QUASICIRCLE BOUNDARIES AND EXOTIC ALMOST-ISOMETRIES

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Abstract. We consider properly discontinuous, isometric, convex cocompact actions of surface groups Γ on a CAT(-1) space X. We show that the limit set of such an action, equipped with the canonical visual metric, is a (weak) quasicircle in the sense of Falconer and Marsh. It follows that the visual metrics on such limit sets are classified, up to bi-Lipschitz equivalence, by their Hausdorff dimension. This result applies in particular to boundaries at infinity of the universal cover of a locally CAT(-1) surface. We show that any two periodic CAT(-1) metrics on $\mathbb{H}^2$ can be scaled so as to be almost-isometric (though in general, no equivariant almost-isometry exists). We also construct, on each higher genus surface, k-dimensional families of equal area Riemannian metrics, with the property that their lifts to the universal covers are pairwise almost-isometric but are **not** isometric to each other. Finally, we exhibit a gap phenomenon for the optimal multiplicative constant for a quasi-isometry between periodic CAT(-1) metrics on $\mathbb{H}^2$.

1. Introduction

Consider a Riemannian manifold (M, g) with curvature ≤ -1 (or more generally, equipped with a locally CAT(-1) metric). The fundamental group $\pi_1(M)$ acts via deck transformations on the universal cover X, and the metric g lifts to a $\pi_1(M)$-invariant metric $\tilde{g}$ on X. An important theme has been the study of the dynamics of $\pi_1(M)$ on the boundary at infinity ∂X. If one equips the boundary at infinity with the canonical visual metric d_∂ (see Definition 2.2), then it is well-known that the boundary at infinity exhibits some fractal-like behavior. More generally this phenomenon occurs if $\pi_1(M)$ acts properly discontinuously and convex cocompactly on a CAT(-1) space X. Our first result follows this general philosophy:

Theorem 1.1. *Let Γ be a surface group, and let (X, d) be any proper CAT(-1) space on which Γ acts isometrically, properly discontinuously, and convex cocompactly. Let Λ be the limit set of the Γ-action on X and let d_∂ denote the canonical visual metric on Λ. Then (Λ, d_∂) is a (weak) quasicircle in the sense of Falconer-Marsh.*

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Remark 1.2. The term *quasicircle* is also used in geometric function theory as any metric space that is the image of the round circle under a quasisymmetric map. This notion does *not* coincide with Falconer-Marsh quasicircles. In fact, Falconer-Marsh quasicircles form a strict subset of the images of quasisymmetric maps. See Remark 2.8 for more information.

As mentioned above, Theorem 1.1 applies in particular if (X, d) is the universal cover of a closed surface equipped with a locally CAT(-1) metric and Γ acts by deck transformations. The results below are obtained from this special case of Theorem 1.1.

Corollary 1.3. *Let $(M_1, d_1), (M_2, d_2)$ be any pair of closed surfaces equipped with locally CAT(-1) metrics, and let $(X_i, \tilde{d}_i)$ be their universal covers. Then one can find real numbers $0 < \lambda_i \leq 1$, with $\max\{\lambda_1, \lambda_2\} = 1$, having the property that $(X_1, \lambda_1\tilde{d}_1)$ is almost-isometric to $(X_2, \lambda_2\tilde{d}_2)$.*

Recall that an *almost-isometry* is a quasi-isometry with multiplicative constant = 1. If M_1, M_2 are locally CAT(-1) manifolds, then the existence of an almost-isometry $X_1 \to X_2$ forces strong constraints on the geometry of M_1 and M_2. In many cases, this forces the universal covers to be isometric (see [9] for more information). On the other hand, it follows immediately from Corollary 1.3 that there are examples of almost-isometric universal covers that are not isometric – take for example d_1 to be a Riemannian metric and d_2 to be a non-Riemannian metric.

