the ample cone of the Kronecker moduli space; in the case, the Brill-Noether divisor D_V is also inside the movable cone.

Let $\{E_\alpha, E_\beta\}$ be an associated extremal pair to ξ with orthogonal Chern character ζ, $\{F_{-1}, F_0\}$ be the left mutation of the minimally ranked right completion pair of $\{E_\alpha, E_\beta\}$, $U \in M(\xi)$ be a general element, and K be the Kronecker moduli space containing the Kronecker module appearing in the resolution of U.

7.1 The Zero Dimensional Kronecker Moduli Space Case

Theorem 7.1.1. *Let ξ^+ be a primary orthogonal Chern character to $\{\alpha, \beta\}$ for the Chern character ξ with* $\dim(K) = 0$ *and let $V \in M(\xi^+)$ be the element. Then the Brill-Noether divisor*

$$D_V = \{U' \in M(\xi) : h^1(U' \otimes V) \neq 0\}$$

is on an edge of the effective cone of $M(\xi)$. Using the isomorphism $\mathrm{NS}\,(M(\xi)) \cong \xi^{\perp}$, *$D_V$ corresponds to ξ^+.*

Proof. The general element $U \in M(\xi)$ fits into the short exact sequence

$$0 \to E_\alpha^*(K)^{m_3} \bigoplus F_0^{*m_2} \to F_{-1}^{*m_1} \bigoplus \left(E_\beta^*\right)^{m_0} \to U \to 0$$

In order to show that D_V is an effective Brill-Noether divisor, we have to show that V is cohomologically orthogonal to U. This means showing that $U \otimes V = \mathscr{H}om(U^*, V)$ has no cohomology. How we show this orthogonality depends upon which if any of the m_i vanish. Note that $\dim(K) = 0$ implies that at most one of the m_i is zero because if two are zero then $\dim(K) = \dim(M(\xi))$.

Assume that none of the m_i are zero. Then the general element, $U \in M(\xi)$, fits into the triangle

$$\left(E_\beta^*\right)^{m_0} \to U \to E_\alpha^*(K)^{m_3}[1] \bigoplus W$$

where W is the complex $(F_0^*)^{m_2} \to \left(F_{-1}^*\right)^{m_1}$ sitting in degrees -1 and 0. Similarly, the general element, $V \in M(\xi^+)$, fits into the triangle

$$E_1^{n_2} \to V \to E_0^{n_1}.$$

By choice of resolving exceptional bundles, $\mathscr{H}om(E_\beta^*, V)$ and $\mathscr{H}om(E_\alpha^*(K), V) = 0$ have no cohomology. Thus, to construct the divisor it suffices to show that $\mathscr{H}om(W^*, V)$ has no cohomology. As $\mathscr{H}om(W^*, E_\beta)$ has no cohomology, this is equivalent to $\mathscr{H}om(W^*, R_\beta V)$ having no cohomology. We reduce further to showing that $\mathscr{H}om(W^*, R_\beta V[1])$ has no cohomology as shifting merely shifts the cohomology. Then if f and e are the Kronecker modules corresponding to W^* and $R_\beta V[1]$, respectively, the vanishing of these cohomologies is equivalent to the vanishing of the Hom(f, e), but that vanishing follows directly from Thm. 6.1 of (6). Thus, we have the orthogonality that we needed.

If one of the m_i is zero (in which case we have not constructed a Kronecker fibration explicitly), then V is one of the exceptional bundles E_β, E_1, E_0, or E_α. Then V is cohomologically orthogonal to all three bundles that appear in the resolution of U so it is automatically cohomologically orthogonal.

Thus, we have shown the cohomological orthogonality in either case. The class of D_V and the fact that it is effective is computed using Prop. 2.8.3.

To show it lies on an edge, we construct a moving curve by varying a map in the resolution.

If $m_3 \neq 0$ and $m_0 \neq 0$, fix every map except $E_\alpha^*(K)^{m_3} \to \left(E_\beta^*\right)^{m_0}$, let

$$S = \mathbb{P}\mathrm{Hom}\left(E_\alpha^*(K)^{m_3}, \left(E_\beta^*\right)^{m_0}\right),$$

and let $\mathcal{U}/S$ be the universal cokernel sheaf (of the fixed map plus the varying part).

If $m_3 = 0$, fix every map except $F_0^{*m_2} \to \left(E_\beta^*\right)^{m_0}$, let

$$S = \mathbb{P}\mathrm{Hom}\left(F_0^{*m_2}, \left(E_\beta^*\right)^{m_0}\right),$$

and let $\mathcal{U}/S$ be the universal cokernel sheaf (of the fixed map plus the varying part).

If $m_0 = 0$, fix every map except $E_\alpha^*(K)^{m_3} \to (F_0^*)^{m_1}$, let

$$S = \mathbb{P}\mathrm{Hom}\left(E_\alpha^*(K)^{m_3}, (F_0^*)^{m_1}\right),$$

and let $\mathcal{U}/S$ be the universal cokernel sheaf (of the fixed map plus the varying part).

In any case, we have our $\mathcal{U}$ and our S. Because $M(\xi)$ is positive dimensional and the general sheaf in it has a resolution of this form, S is nonempty. Then $\mathcal{U}$ is a complete family of prioritary sheaves whose fixed Chern character lies above the δ surface. Define the open set $S' \subset S$ by

$$S' := \{s \in S : \mathcal{U}_s \text{ is stable}\}$$

Thus, by assumption, the complement of S' has codimension at least 2 which allows us to find a complete curve in S' containing the point corresponding to U for the general element $U \in M(\xi)$. Notice that this is a moving curve by the codimension statement. Any curve in S' is disjoint from D_V which makes the curve dual to it.

This curve makes D_V be on an edge. As the resolution only provides one moving curve, this resolution only shows that it lies on an edge of the cone, not that it is an extremal ray. □

7.2 The Positive Dimensional Kronecker Moduli Space Case

Theorem 7.2.1. *Let ξ^+ be a primary orthogonal Chern character to $\{\alpha, \beta\}$ for the Chern character ξ with $\dim(K) > 0$ and $V \in M(\xi^+)$ be a general element. Then the Brill-Noether divisor*

$$D_V = \{U' \in M(\xi) : h^1(U' \otimes V) \neq 0\}$$

lies on the edge of the effective cone of $M(\xi)$*. Using the isomorphism* $\mathrm{NS}\,(M(\xi)) \cong \xi^{\perp}$*,* D_V *corresponds to* ξ^{+}*.*

Proof. Recall that we have a dominant rational map $\pi : M(\xi) \dashrightarrow K$, There are two possibilities; either π is a birational map or π has positive dimensional fibers.

Birational Case In this case, either zero or two of the m_i can vanish. If zero vanish, we show that V is cohomologically orthogonal to U by the same arguments as the previous theorem. If two vanish, then V is again one of the exceptional bundles so orthogonality is immediate. The class of D_V and the fact that it is effective is computed using Prop. 2.8.3, and D_V is the exceptional locus of π. Using that fact, we get a dual moving curve to D_V by varying the Kronecker module. Formally, because K is Picard rank one, there is a moving curve C. Then $[\pi^*(C)]$ is a moving curve which is dual to the exceptional locus of π (i.e. dual to D_V). Thus, D_V is on the edge of the effective cone.

Positive dimensional fiber case In this case, none of the m_i is zero.

We first show cohomological orthogonality. This means showing that $U \otimes V = \mathscr{H}om(U^*, V)$ has no cohomology. In this case, the general element, $U \in M(\xi)$, fits into the triangle

$$\left(E_{\beta}^{*}\right)^{m_0} \to U \to E_{\alpha}^{*}(K)^{m_3}[1] \bigoplus W$$

where W is the complex $\left(F_0^*\right)^{m_2} \to \left(F_{-1}^*\right)^{m_1}$ sitting in degrees -1 and 0. Similarly, the general element, $V \in M(\xi^+)$, fits into the triangle

$$E_1^{n_2} \to V \to E_0^{n_1}.$$

By choice of resolving exceptional bundles, $\mathscr{H}om(E_{\beta}^*, V)$ and $\mathscr{H}om(E_{\alpha}^*(K), V)$ have no cohomology. Thus, to construct the divisor it suffices to show that $\mathscr{H}om(W^*, V)$ has no cohomology. As $\mathscr{H}om(W^*, E_{\beta})$ has no cohomology, we have that this is equivalent to $\mathscr{H}om(W^*, R_{\beta}V)$ having no cohomology. We reduce further to showing that $\mathscr{Hom}(W^*, R_\beta V[1])$ has no cohomology as shifting merely shifts the cohomology. Then if f and e are the Kronecker modules corresponding to W^* and $R_\beta V[1]$, respectively, the vanishing of these cohomologies is equivalent to the vanishing of the $\mathrm{Hom}(f, e)$, but that vanishing follows directly from Thm. 6.1 of (6)

We have now established the cohomological orthogonality. The class of D_V and the fact that it gives an effective divisor are computed using Prop. 2.8.3.

