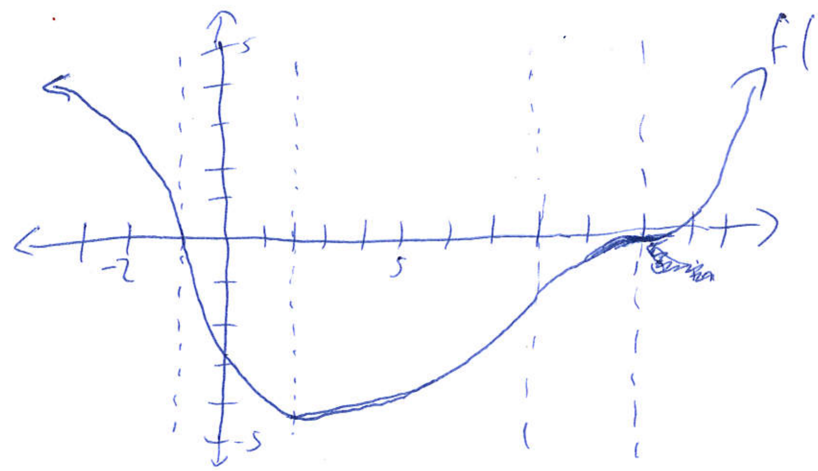
Math 180
Worksheet 10
Sections 4.3-4.4

1.) Sketch a possible graph of a continuous function with the following properties:

$$f' < 0 \text{ and } f'' < 0, \text{for } x < -1$$
$$f' < 0 \text{ and } f'' > 0, \text{for } -1 < x < 2$$
$$f' > 0 \text{ and } f'' > 0, \text{for } 2 < x < 8$$
$$f' > 0 \text{ and } f'' < 0, \text{for } 8 < x < 10$$
$$f' > 0 \text{ and } f'' > 0, \text{for } x > 10$$

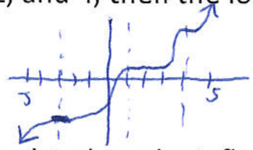
inf. pt   local min   inf. pt   inf. pt

x	$(-\infty,-1)$	$(-1,2)$	$(2,8)$	$(8,10)$	$(10,\infty)$
$f'(x)$	−	−	+	+	+
inc/dec	dec	dec	inc	inc	inc
$f''(x)$	−	+	+	−	+
concave up/down	down	up	up	down	up



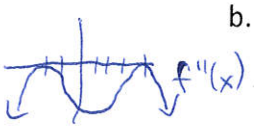
2.) Determine if the following statements are true and give an explanation or counterexample:

a. If the zeros of f' are -3, 1, and 4, then the local extrema of f are located at these points. False



The function might "plateau" instead of having a local extrema.

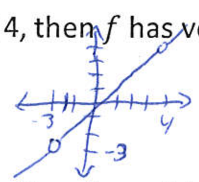
b. If the zeros of f'' are -2 and 4, then the inflection points of f are located at these points. False,



those might ~~be~~ be places f'' switches sign.

c. If the zeros of the denominator of f are -3 and 4, then f has vertical asymptotes at these points.

False, $f(x) = \frac{(x+3)(x-4)x}{(x+3)(x-4)}$



d. If a rational function has a finite limit as $x \to \infty$, then it must have a finite limit as $x \to -\infty$.

True, if ~~the~~ degree of the numerator is higher than (less than or equal to) the degree of the denominator $\lim_{x\to\infty} f(x)$ & $\lim_{x\to-\infty} f(x)$ both ~~do not have~~ (have) ~~a finite lim~~ aren't (are) finite

3.) Follow the steps discussed in class and in the textbook to identify all key information and make a sketch of the function $f(x) = e^{-x}\sin x$ on the interval $[-\pi, \pi]$.

