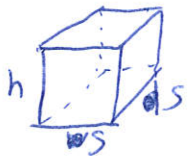
Answer Key

Math 180
Worksheet 11
Section 4.4-4.5

1.) Of all boxes with a square base and a volume of 100 m^3, which one has the minimum surface area? (Give the dimensions of the box)



Surface Area $= S = 4hs + 2s^2$

Volume $= V = 100m^3 = s^2h$

$$h = \frac{100}{s^2}$$

$$S = 4\frac{100}{s^2}s + 2s^2 = \frac{400}{s} + 2s^2$$

$$\frac{dS}{ds} = \frac{-400}{s^2} + 4s = 0$$

$$4s = \frac{400}{s^2}$$

$$s^3 = 100$$

$$s = \sqrt[3]{100}$$

$$\frac{d^2S}{ds^2} = \frac{800}{s^3} + 4$$

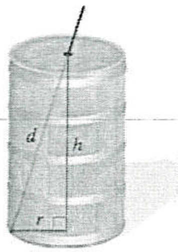
$$\frac{d^2S}{ds^2}(\sqrt[3]{100}) = \frac{800}{100} + 4 = 12$$

so $\sqrt[3]{160} = s$ is a local min

Thus the dimensions of the box with minimum surface area is $\sqrt[3]{100}\,m \times \sqrt[3]{100}\,m \times \sqrt[3]{100}\,m$ & its surface area is $6\sqrt[3]{100}$.

2.) Several mathematical stories originated with the second wedding of the mathematician and astronomer Johannes Kepler. Here is one of them:

While shopping for wine for his wedding, Kepler noticed that the price of a barrel of wine (here assumed to be a cylinder) was determined solely by the length d of a dipstick that was inserted diagonally through a centered hole in the top of the barrel to the edge of the base of the barrel (see figure).



Kepler realized that this measurement does not determine the volume of the barrel and that for a fixed value of d, the volume varies with the radius r and height h of the barrel.

For a fixed value of d, what is the ratio r/h that maximizes the volume of the barrel?

Volume $= V = \pi r^2 h$

$d^2 = h^2 + r^2 \rightarrow r^2 = d^2 - h^2$

$V = \pi(d^2 - h^2)h = \pi d^2 h - \pi h^3$

$\frac{dV}{dh} = \pi d^2 - 3\pi h^2$

$\frac{dV}{dh} = 0 \Rightarrow \pi d^2 = 3\pi h^2 \Rightarrow h^2 = \frac{d^2}{3} \Rightarrow h = \sqrt{\frac{d^2}{3}} = \frac{d}{\sqrt{3}}$

$r^2 = d^2 - \frac{d^2}{3} = \frac{2}{3}d^2 \Rightarrow r = \sqrt{\frac{2}{3}}\, d$

$\frac{d^2V}{dh^2} = -3\pi 2h = -6\pi h < 0$ so $h = \frac{d}{\sqrt{3}}$ is a local max

Thus, the ratio of $\frac{r}{h}$ that maximizes volume is

$\frac{r}{h} = \frac{\frac{\sqrt{2}}{\sqrt{3}} d}{\frac{1}{\sqrt{3}} d} = \sqrt{2}$

3.) Consider the function $f(x) = e^{-x}$ near the point $a = 0$.

a. Find the linear approximation function $L(x)$.

$L(x) = f(a) + f'(a)(x-a)$

$f'(x) = -e^{-x}$

$f(0) = e^{-0} = 1,\ f'(0) = -e^{-0} = -1 \Rightarrow L(x) = 1 - 1(x-0) = 1-x$

b. Use L to approximate e^{-2} and e^{1}.

$L(2) = 1-2 = -1$

$L(-1) = 1-(-1) = 2$

c. Using derivatives of f, are your answers in part b overestimates or underestimates?

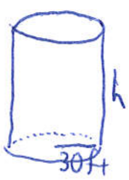
$f''(x) = e^{-x},\ f''(0) = e^{-0} = 1 > 0$

So f is concave up so both approximations underestimate

d. Find the error for each estimate.

$100\% \left|\dfrac{e^{-2}+1}{e^{-2}}\right| = 838.91\%$ $\qquad 100\% \left|\dfrac{e^{1}-2}{e^{1}}\right| = 26.42\%$

4.) Approximate the change in the surface area of a right circular cylinder of fixed radius $r = 30\text{ft}$ when its height decreases from $h = 12\text{ft}$ to $h = 11.8\text{ft}$.



Surface Area $= S = 2\pi(30ft)^2 + 2\pi(30ft)h$

$\dfrac{dS}{dh} = 60\pi \dfrac{ft^2}{ft}$

$\cancel{S(12ft) = 900\pi ft^2 + 60\pi(12)ft^2 = 1620\pi ft^2}$

$\dfrac{dS}{dh}(12ft) = 60\pi\ ft$

$\Delta S \approx \dfrac{dS}{dh}(12ft)\cdot(12ft - 11.8ft) = (60\pi\ ft)(.2ft) = 12\pi ft^2$