Corollary 1.4. *Let M be a closed surface and let g_1, g_2 be two Riemannian metrics on M with curvatures ≤ -1. Equip ∂X with the corresponding canonical visual metrics ρ_1 and ρ_2. Then the following three statements are equivalent:*

1. *The topological entropies of the two geodesic flows on T^1M are equal.*
2. *The boundaries $(\partial X, \rho_1)$ and $(\partial X, \rho_2)$ are bi-Lipschitz equivalent.*
3. *The universal covers $(X, \tilde{g}_1)$ and $(X, \tilde{g}_2)$ are almost-isometric.*

So, after possibly scaling one of the metrics, we can ensure that the universal covers of any two Riemannian surfaces are almost-isometric. Of course, when scaling, we also change the geometry of the metric, e.g. the area. Our next result shows that one can arrange for examples with almost-isometric universal covers, while still keeping control of the area of the surface.

Theorem 1.5. *Let M be a closed surface of genus ≥ 2, and $k \geq 1$ an integer. One can find a k-dimensional family $\mathcal{F}_k$ of Riemannian metrics on M, all of curvature ≤ -1, with the following property. If g, h are any two distinct metrics in $\mathcal{F}_k$, then*

- *$Area(M, g) = Area(M, h)$.*
- *the lifted metrics $\tilde{g}, \tilde{h}$ on the universal cover X are almost-isometric.*
- *the lifted metrics $\tilde{g}, \tilde{h}$ on the universal cover X are* **not** *isometric.*

In the above Theorem 1.5, we think of the almost isometries between the lifted metrics on X as being *exotic* since they cannot be realized equivariantly with respect to the two natural $\pi_1(M)$-actions on X. Indeed, the existence of a $\pi_1(M)$-equivariant almost-isometry between the two lifted metrics on X implies that the two metrics on M have equal marked length spectra (see [9], for example), and are therefore isometric by [5, 15].

As a final application, we exhibit a *gap phenomenon* for the optimal multiplicative constant for quasi-isometries between certain periodic metrics.

Corollary 1.6. *Let $(M_1, d_1), (M_2, d_2)$ be any pair of closed surfaces equipped with locally CAT(-1) metrics, and assume that their universal covers $(X_i, \tilde{d}_i)$ are* **not** *almost-isometric. Then there exists a constant $\epsilon > 0$ with the property that any (C, K)-quasi-isometry from (X_1, d_1) to (X_2, d_2) must satisfy $C \geq 1 + \epsilon$.*

The layout of this paper is as follows. In Section 2, we review the basic definitions, and summarize the results from the literature that we will need. We also show how to deduce Corollaries 1.3, 1.4, and 1.6 from Theorem 1.1. In Section 3, we give a proof of Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.5. Finally, we provide some concluding remarks in Section 5.

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2. Background material

In this section, we review some basic definitions we will need, and also provide descriptions of some results we will need in our proofs.

2.1. Convex cocompact actions.

We briefly summarize the statements we need concerning convex cocompact actions, and refer the reader to [2, Section 1.8] for more details. Given a properly discontinuous isometric action of Γ on a proper CAT(-1) space X, we have an associated *limit set* in the boundary at infinity ∂X. This set Λ_Γ is obtained by taking the closure $\overline{\Gamma \cdot p}$ of the Γ-orbit of a point $p \in X$ inside the compactification $\overline{X} := X \cup \partial X$, and setting $\Lambda_\Gamma := \overline{\Gamma \cdot p} \cap \partial X$.

Definition 2.1. The Γ-action on X is *convex cocompact* if it satisfies any of the following equivalent conditions:

(1) the map $\Phi : \Gamma \to X$ given by $\Phi(g) := g(x)$ is quasi-isometric.

(2) the orbit of a point $\Gamma \cdot p$ is a quasi-convex subset of X (i.e. every geodesic joining a pair of points in $\Gamma \cdot p$ lies within a uniform neighborhood of $\Gamma \cdot p$).
(3) the action of Γ on the convex hull of its limit set $\text{Co}(\Lambda_\Gamma)$ is cocompact.

The equivalence of statements (1) and (2) can be found in [2, Corollary 1.8.4], while the equivalence of (2) and (3) is shown in [2, Proposition 1.8.6]. It follows from these conditions that Γ is δ-hyperbolic, and therefore we can compare the boundary ∂X with the Gromov boundary $\partial\Gamma$. For our purposes, an important consequence of the action being convex cocompact is that the limit set Λ_Γ is homeomorphic to $\partial\Gamma$. So in the special case where Γ is a surface group, the limit set Λ_Γ is homeomorphic to S^1.