As the general U has the given resolution, the fibers of map to K are covered by varying the other maps of the resolution as we did in the last proof. As these moving curves sit inside fibers, they are dual to D_V since V is dual to the Kronecker modules in K.

If we can vary two different maps in the resolution other than the Kronecker module independently, than D_V has the same class as the pullback of an ample divisor of K. This immediately implies that D_V is in the moving cone. We can vary two maps independently if the no m_i is zero and no subcomplex of the resolution has enough dimensions to account for all of the dimensions of our moduli space. $\square$

These theorems together give an effective divisor on $M(\xi)$. These conjecturally might give a spanning set of effective divisors for effective cone of $M(\xi)$, but even if this method does not do that, it gives a way to construct effective divisors on many of these moduli spaces. In addition, this same method works to give secondary extremal rays when the rank of the moduli space is at least three (for rank less than three all secondary rays have special meaning).

CHAPTER 8

EXAMPLES

The method of the previous chapter constructs Brill-Noether divisors on the faces of the effective cone for moduli spaces of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$. In this chapter, we work out a series of examples showing the usefulness of our theorems. We work out the effective cones of the first fifteen Hilbert schemes of points as well as some series of extremal rays that occur for infinitely many Hilbert schemes of points on $\mathbb{P}^1 \times \mathbb{P}^1$. Lastly, we provide an extremal edge for the effective cone of a moduli space of rank two sheaves with nonsymmetric slope so that we see the theorems are useful in that setting as well.

8.1 The Effective Cones of Hilbert Schemes of at Most Sixteen Points

The most classical example of a moduli space of sheaves on $\mathbb{P}^1 \times \mathbb{P}^1$ is Hilbert scheme of n points on it. For these Hilbert schemes, the Picard group has a classical basis, $\{B, H_1, H_2\}$. Each element of this basis has an extremely geometric interpretation. B is the locus of nonreduced schemes or equivalently the schemes supported on $n-1$ or fewer closed points. H_1 is the schemes whose support intersects a fixed line of type $(1,0)$. Similarly, H_2 is the schemes whose support intersects a fixed line of type $(0,1)$.

Using this basis, every ray in the Néron-Severi space is spanned by a ray of the form B, aH_1+bH_2+B, aH_1+bH_2, or $iH_1+jH_2-\frac{B}{2}$. Then we fix the notation for the last two types of ray as

$$Y_{a,b} = aH_1 + bH_2 \text{ and}$$

$$X_{i,j} = iH_1 + jH_2 - \frac{1}{2}B.$$

Using this notation, we list the extremal rays of the effective cones of $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]}$ for $n \leq 16$, explicitly work out the case of $n=7$, prove that some sequences of rays are extremal for varying n, and then finally explicitly work out each remaining extremal ray for $n \leq 16$. There are all new results except for the cases of $n \leq 5$. Cross-sections of these cones are given in 9.

n	Extremal Rays
2	B, $X_{1,0}$, and $X_{0,1}$
3	B, $X_{2,0}$, and $X_{0,2}$
4	B, $X_{3,0}$, $X_{1,1}$, and $X_{0,3}$
5	B, $X_{4,0}$, $X_{\frac{4}{3},\frac{4}{3}}$, and $X_{0,4}$
6	B, $X_{5,0}$, $X_{2,1}$, $X_{1,2}$, and $X_{0,5}$
7	B, $X_{6,0}$, $X_{\frac{12}{5},\frac{6}{5}}$, $X_{2,\frac{3}{2}}$, $X_{\frac{3}{2},2}$, $X_{\frac{6}{5},\frac{12}{5}}$, and $X_{0,6}$
8	B, $X_{7,0}$, $X_{3,1}$, $X_{1,3}$, and $X_{0,7}$
9	B, $X_{8,0}$, $X_{\frac{24}{7},\frac{8}{7}}$, $X_{2,2}$, $X_{\frac{8}{7},\frac{24}{7}}$, and $X_{0,8}$
10	B, $X_{9,0}$, $X_{4,1}$, $X_{\frac{5}{2},2}$, $X_{2,\frac{5}{2}}$, $X_{1,4}$, and $X_{0,9}$
11	B, $X_{10,0}$, $X_{\frac{40}{9},\frac{10}{9}}$, $X_{4,\frac{4}{3}}$, $X_{\frac{12}{5},\frac{12}{5}}$, $X_{\frac{4}{3},4}$, $X_{\frac{10}{9},\frac{40}{9}}$,and $X_{0,10}$
12	B, $X_{11,0}$, $X_{5,1}$, $X_{3,2}$, $X_{2,3}$, $X_{1,5}$, and $X_{0,11}$
13	B, $X_{12,0}$, $X_{\frac{60}{11},\frac{12}{11}}$, $X_{\frac{9}{2},\frac{3}{2}}$, $X_{\frac{7}{2},2}$, $X_{\frac{8}{3},\frac{8}{3}}$, $X_{2,\frac{7}{2}}$, $X_{\frac{3}{2},\frac{9}{2}}$, $X_{\frac{12}{11},\frac{60}{11}}$, and $X_{0,12}$
14	B, $X_{13,0}$, $X_{6,1}$, $X_{\frac{10}{3},\frac{7}{3}}$, $X_{\frac{7}{3},\frac{10}{3}}$, $X_{1,6}$, and $X_{0,13}$
15	B, $X_{14,0}$, $X_{\frac{84}{13},\frac{14}{13}}$, $X_{4,2}$, $X_{2,4}$, $X_{\frac{14}{13},\frac{84}{13}}$, and $X_{0,14}$
16	B, $X_{15,0}$, $X_{7,1}$, $X_{\frac{9}{2},2}$, $X_{3,3}$, $X_{2,\frac{9}{2}}$, $X_{1,7}$, and $X_{0,15}$

8.2 The Effective Cone of the Hilbert Scheme of 7 Points

It is worth showing how the theorem is applied in one these cases to compute the effective cone. Recall that the general strategy to compute an effective cone has two steps. First, provide effective divisors. Second, provide moving curves which are dual to the effective divisors.

We use our main theorem to do this for the primary extremal rays of the effective cone; we have to deal with the secondary extremal rays separately. There is a single secondary extremal ray which is spanned by B. B is clearly an effective divisor as it is the locus of nonreduced schemes. In order to show that B spans an extremal ray, we just have to construct two distinct dual moving curves.

We now construct these moving curves, C_1 and C_2. We construct C_1 by fixing 6 general points and then varying a seventh point along a curve of type $(1,0)$. Similarly, we construct C_2 by fixing 6 general points and then varying a seventh point along a curve of type $(0,1)$. Any set of 7 distinct points lies on at least one curve of type C_1, so it is a moving curve. Similarly, C_2 is a moving curve.

We now show that C_1 and C_2 are dual to B. Starting with six general points, we can find a line l of type $(1,0)$ that does not contain any of the points. We get a curve C' of type C_1 in the Hilbert scheme by varying the seventh point along l. As l does not contain any of the six general points, every point in C' corresponds to seven distinct points, so C' does not intersect B. Thus, we get

$$C_1 \cdot B = C' \cdot B = 0.$$

Similarly, we get that

$$C_2 \cdot B = 0.$$

The only thing left to do in order to show that B spans an extremal ray is to show that C_1 and C_2 have distinct classes. Starting with six general points, we find lines l and l' of type $(1,0)$ that does not contain any of the points. Again, we get a curve C' of type C_1 in the Hilbert scheme by varying the seventh point along l. Analogously, we get a divisor H' of type H_1 as the locus of schemes whose support intersects l'. As l' does not contain any of the general fixed points and does not intersect l, we get that H_1 and C' are disjoint. Thus,

$$C_1 \cdot H_1 = C' \cdot H' = 0.$$

Using the same six general points, we find a line l_0 of type $(0, 1)$ that does not contain any of the points. We get a curve C_0 of type C_2 by varying the seventh point along l_0. Then l_0 does not contain any of the six general points by construction but does intersect l' in exactly one point. Thus,

$$C_2 \cdot H_1 = C_0 \cdot H' = 1.$$

As $C_1 \cdot H_1 \neq C_2 \cdot H_1$, we know that C_1 and C_2 are distinct classes. This observation completes the proof that B spans an extremal ray.