1. domain: $[-\pi, \pi]$

2. $f(-x) = e^{-(-x)}\sin(-x) = e^{x}\sin(-x) = \underline{-e^{x}\sin(x)}$

$f(-x) \neq f(x)$, $f(-x) \neq -f(x)$

So it is neither even nor odd

3. $f(x) = 0 = e^{-x}\sin(x)$

$\Rightarrow \sin(x) = 0$ (since $e^{-x} \neq 0$)

$\Rightarrow x = -\pi, 0,$ or π

$f(0) = e^{-0}\sin(0) = 1 \cdot 0 = 0$

4. $f(x)$ is continuous on $[-\pi, \pi]$

So there is no holes or vertical asymptotes

There are no horizontal asymptotes because the domain has finite length

5. $f'(x) = -e^{-x}\sin(x) + e^{-x}\cos(x)$

$f'(x) = e^{-x}(\cos(x) - \sin(x))$

$f''(x) = -e^{-x}(\cos(x) - \sin(x)) + e^{-x}(-\sin(x) - \cos(x))$

$f''(x) = e^{-x}(-\cos(x) + \sin(x) - \sin(x) - \cos(x))$

$f''(x) = -2e^{-x}\cos(x)$

6. critical values of $f(x)$

$f'(x) = 0 = e^{-x}(\cos(x) - \sin(x))$

$\Rightarrow \cos(x) - \sin(x) = 0$ (since $e^{-x} \neq 0$)

$\Rightarrow \cos(x) = \sin(x) \Rightarrow \tan(x) = 1$

$\Rightarrow x = \frac{-3\pi}{4}$ or $\frac{\pi}{4}$

Critical values of $f''(x)$

$f''(x) = 0 = -2\cos(x)e^{-x}$

$\Rightarrow \cos(x) = 0$ (since $-2e^{-x} \neq 0$)

$\Rightarrow x = -\frac{\pi}{2}, \frac{\pi}{2}$

7. local extrema

	$(-\pi, \frac{-3\pi}{4})$	$(\frac{-3\pi}{4}, \frac{\pi}{4})$	$(\frac{\pi}{4}, \pi)$
$f'(x)$	$-$	$+$	$-$

So ~~$x = \frac{-3\pi}{4}$ is~~ a local min occurs at $x = \frac{-3\pi}{4}$ & a local max occurs at $x = \frac{\pi}{4}$

$f(\frac{-3\pi}{4}) = e^{-(\frac{-3\pi}{4})}\sin(\frac{-3\pi}{4}) = e^{3\pi/4}\frac{-1}{\sqrt{2}} = -e^{\frac{3\pi}{4}}\frac{1}{\sqrt{2}}$

$f(\frac{\pi}{4}) = e^{-\pi/4}\sin(\frac{\pi}{4}) = e^{\pi/4}\frac{1}{\sqrt{2}}$

So the local min at $x = \frac{-3\pi}{4}$ is $-e^{\frac{3\pi}{4}}\frac{1}{\sqrt{2}}$ & the local max at $x = \frac{\pi}{4}$ is $e^{\pi/4}\frac{1}{\sqrt{2}}$

8. $f(-\pi) = e^{-(-\pi)}\sin(-\pi) = e^{\pi} \cdot 0 = 0$

$f(\pi) = e^{-\pi}\sin(\pi) = e^{-\pi} \cdot 0 = 0$

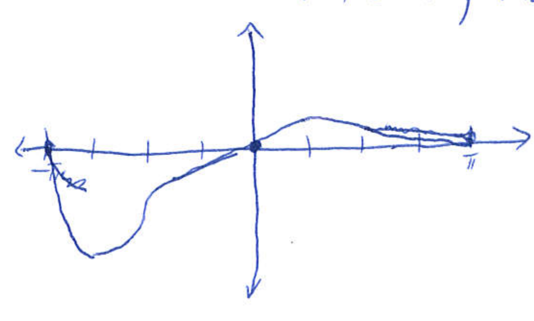
So the absolute max is $e^{\pi/4}\frac{1}{\sqrt{2}}$ & occurs at $x = \frac{\pi}{4}$. The absolute min. is $-e^{-3\pi/4}\frac{1}{\sqrt{2}}$ & occurs at $x = \frac{-3\pi}{4}$.