2.2. **Metrics on the boundary.** We refer the reader to [2] for more details concerning this subsection.

Let X be a CAT(-1) space with boundary at infinity ∂X. Fix a basepoint $w \in X$. The Gromov product $(\cdot|\cdot)_w : X \times X \to \mathbb{R}$ is defined by

$$(p|q)_w := \frac{1}{2}\left(d(w,p) + d(w,q) - d(p,q)\right)$$

for each $x, y \in X$, and extends to $\partial X \times \partial X$ by

$$(x,y)_w := \lim_{n\to\infty} (x_n|y_n)_w$$

where $\{x_n\}$ and $\{y_n\}$ are sequences in X converging to x and y. Gromov products induce a (family of) canonical visual metric(s) on the boundary ∂X defined as follows.

Definition 2.2. Fix a basepoint $w \in X$. Then the metric d_w is defined by

$$d_w(x,y) := e^{-(x|y)_w}$$

This gives a family of bi-Lipschitz equivalent metrics, obtained by varying the choice of the basepoint w. By an abuse of language, we will refer to this bi-Lipschitz class of metrics on ∂X as *the canonical visual metric.*

Remark 2.3. If we let γ_{xy} denote the geodesic joining x and y, there is a universal constant C with the property that, for all pairs of points $x, y \in \partial X$,

$$|(x|y)_w - d(w, \gamma_{xy})| < C.$$

It follows that the distance function d_w is bi-Lipschitz equivalent to the function $\hat{d}_w$ defined via

$$\hat{d}_w(x,y) := e^{-d(w,\gamma_{xy})},$$

(even though $\hat{d}_w$ might not define a metric).

Let Γ be a properly discontinuous group of isometries of X acting convex cocompactly. Let $\Lambda \subseteq \partial X$ be the limit set of ∂X, $s = \text{Hdim}(\partial X)$ be the Hausdorff dimension of the canonical visual metric on Λ, and H^s be the corresponding Hausdorff measure. Then H^s is a finite measure, and fully supported on Λ. The Hausdorff dimension and measure can be estimated using the following result of Patterson and Sullivan [17] – originally established in the hyperbolic space setting, but whose proof extends verbatim to the CAT(-1) setting (see Bourdon [2, Theorem 2.7.5]).

Theorem 2.4 (Patterson-Sullivan)**.**

(i) $s = \overline{\lim_{n\to\infty}} \frac{1}{n} \log \#\{\gamma \in \Gamma \,|\, d(w, \gamma w) \le n\}$

(ii) There exists a constant $C \ge 1$ such that for each metric ball $B(x,r)$ in ∂X (with center x and radius r),

$$C^{-1} r^s \le H^s\left(B(x,r)\right) \le C r^s.$$

Remark 2.5. If (M,g) is a closed Riemannian manifold with sectional curvatures ≤ -1, then its universal Riemannian covering X is a CAT(-1) space equipped with a geometric action of $\Gamma = \pi_1(M)$ by deck transformations. In this case, $s = h(g)$, where the latter denotes the topological entropy of the geodesic flow on the unit tangent bundle $T^1(M)$.

To see this, let W denote a bounded fundamental domain for the Γ-action, $V = \mathrm{vol}(W)$, and $D = \mathrm{diam}(W)$. The following basic estimates

$$\mathrm{vol}(B(w,n)) \le V \cdot \#\{\gamma \in \Gamma \,|\, d(w,\gamma w) \le n + D\}$$

and

$$V \cdot \#\{\gamma \in \Gamma \,|\, d(w,\gamma w) \le n\} \le \mathrm{vol}(B(w, n+D))$$

imply that the Hausdorff dimension s and the volume growth entropy

$$h_{vol}(g) := \lim_{r\to\infty} \frac{1}{r} \log \mathrm{vol}(B(w,r))$$

coincide. Then by Manning's theorem [14], $s = h_{\mathrm{vol}}(g) = h(g)$.

2.3. **(Weak) Quasicircles according to Falconer-Marsh.**

Next let us briefly review some notions and results of Falconer and Marsh [6].