While constructing the primary extremal rays, we will construct two moving curves dual to B. These curves will show that B is the only secondary extremal ray. Also as the slope of the ideal sheaf is $(0, 0)$, the effective cone is symmetric in the coordinates of H_1 and H_2 so we only deal with the primary rays spanned by $X_{i,j}$ where $i \geq j$. Keeping that in mind, we move to computing the primary extremal rays using our theorem.

One way to think about the main results of this paper are that they give an algorithm to compute the primary extremal rays of the effective cone of $M(\xi)$. That algorithm breaks down roughly into four steps: find the extremal pairs, use the extremal pairs to resolve the general object of $M(\xi)$, use those resolutions to construct maps to moduli spaces of Kronecker modules, and analyze these maps to find divisors spanning extremal rays. Let's follow those steps in this specific case.

Step 1

As we noted above, the first step is to find all of the extremal pairs. Proceed by finding all controlling exceptional bundles, finding the controlling pairs, and then finding which are extremal pairs.

Controlling exceptional bundles are those controlling the δ-surface over the locus

$$\left\{X \in \left(1, \mu, \frac{1}{2}\right) \subset K(\mathbb{P}^1 \times \mathbb{P}^1) : \chi(X \otimes \mathcal{I}_z) = 0 \text{ for } \mathcal{I}_z \in \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]}\right\}.$$

Using Mathematica, we find that these controlling exceptional bundles are

$$\{\cdots,\{0,4,1\},\{0,5,1\},\{0,6,1\},\{0,7,1\},\{0,8,1\},\{0,9,1\},\{0,10,1\},\{0,11,1\},\{0,12,1\},\{0,13,1\},$$

$$\{0,14,1\},\{1,27,5\},\{1,13,3\},\{2,11,3\},\{1,2,1\},\{1,3,1\},\{1,4,1\},\{6,12,5\},\{2,1,1\},\{2,2,1\},\cdots\}$$

where we record an exceptional bundle with Chern character $(r,(\mu_1,\mu_2),\Delta)$ as (μ_1,μ_2,r) and we truncate the list when bundles can no longer possibly matter. We will see that the ones we have truncated do not matter as our first resolution will be dual to B.

There are many, many controlling exceptional pairs, but we do not need to see all of them.

Finally, we check to see which of these are extremal pairs. They are whittled down by eliminating each pair that does not have each of the properties of an extremal pair. The only four controlling pairs that are extremal pairs are

$$\{\mathcal{O}(6,0),\mathcal{O}(7,0)\},\ \{\mathcal{O}(3,1),\mathcal{O}(6,0)\},\ \{\mathcal{O}(2,1),\mathcal{O}(3,1)\},\text{ and }\{\mathcal{O}(2,1),\mathcal{O}(2,2)\}.$$

Each extremal pair controls an extremal ray of the effective cone. Recall that given an extremal pair $\{A,B\}$, the extremal ray it corresponds to is spanned by the primary orthogonal Chern character of the pair: $\mathrm{ch}(A)$, $\mathrm{ch}(B)$, or

$$p=\{X\in K(\mathbb{P}^1\times\mathbb{P}^1):Q_{\xi,A}(X)=Q_{\xi,B}(X)=\chi(\mathcal{I}_z\otimes X)=0\}.$$

Then the primary orthogonal Chern character for our exceptional pairs are $(1,(6,0),0)$, $(1,(6,0),0)$, $(5,(12,6),12)$, and $(2,(4,3),5)$, respectively. These Chern characters correspond to the extremal rays $X_{6,0}$, $X_{6,0}$, $X_{\frac{12}{5},\frac{6}{5}}$, and $X_{2,\frac{3}{2}}$, respectively. Notice that one of the rays is repeated twice. This repetition is because we need all of these extremal pairs to share each of their elements with another extremal pair in order to link neighboring extremal rays with moving curves.

Step 2

The next step in computing the effective cone is to turn the extremal pairs into resolutions of the general element of the Hilbert scheme. We will use Thm. 5.1.2 and Thm. 5.3.2 to get these resolutions. To apply those theorems, we have to complete the pairs to coils as described in Chapter 5. This approach gives the coils

$$\{\mathcal{O}(-7,-1), \mathcal{O}(-6,-1), \mathcal{O}(-7,-0), \mathcal{O}(-6,0)\}$$

$$\{\mathcal{O}(-7,-2), \mathcal{O}(-4,-1), \mathcal{O}(-3,-1), \mathcal{O}(-6,0)\},$$

$$\{\mathcal{O}(-4,-3), \mathcal{O}(-4,-2), \mathcal{O}(-3,-2), \mathcal{O}(-3,-1)\}, \text{ and}$$

$$\{\mathcal{O}(-4,-3), \mathcal{O}(-3,-2), \mathcal{O}(-3,-1), \mathcal{O}(-2,-2)\},$$

respectively.

Given these coils, we get the resolutions we wanted using the spectral sequence as in the proofs of the relevant theorems. Following the proof, we get the resolutions

$$0 \to \mathcal{O}(-7,-1)^7 \to \mathcal{O}(-6,-1)^7 \bigoplus \mathcal{O}(-7,0) \to \mathcal{I}_z \to 0,$$

$$0 \to \mathcal{O}(-7,-2) \to \mathcal{O}(-4,-1) \bigoplus \mathcal{O}(-3,-1) \to \mathcal{I}_z \to 0,$$

$$0 \to \mathcal{O}(-4,-3) \bigoplus \mathcal{O}(-4,-2)^2 \to \mathcal{O}(-3,-2)^3 \bigoplus \mathcal{O}(-3,-1) \to \mathcal{I}_z \to 0, \text{ and}$$

$$0 \to \mathcal{O}(-4,-3) \bigoplus \mathcal{O}(-3,-2) \to \mathcal{O}(-3,-1) \bigoplus \mathcal{O}(-2,-2)^2 \to \mathcal{I}_z \to 0,$$

respectively.

Step 3

We now get to the third step in the process, turning the resolutions into maps to Kronecker moduli spaces. There are no Kronecker modules that are used in the first two resolutions and the Kronecker module in each of the last two resolutions are

$$\mathcal{O}(-4,-2)^2 \to \mathcal{O}(-3,-2)^3 \text{ and}$$

$$\mathcal{O}(-3,-2) \to \mathcal{O}(-3,-1),$$

respectively.

This means that we have the maps

$$\pi_1 : \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[7]} \dashrightarrow \mathsf{Kr}_{\hom(\mathcal{O}(3,2),\mathcal{O}(4,2))}(3,2), \text{ and}$$

$$\pi_2 : \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[7]} \dashrightarrow \mathsf{Kr}_{\hom(\mathcal{O}(3,1),\mathcal{O}(3,2))}(1,1),$$

respectively.

Note that the dimensions of these Kronecker moduli spaces are 0 and 1, respectively, so we will only consider the map in the last case.

Step 4

The fourth and final step is actually computing the effective divisors and their dual moving curves. Let $D = aH_1 + bH_2 - c\frac{B}{2}$ be a general effective divisor.

In the first case, the Brill-Noether divisor is D_V where $V = \mathcal{O}(6,0)$. The moving curve comes from a pencil of maps $\mathcal{O}(-7,-1)^7 \to \mathcal{O}(-6,-1)^7$. The restriction this moving curve places on D is that $b \geq 0$. In particular, B and $X_{6,0}$ are dual to this moving curve.

In the second case, the Brill-Noether divisor is D_V where $V = \mathcal{O}(6,0)$. The moving curve comes from a pencil of maps $\mathcal{O}(-7,-2) \to \mathcal{O}(-4,-1)$. The restriction this moving curve places on D is that $3b \geq 6 - a$. In particular, $X_{6,0}$ and $X_{\frac{12}{5},\frac{6}{5}}$ are dual to this moving curve.

For π_1, the Brill-Noether divisor is D_V where V is the exceptional bundle $E_{\frac{12}{5},\frac{6}{5}}$. Notice that in this case, the Kronecker fibration is a map to a point. This implies that the divisor D_V is rigid. The two types of moving curve come from pencils of maps $\mathcal{O}(-4,-3) \to \mathcal{O}(-3,-2)^3$ and $\mathcal{O}(-4,-2)^2 \to \mathcal{O}(-3,-1)$. These are dual to D_V by the resolution

$$0 \to \mathcal{O}(3,0)^4 \to E^3_{\frac{8}{3},\frac{2}{3}} \to V \to 0$$

since $\chi((3,2),(4,3)) = 12 + 6 - 2 * 3 * 3 = 0$. The restriction these two moving curves place on D are that $3b \geq 6 - a$ and $4b \geq 12 - 3a$. In particular, $X_{\frac{12}{5},\frac{6}{5}}$ is dual to both moving curves and $X_{2,\frac{3}{2}}$ is dual to the second moving curve.