9.

x	$(-\pi, \frac{-3\pi}{4})$	$(\frac{-3\pi}{4}, \frac{-\pi}{2})$	$(\frac{-\pi}{2}, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$
$f'(x)$	$-$	$+$	$+$	$-$	$-$
inc/dec	dec	inc.	inc.	dec.	dec.
$f''(x)$	$+$	$+$	$-$	$-$	$+$
concave up/down	up	up	down	down	up

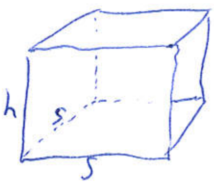
10. $x = \frac{-\pi}{2}$ & $\frac{\pi}{2}$ are inflection points.

11.



4.) Of all boxes with a square base and a volume of 100 m^3, which one has the minimum surface area? (Give the dimensions of the box)

V = volume A = surface area



h = height

s = side length of the sq. base

$V = hs^2 = 100 m^3 \Rightarrow h = \frac{100}{s^2}$

$A(s) = 2s^2 + 4hs$

$A(s) = 2s^2 + 4\left(\frac{100}{s^2}\right)s = 2s^2 + 400\frac{1}{s} = 2s^2 + 400s^{-1}$

$\frac{dA}{ds}(s) = 4s - 400s^{-2}$

$\frac{dA}{ds}(s) = 0 \Rightarrow 4s = 400\frac{1}{s^2} \Rightarrow s^3 = 100 \Rightarrow s = \sqrt[3]{100}$

So $s = \sqrt[3]{100}$ is a critical pt. of $A(s)$

$A''(s) = 4 + 800s^{-3} \Rightarrow A(\sqrt[3]{100}) = 4 + \frac{800}{100} = 4 + 8 = 12$

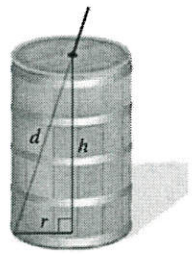
So by the 2nd derivative test, $\sqrt[3]{100} = s$ is a local min

$s = \sqrt[3]{100} \Rightarrow h = \frac{100}{100^{2/3}} = 100^{1/3}$

Thus, the box with minimum surface area has dimensions $(\sqrt[3]{100}\ m) \times (\sqrt[3]{100}\ m) \times (\sqrt[3]{100}\ m)$

5.) (Challenge) Several mathematical stories originated with the second wedding of the mathematician and astronomer Johannes Kepler. Here is one of them:

While shopping for wine for his wedding, Kepler noticed that the price of a barrel of wine (here assumed to be a cylinder) was determined solely by the length d of a dipstick that was inserted diagonally through a centered hole in the top of the barrel to the edge of the base of the barrel (see figure).



Kepler realized that this measurement does not determine the volume of the barrel and that for a fixed value of d, the volume varies with the radius r and height h of the barrel.

For a fixed value of d, what is the ratio r/h that maximizes the volume of the barrel?

$V = \pi r^2 h \qquad d^2 = r^2 + h^2$

$r^2 = d^2 - h^2$

$V = \pi(d^2 - h^2)h = \pi d^2 h - \pi h^3$

$V' = \frac{dV}{dh} = \pi d^2 - 3\pi h^2$, $V'' = -6\pi h < 0$ so local max by 2nd der. test

$V' = 0 \Rightarrow \pi d^2 - 3\pi h^2 = 0 \Rightarrow d^2 - 3h^2 = 0 \Rightarrow d^2 = 3h^2$

So $3h^2 = r^2 + h^2 \Rightarrow 2h^2 = r^2 \Rightarrow 2 = \left(\frac{r}{h}\right)^2 \Rightarrow \sqrt{2} = \frac{r}{h}$

Thus, $\frac{r}{h} = \sqrt{2}$ maximizes the volume of the barrel