Definition 2.6 (Falconer-Marsh)**.** A metric space (C,d) is a *quasicircle* if

(i) C is homeomorphic to S^1,

(ii) (expanding similarities) There exist $a, b, r_0 > 0$ with the following property. For any $r < r_0$ and $N \subseteq C$ with $\mathrm{diam}(N) = r$, there exists an expanding map $f : N \to C$ with expansion coefficient between $\frac{a}{r}$ and $\frac{b}{r}$, i.e. for all distinct $x, y \in N$, we have the estimate

$$\frac{a}{r} \le \frac{d(f(x), f(y))}{d(x,y)} \le \frac{b}{r}$$

Note that a, b are independent of the size of N, and of the choice of points in N.

(iii) (contracting similarities) There exist $c, r_1 > 0$ with the following property. For any $r < r_1$ and ball $B \subseteq C$ with radius r, there exists a map $f : C \to B \cap C$ contracting no more than a factor cr.

Let $s = \mathrm{Hdim}(C)$ be the Hausdorff dimension of C, and H^s the corresponding Hausdorff measure. The following alternate property to (iii) is implied by (ii) and (iii):

(iiia) For any open $U \subseteq C$ one has $0 < H^s(U) < \infty$, and $H^s(U) \to 0$ as $\mathrm{diam}(U) \to 0$.

The main result of Falconer-Marsh [6] is that two quasicircles C_1 and C_2 are bi-Lipschitz equivalent if and only if their Hausdorff dimensions are equal. While stated this way, their proof only uses conditions (i), (ii), and (iiia). For this reason, we will say that a metric space (C, d) is a *weak quasicircle* when conditions (i), (ii), and (iiia) are satisfied.

Theorem 2.7 (Falconer-Marsh)**.** *Two weak quasicircles C_1 and C_2 are bi-Lipschitz equivalent if and only if their Hausdorff dimensions are equal.*

Remark 2.8. In geometric function theory, the term quasicircle is used for the image of S^1 under a quasisymmetric map f. We will call these *quasisymmetric circles.* We will now sketch the proof that Falconer-Marsh quasicircles form a strict subset of the quasisymmetric circles.

To see that any Falconer-Marsh quasicircle X is a quasisymmetric circle, recall that X is bi-Lipschitz equivalent to any Falconer-Marsh quasicircle with the same Hausdorff dimension as X. On the other hand, by Theorem 1.1, the visual boundary Λ of a closed negatively curved surface S is a Falconer-Marsh quasicircle. By scaling the metric on S we can arrange that $\mathrm{Hdim}(\Lambda) = \mathrm{Hdim}(X)$, so that Λ and X are bi-Lipschitz equivalent. Finally, recall that the visual boundary of a negatively curved surface is a quasisymmetric circle. Therefore X is a quasisymmetric circle.

To produce quasisymmetric circles that are not Falconer-Marsh quasicircles, note that by Property (ii) of Falconer-Marsh quasicircles (see Definition 2.6) any nonempty open subset U of a Falconer-Marsh quasicircle X has the same Hausdorff dimension as X. However, quasisymmetric circles need not have this property (see [16]).

2.4. **Almost-isometries and the work of Bonk-Schramm.**

Now let us recall some results of Bonk and Schramm [1] that we will need.

Definition 2.9. A map between $f : X \to Y$ between metric spaces (X, d_X) and (Y, d_Y) is *quasi-isometric* if there exists constants C, K such that

$$\frac{1}{C} d_X(x, y) - K \leq d_Y\left(f(x), f(y)\right) \leq C d_X(x, y) + K.$$

A map $f : X \to Y$ is *coarsely onto* if Y lies in a bounded neighborhood of $f(X)$. If the quasi-isometric map f is coarsely onto, then we call it a *quasi-isometry*, and we say that X, Y are *quasi-isometric.* A map is almost-isometric if it is quasi-isometric with multiplicative constant $C = 1$. An almost-isometric map $f : X \to Y$ which is coarsely onto is called an *almost-isometry*, in which case we say that X, Y are *almost-isometric.*

Special cases of the results in [1] relate the existence of almost-isometries with metric properties of ∂X, as follows.