For π_2, the Brill-Noether divisor is D_V where V is a bundle $F_{2,\frac{3}{2}}$ that has Chern character $(2,(4,3),5)$. The two types of moving curve covering each fiber come from pencils of maps $\mathcal{O}(-4,-3) \to K$ and $\mathcal{O}(-4,-3) \to \mathcal{O}(-2,-2)^2$. These are dual to D_V by the resolution

$$0 \to V \to E_{\frac{7}{3},\frac{4}{3}} \to \mathcal{O}(3,1) \to 0$$

since $\chi((1,1),(1,1)) = 2*1*1-1*1-1*1+1*1 = 0$. The restriction these two moving curve place on D are that $4b \geq 12-3a$ and $2b \geq 7-2a$. In particular, $X_{2,\frac{3}{2}}$ is dual to both moving curves, $X_{\frac{12}{5},\frac{6}{5}}$ is dual to the first moving curve, and $X_{\frac{3}{2},2}$ is dual to the second moving curve.

We have now exhibited 7 effective divisors $\left(B, X_{6,0}, X_{\frac{12}{5},\frac{6}{5}}, X_{2,\frac{3}{2}}, X_{\frac{3}{2},2}, X_{\frac{6}{5},\frac{12}{5}}, X_{0,6},\right)$ and 7 moving curves that are dual to each pair of extremal rays that span a face of the effective cone. Taken together, these divisors and moving curves determine the effective cone.

8.3 Infinite Series of Extremal Rays

As another example of the power of the methods produced in this paper, we can construct an extremal ray in the Hilbert scheme of n points for infinite sequences of n. We provide two extremal rays for three such sequences and one extremal ray for a fourth sequence. The strategy for each proof is to first find an extremal pair, then use the process outlined by our theorems to show that they give the desired extremal ray(s).

The first sequence we look at is actually just all n. For this sequence, we prove what the edges of the effective cone that share the secondary extremal ray are.

Proposition 8.3.1. *The edge spanned by* $X_{n-1,0}$ *and* B *is an extremal edge of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]}$. *Similarly, the edge spanned by* $X_{n-1,0}$ *and* B *is an extremal edge of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]}$.

Proof. This was proved for $n \leq 5$ in (31). It is immediate from the symmetry of the effective cone in terms of a and b that the second statement is immediate from the first statement. We now prove the first statement.

The first step in proving this edge is an extremal ray is finding an extremal pair. To find an extremal pair, we first have to find the two controlling exceptional bundles that will make up the pair. The vector bundle whose Brill-Noether divisor will span the ray $X_{n-1,0}$ is $\mathcal{O}(n-1,0)$. To find our controlling exceptional bundles, we need to find exceptional bundles cohomologically orthogonal to $\mathcal{O}(n-1,0)$. Then,

$$\chi(\mathcal{O}(n-1,1),\mathcal{O}(n-1,0)) = 0 \text{ and } \chi(\mathcal{O}(n,0),\mathcal{O}(n-1,0)) = 0.$$

Then it is easy to see that $\mathcal{O}(n-1,1)$ and $\mathcal{O}(n,0)$ are controlling exceptional bundles for the Hilbert scheme, and the pair $\{\mathcal{O}(n-1,1),\mathcal{O}(n,0)\}$ is an extremal pair.

Once we have the extremal pair, we need to turn it into a resolution of the general object of the Hilbert scheme. We complete the pair to a coil as prescribed by Thm. 5.3.2. This completion gives the coil

$$\{\mathcal{O}(-n,-1), E_{\frac{-2-3(n-1)}{3},\frac{-2}{3}}, \mathcal{O}(-n+1,-1), \mathcal{O}(-n,0)\}.$$

Next, we use the Beilinson spectral sequence to resolve the general ideal sheaf. The spectral sequence gives the resolution

$$0 \to \mathcal{O}(-n,-1)^n \to \mathcal{O}(-n+1,-1)^n \bigoplus \mathcal{O}(-n,0) \to \mathcal{I}_Z \to 0.$$

The moving curves are pencils in the space $\mathrm{Hom}(\mathcal{O}(-n,-1),\mathcal{O}(-n+1,-1))$. The restriction this moving curve places on D is that $b \geq 0$. In particular, B and $X_{n-1,0}$ are dual to this moving curve.

B is known to be an effective divisor. The ray corresponding to $X_{n-1,0}$ is spanned by the effective Brill-Noether divisor D_V where $V = \mathcal{O}(n-1,0)$ by Thm. 7.2.1. By symmetry, it is clear that B spans an extremal ray. We have not yet shown that $X_{n-1,0}$ spans an extremal ray because we have only provided one moving curve dual to it. The next two propositions will complete the proof that it spans an extremal ray by providing a second dual moving curve. The first proposition provides the dual moving curve in the case that n is even while the second proposition does so in the case that n is odd. □

The next proposition provides another edge of the effective cone in the case that n is even, i.e. $n = 2k$. This edge will share an extremal ray with the edge provided by the previous theorem. It will provide the second dual moving curve we needed to complete the previous proposition in the case that n is even.

Proposition 8.3.2. *The edge spanned by* $X_{2k-1,0}$ *and* $X_{k-1,1}$ *is an extremal edge of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[2k]}$ *for* $k > 0$. *Similarly, the edge spanned by* $X_{0,2k-1}$ *and* $X_{1,k-1}$ *is an extremal edge of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[2k]}$ *for* $k > 0$.

Proof. This is proved for $k = 1$ and $k = 2$ in (31). The proof now proceeds analogously as the previous proof. Again, it is immediate from the symmetry of the effective cone in terms of a and b that the second statement immediately follows from the first statement. We now prove the first statement.

The first step in proving this edge is an extremal ray is finding an extremal pair. To find an extremal pair, we first have to find the two controlling exceptional bundles that will make up the pair. The vector bundle whose Brill-Noether divisor will span the ray $X_{2k-1,0}$ is $\mathcal{O}(2k-1, 0)$. The vector bundle whose Brill-Noether divisor will span the ray $X_{k-1,1}$ is $\mathcal{O}(k-1, 1)$. To find our controlling exceptional bundles, we need to find exceptional bundles cohomologically orthogonal to $\mathcal{O}(2k-1, 0)$ and $\mathcal{O}(k-1, 1)$. Then, we have that

$$\chi(\mathcal{O}(k, 1), \mathcal{O}(2k-1, 0)) = 0,\ \chi(\mathcal{O}(2k-1, 0), \mathcal{O}(2k-2, 0)) = 0,$$

$$\chi(\mathcal{O}(k, 1), \mathcal{O}(k-1, 1)) = 0, \text{ and } \chi(\mathcal{O}(k-1, 1), \mathcal{O}(2k-2, 0)) = 0.$$

Next, it is easy to see that $\mathcal{O}(2k-2, 0)$ and $\mathcal{O}(k, 1)$ are controlling exceptional bundles for the Hilbert scheme and that the pair $\{\mathcal{O}(2k, 0), \mathcal{O}(k, 1)\}$ is an extremal pair.

Once we have the extremal pair, we need to turn it into a resolution of the general object of the Hilbert scheme. We complete the pair to a coil as prescribed by Thm. 5.1.2. This completion gives the coil

$$\{\mathcal{O}(-2k,-2),\mathcal{O}(-2k+1,-2),\mathcal{O}(-k-1,-1),\mathcal{O}(-k,-1)\}.$$

Next, we use the Beilinson spectral sequence to resolve the general ideal sheaf. The spectral sequence gives the resolution

$$0\to\mathcal{O}(-2k,-2)\to\mathcal{O}(-k,-1)^2\to\mathcal{I}_z\to 0.$$

Using this resolution, the third step is again finding a map to a moduli space of Kronecker modules. The Kronecker module in this resolution is $\mathcal{O}(-2k,-2)\to\mathcal{O}(-k,-1)^2$. Then we get a map

$$\pi:\left(\mathbb{P}^1\times\mathbb{P}^1\right)^{[n]}\dashrightarrow \mathrm{Kr}_{\hom(\mathcal{O}(-2k,-2),\mathcal{O}(-k,-1))}(1,2).$$

Using this map, we can finally compute the desired part of the effective cone. By a straightforward dimension count, we know that this map is birational. Then the two effective Brill-Noether divisors D_V and $D_{V'}$ where $V=\mathcal{O}(2k-1,1)$ and $V'=\mathcal{O}(k-1,1)$ are contracted by this map. Next, any moving curve in the Kronecker moduli space is dual to these contracted divisors. Thus, a pencil in the space $\mathrm{Hom}(\mathcal{O}(-2k,-2),\mathcal{O}(-k,-1))$ provides a dual moving curve showing that these divisors are on an edge of the effective cone. Alternatively, we could show that this moving curve gives the restriction $kb\geq 2k-1-a$. Coupled with the previous proposition, it is clear that $X_{2k-1,0}$ spans an extremal ray.