Theorem 2.10 (Bonk-Schramm). *Let X, Y be a pair of CAT(-1) spaces. Then X, Y are almost-isometric if and only if the canonical visual metrics on the boundaries ∂X, ∂Y are bi-Lipschitz homeomorphic to each other.*

The fact that an almost-isometry between X and Y induces a bi-Lipschitz homeomorphism between the boundaries ∂X and ∂Y appears in [1, proof of Theorem 6.5] – where it should be noted that, in the notation of their proof, our more restrictive context corresponds to $\epsilon = \epsilon' = 1$ and $\lambda = 1$. As for the converse, the interested reader should consult [1, Theorem 7.4] to see that a bi-Lipschitz map between boundaries ∂X and ∂Y induces an almost-isometry between the metric spaces $\mathrm{Con}(X)$ and $\mathrm{Con}(Y)$. The comment following [1, Theorem 8.2] applies in our context where $a = 1$, so that $\mathrm{Con}(X)$ and $\mathrm{Con}(Y)$ are almost isometric to X and Y, respectively, whence X and Y are almost-isometric.

2.5. **Proof of Corollaries 1.3, 1.4, and 1.6.** The proof of all three corollaries are now completely straightforward.

Proof of Corollary 1.4. Theorem 1.1 gives us that $(\partial X, \rho_1)$ and $(\partial X, \rho_2)$ are weak quasicircles. Then Falconer and Marsh's Theorem 2.7 and Remark 2.5 gives the equivalence of statements (1) and (2), while Bonk and Schramm's Theorem 2.10 gives the equivalence of statements (2) and (3). □

Proof of Corollary 1.3. When two metrics on X are related by a scale factor λ, it easily follows from the formula for the canonical visual metric that the Hausdorff dimension of the boundary at infinity scales by $1/\lambda$. Corollary 1.3 immediately follows by combining our Theorem 1.1, Falconer and Marsh's Theorem 2.7, and Bonk and Schramm's Theorem 2.10. □

Proof of Corollary 1.6. Combining our Theorem 1.1, Falconer and Marsh's Theorem 2.7, and Bonk and Schramm's Theorem 2.10, we see that the boundaries at infinity $(\partial X_i, \rho_i)$ must have distinct Hausdorff dimensions. Without loss of generality, let us assume $\mathrm{Hdim}(\partial X_1, \rho_1) < \mathrm{Hdim}(\partial X_2, \rho_2)$. If $\phi : X_1 \to X_2$ is a (C, K)-quasi-isometry, we want to obtain a lower bound on C. But the quasi-isometry induces a homeomorphism $\partial\phi : \partial X_1 \to \partial X_2$ which, from the definition of the canonical visual metrics, has the property that

$$\rho_2\left(\partial\phi(x), \partial\phi(y)\right) \leq e^K \cdot \rho_1(x, y)^C$$

for all $x, y \in \partial X_1$. It easily follows that $\mathrm{Hdim}(\partial X_2, \rho_2) \leq C \cdot \mathrm{Hdim}(\partial X_1, \rho_1)$, giving us the desired inequality

$$1 < \frac{\mathrm{Hdim}(\partial X_2, \rho_2)}{\mathrm{Hdim}(\partial X_1, \rho_1)} \leq C.$$

□

3. Surface boundaries are (weak) quasicircles

This section is devoted to the proof of Theorem 1.1. Looking at Definition 2.6, property (i) is obvious (see the discussion in Section 2.1), while property (iiia) is an immediate consequence of Theorem 2.4. It remains to establish property (ii).

Let Γ be a surface group, and let (X, d) be a CAT(-1) space. Suppose Γ acts on X isometrically, properly discontinuously, and convex cocompactly. Let $\Lambda \subseteq \partial X$ be the limit set of Γ and write $Y := \mathrm{Co}(\Lambda)$ for the convex hull of Λ.

Lemma 3.1. *There exists $K > 0$ such that the following holds. Let η be any geodesic in X with endpoints in Λ (and hence $\eta \subset Y$) and let $N_K(\eta)$ be the K-neighborhood of η. Then $N_K(\eta) \cap Y$ separates Y.*

Proof. Let Σ be a closed hyperbolic surface with fundamental group Γ and fix a homotopy equivalence $f : Y/\Gamma \to \Sigma$. Lift f to a map on universal covers $\tilde{f} : Y \to \tilde{\Sigma}$. Then $\tilde{f}$ is a (C, L)-quasi-isometry for some $C \geq 1$ and $L > 0$. Further we can choose $D > 0$ such that a (C, L)-quasigeodesic in $\tilde{\Sigma}$ is Hausdorff distance at most D from a geodesic. We will prove the lemma with $K := 2CD + L$.

To see this, let η be a geodesic in Y. Then $\tilde{f} \circ \eta$ is a (C, L)-quasigeodesic and hence is contained in the D-neighborhood of a geodesic γ. Since γ separates $\tilde{\Sigma}$, it follows that $\tilde{f}^{-1}(N_D(\gamma))$ separates Y. Now let $x \in \tilde{f}^{-1}(N_D(\gamma))$. Then we have

$$d(x, \eta) \leq Cd(\tilde{f}(x), \tilde{f} \circ \eta) + L \leq 2CD + L,$$

which is the desired bound. □

Note that for K as in Lemma 3.1, $Y \backslash N_K(\eta)$ will consist of two unbounded components because $\partial N_K(\eta) = \partial \eta$ consists of two distinct points in $\Lambda \cong S^1$, so that $\Lambda \backslash \partial N_K(\eta)$ consists of two components.

Proposition 3.2 (Definite expansion)**.** *Fix a bounded fundamental domain W for the Γ-action on Y and fix $w \in W$. Further choose $K > 0$ as in Lemma 3.1. Then there exists $A > 0$ with the following property.*

- *Let η be a geodesic in Y such that $w \notin N_K(\eta)$,*
- *set $R := d(w, \eta)$ and let p be the projection of w onto η,*
- *let $\gamma \in \Gamma$ be such that $\gamma p \in W$,*
- *let ξ be any geodesic in the unbounded component of $Y \backslash N_K(\eta)$ not containing w.*

Then

$$R - A \leq d(w, \xi) - d(w, \gamma \cdot \xi) \leq R + A.$$

Proof. As X is CAT(-1), X is also Gromov hyperbolic. Hence, there is a $\delta > 0$ with the property that for each geodesic triangle $\Delta(abc)$, the side $[a, b]$ is contained in the δ-neighborhood of the union of the remaining sides

$[a,c] \cup [b,c]$. Let $D = \mathrm{diam}(W)$. The estimates that follow will show the Lemma holds when $A = 4D + 4K + 12\delta$.

Let q (resp. q') be the projection of w onto ξ (resp. $\gamma\xi$), and let p' be the projection of w onto $\gamma\eta$. Then since $w, \gamma p \in W$,

$$d(w,p') = d(w,\gamma\eta) \leq d(w,\gamma p) \leq D \tag{3.1}$$

and

$$d(\gamma p,p') \leq d(\gamma p,w) + d(w,p') \leq 2D. \tag{3.2}$$

Now we claim that

$$d(p,[w,q]) < 3\delta + K \tag{3.3}$$

and

$$d(p',[w,q']) < 3\delta + K. \tag{3.4}$$

To prove (3.3), choose $x \in [w,q] \cap N_K(\eta)$ (note that $[w,q] \cap N_K(\eta)$ is nonempty because the endpoints of ξ lie in the component of $Y \backslash N_K(\eta)$ that does not contain w). Let u be the projection of x onto η. If $d(p,u) < 2\delta$ then we are done (since $d(x,u) \leq K$), so let us assume that $d(p,u) \geq 2\delta$. Consider the geodesic triangle $\Delta(wpu)$, and let y be the point on $[p,u]$ at distance 2δ from p. By the definition of δ, we know that $d(y,[w,p] \cup [w,u]) < \delta$. However, since $d(y,p) > \delta$ and p is the projection of y onto $[w,p]$, we have $d(y,[w,p]) > \delta$. Therefore we must have $d(y,[w,u]) < \delta$. Since we also know $d(y,p) = 2\delta$, we conclude that $d(p,[w,u]) < 3\delta$. Using convexity of distance functions, we have that (u,x) maximizes the function $d : [w,u] \times [w,x] \to \mathbb{R}$. As $[w,x] \subset [w,q]$,

$$d(p,[w,q]) \leq d(p,[w,x]) \leq d(p,[w,u]) + d(u,x) < 3\delta + K.$$

The proof of (3.4) is analogous to that of (3.3). Figure 1 illustrates this argument in the special case where $K = 0$.