In order to show that $X_{k-1,1}$ is an extremal ray at the other end of the edge, we have to provide another extremal pair. The extremal pair needed is $\{\mathcal{O}(k-2,1),\mathcal{O}(k-1,1)\}$ Then we get the coil

$$\{\mathcal{O}(-k,-3),\mathcal{O}(-k,-2),\mathcal{O}(-k+1,-2),\mathcal{O}(-k+1,-1)\}.$$

The spectral sequence gives the resolution

$$0 \to \mathcal{O}(-k,-3)^2 \bigoplus \mathcal{O}(-k,-2)^{k-3} \to \mathcal{O}(-k+1,-2)^k \to \mathcal{I}_z \to 0.$$

A moving curve is a pencil in the space $\mathrm{Hom}(\mathcal{O}(-k,-2), \mathcal{O}(-k+1,-2))$. The restriction this moving curve places on D is that $kb \geq 4k-3-3a$. In particular, $X_{k-1,1}$ is dual to this moving curve which has a different slope than the other moving curve we constructed through this divisor, so we have shown that it is an extremal ray as promised. □

We now move on to the analogous proposition for odd n.

Proposition 8.3.3. *The edge spanned by* $X_{2k,0}$ *and* $X_{\frac{2k(k-1)}{2k-1},\frac{2k}{2k-1}}$ *is an extremal edge of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[2k+1]}$ *for* $k > 1$*. Similarly, the edge spanned by* $X_{0,2k}$ *and* $X_{\frac{2k}{2k-1},\frac{2k(k-1)}{2k-1}}$ *is an extremal edge of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[2k+1]}$ *for* $k > 1$.

Proof. This is shown for $k = 2$ in (31). Assume $k > 2$. This proof proceeds with all of the same elements as the previous proof, but slightly altered notation due to n being odd. Due to this, we give a much briefer proof.

The first statement implies the second statement by the symmetry of the effective cone so we prove only the first statement. Then it can be shown that the pair $\{\mathcal{O}(2k-1,0), \mathcal{O}(k,1)\}$ is an extremal pair. Then we get the coil

$$\{\mathcal{O}(-2k-1,-2), \mathcal{O}(-2k,-2), \mathcal{O}(-k-1,-1), \mathcal{O}(-k,-1)\}.$$

The spectral sequence gives the resolution

$$0 \to \mathcal{O}(-2k-1,-2) \to \mathcal{O}(-k-1,-1) \bigoplus \mathcal{O}(-k,-1) \to \mathcal{I}_z \to 0.$$

A moving curve is a pencil in the space $\mathrm{Hom}(\mathcal{O}(-2k-1,-2),\mathcal{O}(-k,-1))$. The restriction this moving curve places on D is that $kb \geq 2k-a$. In particular, $X_{2k,0}$ and $X_{\frac{2k(k-1)}{2k-1},\frac{2k}{2k-1}}$ are dual to this moving curve.

Then the two effective Brill-Noether divisors D_V and $D_{V'}$ where V is the exceptional bundle $\mathcal{O}(2k,0)$ and V' is the exceptional bundle $E_{\frac{2k(k-1)}{2k-1},\frac{2k}{2k-1}}$ are shown to be on an edge by this moving curve. Coupled with the previous proposition, it is clear that $X_{2k-1,0}$ spans an extremal ray.

In order to show that $X_{\frac{2k(k-1)}{2k-1},\frac{2k}{2k-1}}$ is an extremal ray at the other end of the edge, we have to provide another extremal pair. The extremal pair needed is $\{\mathcal{O}(k-1,1),\mathcal{O}(k,1)\}$ Then we get the coil

$$\{\mathcal{O}(-k-1,-3),\mathcal{O}(-k-1,-2),\mathcal{O}(-k,-2),\mathcal{O}(-k,-1)\}.$$

The spectral sequence gives the resolution

$$0 \to \mathcal{O}(-k-1,-3)\bigoplus \mathcal{O}(-k-1,-2)^{k-1} \to \mathcal{O}(-k,-2)^{k}\bigoplus \mathcal{O}(-k,-1) \to \mathcal{I}_z \to 0.$$

Then we get a map

$$\pi : \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]} \dashrightarrow Kr_{\mathrm{hom}(\mathcal{O}(-k-1,-2),\mathcal{O}(-k,-2))}(k-1,k).$$

By a dimension count, we see that this Kronecker moduli space is zero dimensional so we disregard it. The moving curves are pencils in the spaces $\mathrm{Hom}(\mathcal{O}(-k-1,-3),\mathcal{O}(-k,-2))$ and $\mathrm{Hom}(\mathcal{O}(-k-1,-2),\mathcal{O}(-k,-1))$. The restrictions these moving curves place on D are that $kb \geq 2k-a$ and $(k+1)b \geq 4k-3a$. In particular, $X_{\frac{2k(k-1)}{2k-1},\frac{2k}{2k-1}}$ is dual to these moving curve which have different slopes, so we have shown that it is an extremal ray as promised. □

The final sequence we look at is $n = 3k+1$. We provide this sequence as an example of the large class of extremal rays that be found in more sporadic sequences.

Proposition 8.3.4. *The ray spanned by* $X_{k-\frac{1}{2},2}$ *is an extremal ray of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[3k+1]}$ *for* $k > 1$. *Similarly, the ray spanned by* $X_{2,k-\frac{1}{2}}$ *is an extremal ray of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[3k+1]}$ *for* $k > 1$.

Proof. This proof again proceeds similarly to the previous proofs so we provide a concise version. Again, the second statement follows from the first by symmetry. The extremal controlling pair is $\{\mathcal{O}(k-1,2), \mathcal{O}(k,2)\}$ This completes to the coil

$$\{\mathcal{O}(-k-1,-4), \mathcal{O}(-k,-3), \mathcal{O}(-k+1,-3), \mathcal{O}(-k,-2)\}.$$

Then the resolution that we get is

$$0 \to \mathcal{O}(-k-1,-4) \bigoplus \mathcal{O}(-k,-3)^{k-1} \to \mathcal{O}(-k+1,-3)^{k-1} \bigoplus \mathcal{O}(-k,-2)^2 \to I_z \to 0.$$

Then we get a map

$$\pi : \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]} \dashrightarrow Kr_{\hom(\mathcal{O}(-k,-3),\mathcal{O}(-k+1,-3))}(k-1,k-1),$$

and V has the resolution

$$0 \to \mathcal{O}(k,1) \to E_{\frac{k-1}{3},\frac{7}{3}} \to V \to 0.$$

Next, the Kronecker modules are dual, so we have cohomological orthogonality. This makes D_V into a divisor. By Prop. 2.8.3, we know its class is $X_{k-\frac{1}{2},2}$. We see that it is an extremal ray by looking at pencils in the spaces $\mathrm{Hom}(\mathcal{O}(-k-1,-4), \mathcal{O}(-k+1,-3))$ and $\mathrm{Hom}(\mathcal{O}(-k,-3), \mathcal{O}(-k,-2))$ which cover the fibers of the Kronecker fibration. The restrictions these moving curve places on D are $(1+k)b \geq 6k - 4a$ and $kb \geq 4k - 1 - 2a$. Note, this implies they are distinct curve classes and $X_{k-\frac{1}{2},2}$ is dual to both moving curves. Thus, the Brill-Noether divisor D_V where V is a bundle with Chern character $(2,(2k-1,4),4k-3)$ spans an extremal ray of the effective cone. □

There are a couple more infinite families that we will mention but not prove. Their proofs follow similar techniques. Working out their proofs is a good exercise to become comfortable with this type of computation.

These families of rays on a edge require some notation.

Definition 8.3.5. The **symmetric value** of the effective cone of $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[N]}$ is the value a for which the ray spanned by $X_{a,a}$ is on the edge of the effective cone.

Note $X_{a,a}$ may or may not be an extremal ray. We now state the symmetric value for four infinite sequences of n.