By (3.3) and (3.4), there exist points z and z' on the segments $[w,q]$ and $[w,q']$ such that

$$d(p,z) < 3\delta + K \tag{3.5}$$

and

$$d(p',z') < 3\delta + K. \tag{3.6}$$

Let v be the projection of p onto ξ, and let v' be the projection of p' onto $\gamma\xi$ (see Figure 2). Note that γv is the projection of γp onto $\gamma\xi$.

As the projection of the segment $[p,z]$ to ξ is the segment $[v,q]$, and since projections decrease distances, (3.5) implies

$$d(v,q) \leq d(p,z) \tag{3.7}$$

Analogous arguments show that

$$d(v',q') \leq d(p',z') \tag{3.8}$$

and

$$d(\gamma v,v') \leq d(\gamma p,p'). \tag{3.9}$$

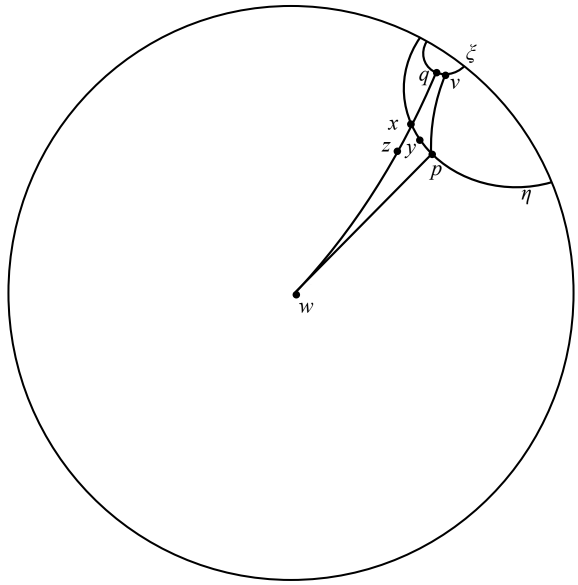

FIGURE 1. Proof that $d(p, [w, q]) < 3\delta$ if $\dim(Y) = 2$.

Use the triangle inequality, (3.2), and (3.5)-(3.9) to estimate

$$\begin{aligned} d(\gamma q, q') &\leq d(\gamma q, \gamma v) + d(\gamma v, q') \\ &= d(q, v) + d(\gamma v, q') \\ &\leq d(p, z) + d(\gamma v, q') \\ &\leq d(p, z) + d(\gamma v, v') + d(v', q') \\ &\leq d(p, z) + d(\gamma p, p') + d(p', z') \\ &< 3\delta + K + 2D + 3\delta + K \\ &= 2D + 2K + 6\delta \end{aligned} \tag{3.10}$$

The triangle inequality and the assumption $d(w, p) = d(w, \eta) = R$ imply

$$d(w, \xi) = d(w, q) \leq d(w, p) + d(p, q) = R + d(p, q). \tag{3.11}$$

The triangle inequality and (3.1) imply

$$d(w, \gamma\xi) \leq d(w, \gamma q) \leq d(w, \gamma p) + d(\gamma p, \gamma q) = D + d(p, q). \tag{3.12}$$

The triangle inequality, (3.5), and the assumption $d(w, p) = R$ imply

$$\begin{aligned} d(w, \xi) &= d(w, q) \\ &= d(w, z) + d(z, q) \\ &\geq d(w, p) - d(p, z) + d(p, q) - d(p, z) \\ &> R + d(p, q) - 6\delta - 2K. \end{aligned} \tag{3.13}$$

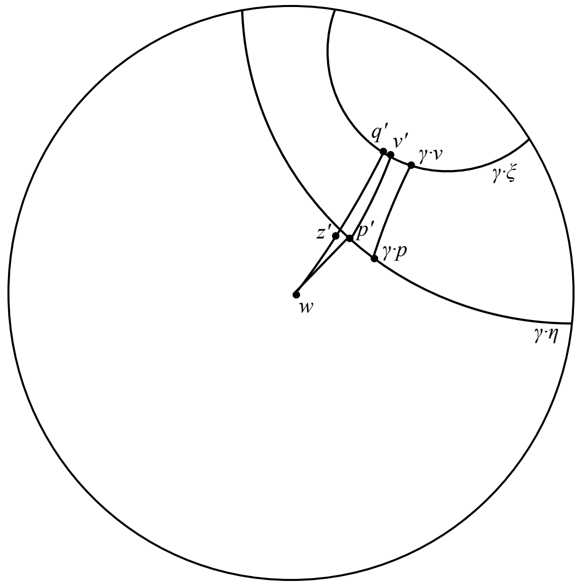

Figure 2. Proof of Proposition 3.2.