Proposition 8.3.6. *The symmetric value of the effective cone of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[n]}$ *is:*

I) $k - 1 - \frac{1}{2k-2}$ *for* $n = k^2 - 2$, $k > 1$

II) $k - 1$ *for* $n = k^2 - 1$ or k^2, $k > 1$

III) $k - 1 + \frac{1}{k+1}$ *for* $n = k^2 + 1$, $k > 1$

IV) $k - \frac{1}{2}$ *for* $n = k^2 + k$, $k > 0$

8.4 Completing the Table

Finally, using our methods, we will give brief proofs of each of the five corners in the table at the beginning of this chapter that do not follow from our general constructions so far. We will only state the propositions and proofs for one of each pair of symmetric extremal rays.

Proposition 8.4.1. $X_{4,\frac{4}{3}}$ *is an extremal ray of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[11]}$.

Proof. Let $Z \in \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[11]}$ be general. The relevant extremal pair is $\{\mathcal{O}(-4,-1), \mathcal{O}(-3,-2)\}$. The resolving coil then will be

$$\{\mathcal{O}(-6,-3), \mathcal{O}(-4,-2), E_{\frac{-11}{3},\frac{-5}{3}}, \mathcal{O}(-3,-2)\}$$

since

$$\chi(\mathcal{O}(-4,-1), \mathcal{I}_Z) = 1 * 1((1+0+4)(1+0+1) - 0 - 11) = -1, \text{ and}$$

$$\chi(\mathcal{O}(-3,-2), \mathcal{I}_Z) = 1 * 1((1+0+3)(1+0+2) - 0 - 11) = 1.$$

Then the resolutions we get from the generalized Beilinson spectral sequence are

$$0 \to \mathcal{O}(-6,-3) \bigoplus \mathcal{O}(-4,-2)^2 \to E_{\frac{-11}{3},\frac{-5}{3}} \bigoplus \mathcal{O}(-3,-2) \to \mathcal{I}_Z \to 0 \text{ and}$$

$$0 \to \mathcal{O}(2,2) \to V \to \mathcal{O}(5,1)^2 \to 0.$$

Next, the Kronecker map is

$$\pi : \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[11]} \dashrightarrow Kr_{\hom\left(E_{\frac{11}{3},\frac{5}{3}}, \mathcal{O}(4,2)\right)}(1,2).$$

Note that the dimension of this Kronecker moduli space is $4 * 2 * 1 - 2^2 - 1^2 + 1 = 4$

Then, the Brill-Noether divisor is D_V where V is a bundle that has Chern character $(3,(12,4),14)$. The two types of moving curves covering each fiber come from pencils of maps $\mathcal{O}(-6,-3) \to K$ and $\mathcal{O}(-6,-3) \to \mathcal{O}(-3,-2)$. The restrictions these two moving curves place on D are that $3b \geq 12 - 2a$ and $6b \geq 20 - 3a$. In particular, $X_{4,\frac{4}{3}}$ is dual to both moving curves, $X_{\frac{40}{9},\frac{10}{9}}$ is dual to the first moving curve, and $X_{\frac{12}{5},\frac{12}{5}}$ is dual to the second moving curve. $\square$

Proposition 8.4.2. $X_{\frac{12}{5},\frac{12}{5}}$ *is an extremal ray of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[11]}$.

Proof. Let $Z \in \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[11]}$ be general. The relevant extremal pair is $\{\mathcal{O}(-2,-3), \mathcal{O}(-3,-2)\}$. The resolving coil then will be

$$\{\mathcal{O}(-4,-4), \mathcal{O}(-3,-3), \mathcal{O}(-2,-3), \mathcal{O}(-3,-2)\}$$

since

$$\chi(\mathcal{O}(-2,-3), \mathcal{I}_Z) = 1 * 1((1+0+2)(1+0+3) - 0 - 11) = 1, \text{ and}$$

$$\chi(\mathcal{O}(-3,-2), \mathcal{I}_Z) = 1 * 1((1+0+3)(1+0+2) - 0 - 11) = 1.$$

Then the resolutions we get from the generalized Beilinson spectral sequence are

$$0 \to \mathcal{O}(-4,-4)^2 \to \mathcal{O}(-3,-3) \bigoplus \mathcal{O}(-2,-3) \bigoplus \mathcal{O}(-3,-2) \to \mathcal{I}_Z \to 0 \text{ and}$$

$$0 \to E^2_{\frac{7}{3},\frac{7}{3}} \to E_{\frac{26}{11},\frac{26}{11}} \to V \to 0.$$

Next, the Kronecker map is

$$\pi : \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[11]} \dashrightarrow Kr_{\mathrm{hom}(\mathcal{O}(3,3),\mathcal{O}(4,4))}(1,2).$$

Note that the dimension of this Kronecker moduli space is $2*2*1-2^2-1^2+1=0$ so the map tells us nothing.

Then, the Brill-Noether divisor is D_V where V is a bundle that has Chern character $(5,(12,12),26)$. There are two types of moving curves coming from pencils of maps $K \to \mathcal{O}(-2,-3)$ and $K \to \mathcal{O}(-3,-2)$. The restrictions these two moving curves place on D are that $3b \geq 20-6a$ and $6b \geq 20-3a$. In particular, $X_{\frac{12}{5},\frac{12}{5}}$ is dual to both moving curves, $X_{4,\frac{4}{3}}$ is dual to the first moving curve, and $X_{\frac{4}{3},4}$ is dual to the second moving curve. □

Proposition 8.4.3. *$X_{\frac{9}{2},\frac{3}{2}}$ is an extremal ray of $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[13]}$.*

Proof. Let $Z \in \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[13]}$ be general. The relevant extremal pair is $\{\mathcal{O}(-5,-1), \mathcal{O}(-4,-2)\}$. The resolving coil then will be

$$\{\mathcal{O}(-7,-3), \mathcal{O}(-5,-2), E_{\frac{-14}{3},\frac{-5}{3}}, \mathcal{O}(-4,-2)\}$$

since

$$\chi(\mathcal{O}(-5,-1), \mathcal{I}_Z) = 1 * 1((1+0+5)(1+0+1) - 0 - 13) = -1, \text{ and}$$

$$\chi(\mathcal{O}(-4,-2), \mathcal{I}_Z) = 1 * 1((1+0+4)(1+0+2) - 0 - 13) = 2.$$

Then the resolutions we get from the generalized Beilinson spectral sequence are

$$0 \to \mathcal{O}(-7,-3) \bigoplus \mathcal{O}(-5,-2)^3 \to E_{\frac{-14}{3},\frac{-5}{3}} \bigoplus \mathcal{O}(-4,-2)^2 \to \mathcal{I}_Z \to 0 \text{ and}$$

$$0 \to \mathcal{O}(6,1) \to V \to \mathcal{O}(3,2) \to 0.$$

Next, the Kronecker map is

$$\pi : \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[13]} \dashrightarrow Kr_{\hom\left(E_{\frac{14}{3},\frac{5}{3}}, \mathcal{O}(5,2)\right)}(1,3).$$

Note that the dimension of this Kronecker moduli space is $4 * 1 * 3 - 3^2 - 1^2 + 1 = 3$.

Then, the Brill-Noether divisor is D_V where V is a bundle that has Chern character $(2,(9,3),12)$. There are two types of moving curves coming from pencils of maps $\mathcal{O}(-7,-3) \to K$ and $\mathcal{O}(-7,-3) \to$

$\mathcal{O}(-4,-2)^2$. The restrictions these two moving curves place on D are that $4b \geq 15-2a$ and $7b \geq 24-3a$. In particular, $X_{\frac{9}{2},\frac{3}{2}}$ is dual to both moving curves, $X_{\frac{7}{2},2}$ is dual to the first moving curve, and $X_{\frac{60}{11},\frac{12}{11}}$ is dual to the second moving curve. □

Proposition 8.4.4. $X_{\frac{8}{3},\frac{8}{3}}$ *is an extremal ray of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[13]}$.

Proof. Let $Z \in \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[13]}$ be general. The relevant extremal pair is $\{\mathcal{O}(-2,-3), \mathcal{O}(-3,-2)\}$. The resolving coil then will be

$$\{\mathcal{O}(-4,-5), \mathcal{O}(-5,-4), \mathcal{O}(-4,-4), \mathcal{O}(-3,-3)\}$$

since

$$\chi(\mathcal{O}(-2,-3), \mathcal{I}_Z) = 1 * 1((1+0+2)(1+0+3) - 0 - 13) = -1, \text{ and}$$

$$\chi(\mathcal{O}(-3,-2), \mathcal{I}_Z) = 1 * 1((1+0+3)(1+0+2) - 0 - 13) = -1.$$

Then the resolution we get from the generalized Beilinson spectral sequence is

$$0 \to \mathcal{O}(-4,-5) \bigoplus \mathcal{O}(-5,-4) \to \mathcal{O}(-3,-3)^3 \to \mathcal{I}_Z \to 0,$$

so this is no Kronecker map.

Next, the Brill-Noether divisor is D_V where V is the exceptional bundle $E_{\frac{8}{3},\frac{8}{3}}$. There are two types of moving curves coming from pencils of maps $\mathcal{O}(-4,-5) \to \mathcal{O}(-3,-3)^3$ and $\mathcal{O}(-5,-4) \to \mathcal{O}(-3,-3)^3$. The restrictions these two moving curves place on D are that $5b \geq 24-4a$ and $4b \geq 24-5a$. In particular, $X_{\frac{8}{3},\frac{8}{3}}$ is dual to both moving curves, $X_{\frac{7}{2},2}$ is dual to the first moving curve, and $X_{2,\frac{7}{2}}$ is dual to the second moving curve. □

Proposition 8.4.5. $X_{\frac{10}{3},\frac{7}{3}}$ *is an extremal ray of* $\left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[14]}$.

Proof. Let $Z \in \left(\mathbb{P}^1 \times \mathbb{P}^1\right)^{[14]}$ be general. The relevant extremal pair is $\{\mathcal{O}(-3,-3), \mathcal{O}(-4,-2)\}$. The resolving coil then will be

$$\{\mathcal{O}(-5,-4), \mathcal{O}(-4,-3), \mathcal{O}(-4,-2), \mathcal{O}(-3,-3)\}$$

since

$$\chi(\mathcal{O}(-3,-3), \mathcal{I}_Z) = 1 * 1((1+0+3)(1+0+3) - 0 - 14) = 2, \text{ and}$$

$$\chi(\mathcal{O}(-4,-2), \mathcal{I}_Z) = 1 * 1((1+0+4)(1+0+2) - 0 - 14) = 1.$$

Then the resolution we get from the generalized Beilinson spectral sequence is

$$0 \to \mathcal{O}(-5,-4)^2 \to \mathcal{O}(-4,-2) \bigoplus \mathcal{O}(-3,-3)^2 \to \mathcal{I}_Z \to 0,$$

so this is no Kronecker map.

Next, the Brill-Noether divisor is D_V where V is the exceptional bundle $E_{\frac{10}{3},\frac{7}{3}}$. There are two types of moving curves coming from pencils of maps $\mathcal{O}(-5,-4)^2 \to \mathcal{O}(-4,-2)$ and $\mathcal{O}(-5,-4)^2 \to \mathcal{O}(-3,-3)^2$. The restriction these two moving curves place on D are that $3b \geq 17-3a$ and $4b \geq 16-2a$. In particular, $X_{\frac{10}{3},\frac{7}{3}}$ is dual to both moving curves, $X_{\frac{7}{3},\frac{10}{3}}$ is dual to the first moving curve, and $X_{6,1}$ is dual to the second moving curve. $\square$

8.5 A Rank Two Example

Let $\xi = \left(\log(2), (\frac{1}{2}, 0), 2\right)$ Then, we can find that $\{\mathcal{O}(-1,2), \mathcal{O}(0,1)\}$ is an extremal pair for $M(\xi)$. It gives the resolution

$$0 \to \mathcal{O}(-1,-4) \to \mathcal{O}(0,-2) \bigoplus \mathcal{O}(0,-1)^2 \to U \to 0$$

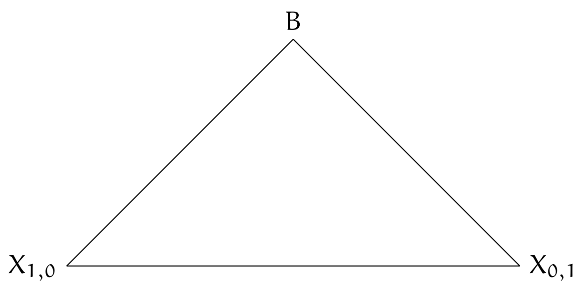
where $U \in M(\xi)$ is general. This gives that the divisor $D_{\mathcal{O}(-1,3)}$ lies on an edge of the effective cone. In this case, we can actually do two dimension counts to see that varying either map gives a moving curve dual to $D_{\mathcal{O}(-1,3)}$ so it in fact spans an extremal ray of the effective cone.

APPENDICES

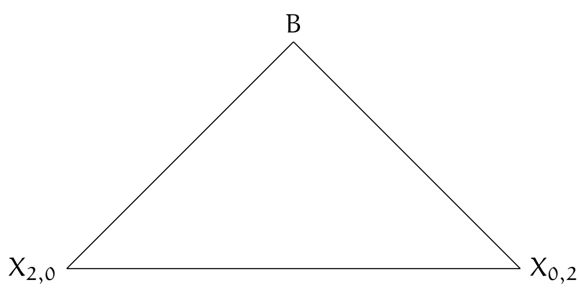
CHAPTER 9

CROSS-SECTIONS OF THE EFFECTIVE CONES OF THE FIRST FIFTEEN HILBERT SCHEMES

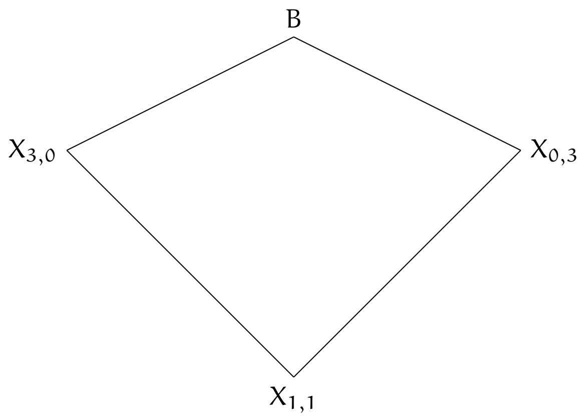
9.0.0.1 $\mathrm{Eff}\left(\mathrm{Hilb}^2\left(\mathbb{P}^1\times\mathbb{P}^1\right)\right)$



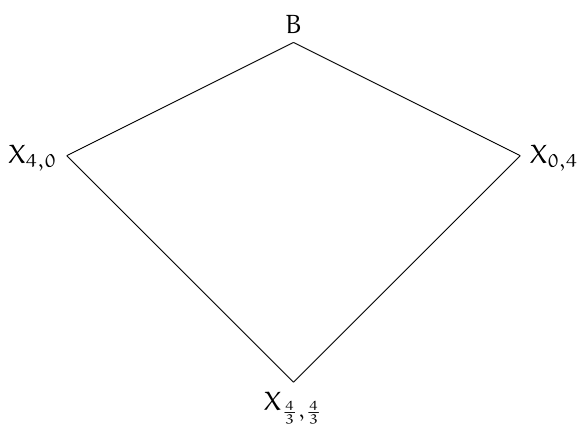
9.0.0.2 $\mathrm{Eff}\left(\mathrm{Hilb}^3\left(\mathbb{P}^1\times\mathbb{P}^1\right)\right)$



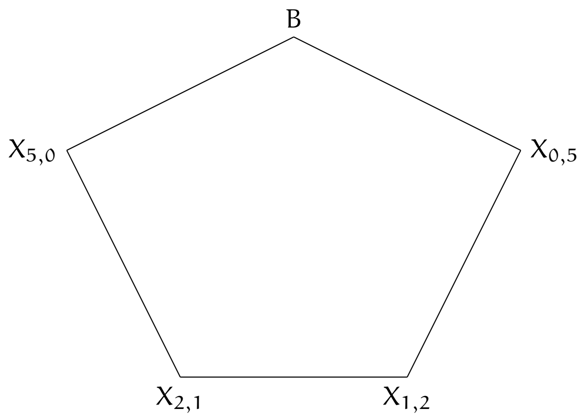
9.0.0.3 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^4\left(\mathbb{P}^1\times\mathbb{P}^1\right)\right)}$



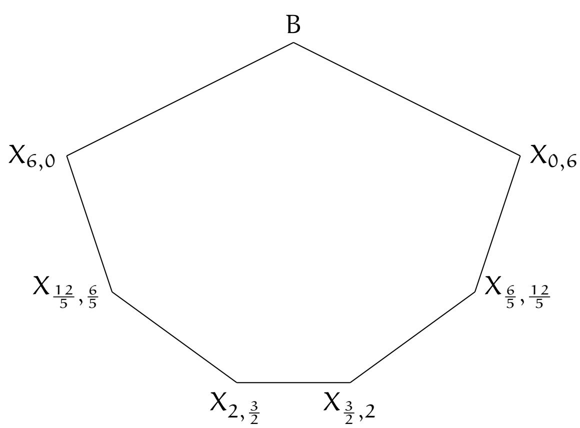
9.0.0.4 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^5\left(\mathbb{P}^1\times\mathbb{P}^1\right)\right)}$



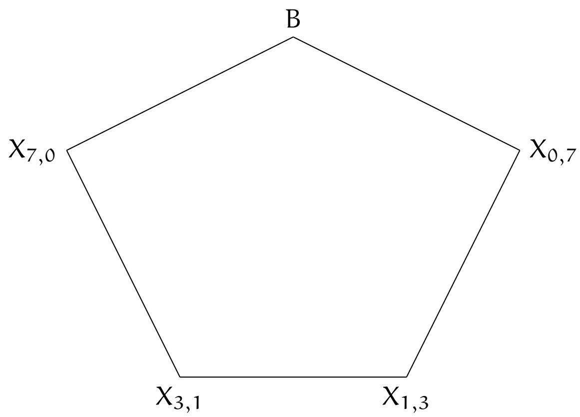
9.0.0.5 $\mathrm{Eff}\left(\mathrm{Hilb}^6\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)$



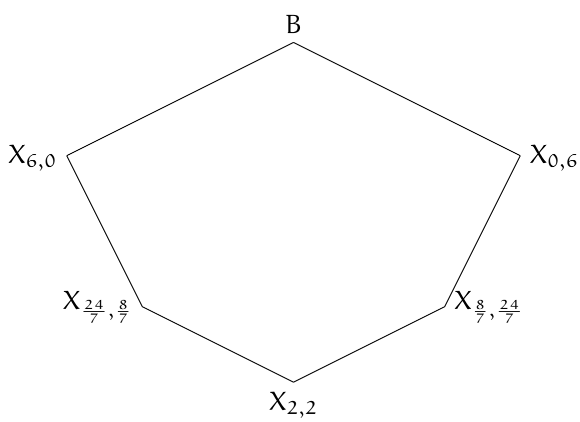
9.0.0.6 $\mathrm{Eff}\left(\mathrm{Hilb}^7\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)$



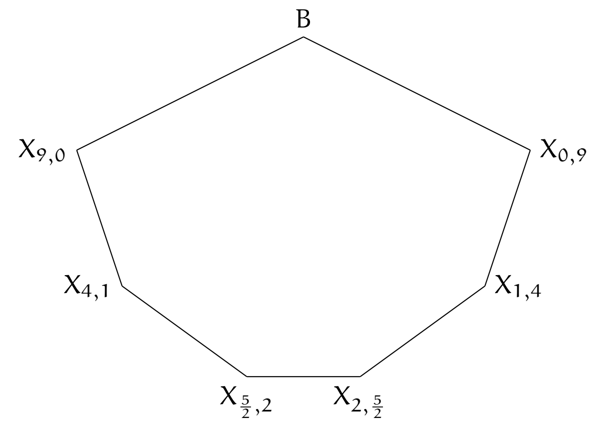
9.0.0.7 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^8\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)}$



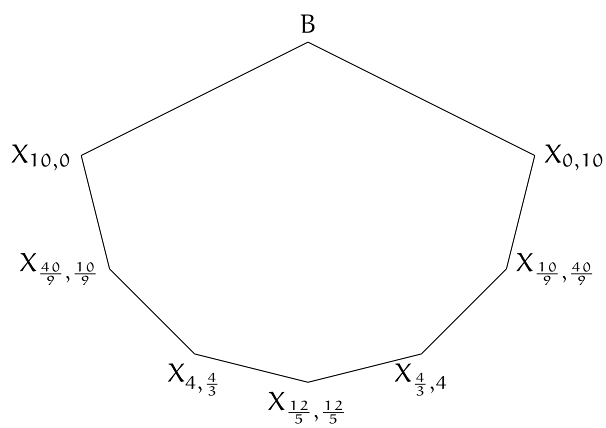
9.0.0.8 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^9\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)}$



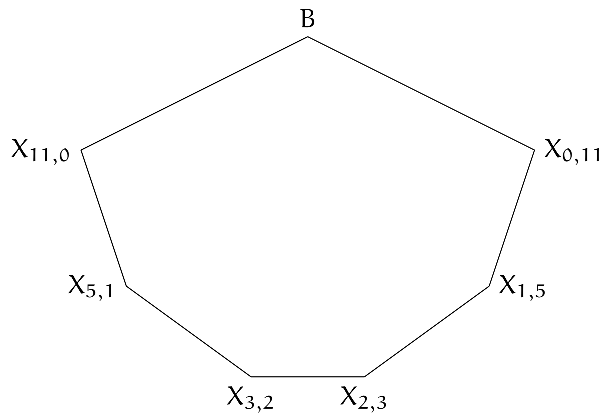
9.0.0.9 $\mathrm{Eff}\left(\mathrm{Hilb}^{10}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)$



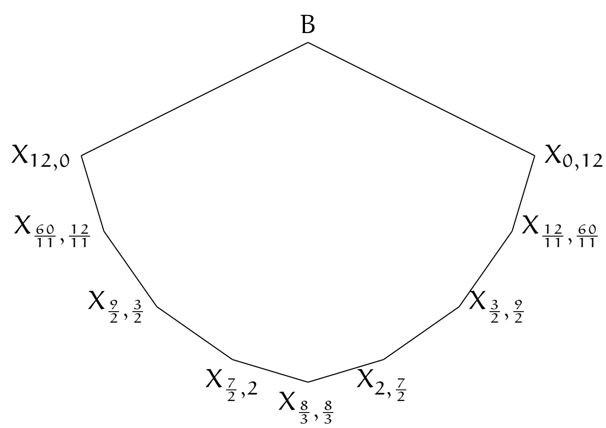
9.0.0.10 $\mathrm{Eff}\left(\mathrm{Hilb}^{11}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)$



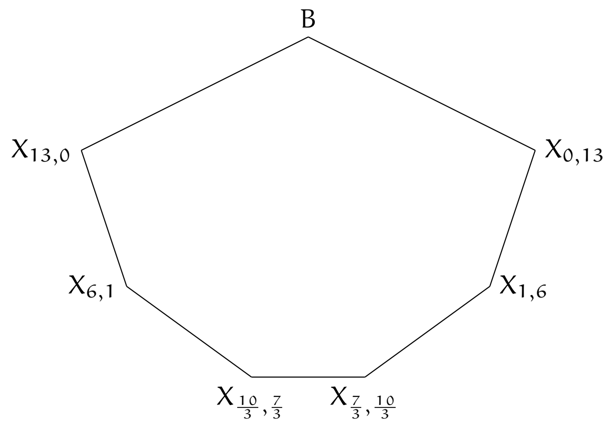
9.0.0.11 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^{12}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)}$



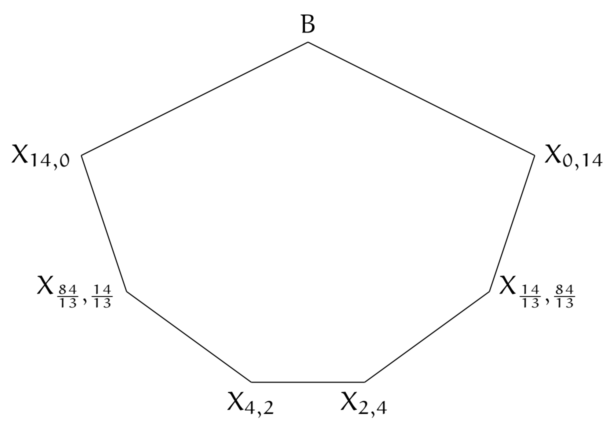
9.0.0.12 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^{13}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)}$



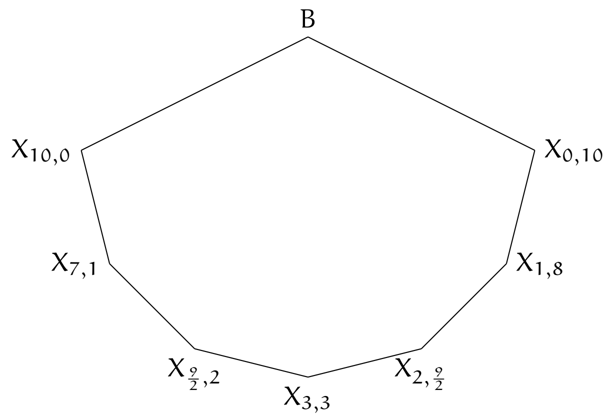
9.0.0.13 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^{14}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)}$



9.0.0.14 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^{15}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)}$



9.0.0.15 $\underline{\mathrm{Eff}\left(\mathrm{Hilb}^{16}\left(\mathbb{P}^1 \times \mathbb{P}^1\right)\right)}$



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