Similarly, the triangle inequality, (3.2), (3.6), and (3.10) imply

$$\begin{aligned} d(w,\gamma\xi) &= d(w,z') + d(z',q') \\ &\geq d(w,p') - d(p',z') + d(p',q') - d(p',z') \\ &> d(p',q') - 6\delta - 2K \qquad (3.14) \\ &\geq d(\gamma p,\gamma q) - d(\gamma p,p') - d(\gamma q,q') - 6\delta - 2K \\ &\geq d(p,q) - 2D - d(\gamma q,q') - 6\delta - 2K \\ &\geq d(p,q) - 2D - (2D + 2K + 6\delta) - 6\delta - 2K \\ &= d(p,q) - (4D + 4K + 12\delta). \end{aligned}$$

Combining (3.11) and (3.14) gives

$$d(w,\xi) - d(w,\gamma\xi) \leq R + (4D + 4K + 12\delta) = R + A$$

Combining (3.12) and (3.13) gives

$$d(w,\xi) - d(w,\gamma\xi) \geq R - (D + 2K + 6\delta) > R - A,$$

concluding the proof of the proposition.

□

Now consider the visual metric associated to the basepoint $w \in X$ defined by

$$d_w(x,y) = e^{-(x|y)_w}$$

for $x, y \in \partial X$. We will prove that (Λ, d_w) has Property (ii) from Definition 2.6. Since d_w and $\hat{d}_w$ are bi-Lipschitz equivalent (see Remark 2.3) and

Property (ii) continues to hold after a change up to bi-Lipschitz equivalence, we will work with $\hat{d}_w$ instead.

Fix a fundamental domain W for the Γ-action on Y and choose $A \geq 0$ as in Proposition 3.2. Let $r_0 := \frac{1}{2}e^{-K}$, $a := e^{-A}$, and $b := e^{A}$.

Given $0 < r < r_0$ and N a ball of radius r in Λ, let η be the geodesic connecting the endpoints of N so that $2r = \mathrm{diam}(N) = e^{-d(w,\eta)}$. Note that since $2r < e^{-K}$, we have $d(w, \eta) \geq -\log(2r) - K > 0$ so that $N_K(\eta)$ does not contain w. Further since $r < \frac{1}{2}$, we know that $\mathrm{diam}(N) < \mathrm{diam}(\partial X \backslash N)$, so that N lies on the side of η not containing w.

Let p be the projection of w onto η and choose γ such that $\gamma p \in W$. Now let $x, y \in N$ be distinct and let ξ be the geodesic joining x and y. By Proposition 3.2, we have

$$R - A \leq d(w, \xi) - d(w, \gamma \cdot \xi) \leq R + A,$$

where $R := d(w, \eta) = -\log(r)$. Further note that

$$\frac{\hat{d}_w(\gamma x, \gamma y)}{\hat{d}_w(x, y)} = e^{d(w,\xi) - d(w, \gamma\cdot\xi)}$$

so we have

$$e^{-A}e^{R} \leq \frac{\hat{d}_w(\gamma x, \gamma y)}{\hat{d}_w(x, y)} \leq e^{A}e^{R},$$

or equivalently

$$\frac{a}{r} \leq \frac{\hat{d}_w(\gamma x, \gamma y)}{\hat{d}_w(x, y)} \leq \frac{b}{r}.$$

This establishes property (ii) in Definition 2.6, and hence completes the proof of Theorem 1.1.

Remark 3.3. Shortly after this preprint was written, we learned from Marc Bourdon that property (ii) was mentioned by Kleiner in his ICM address [13]. A proof that visual boundaries of δ-hyperbolic groups satisfy this property (ii) can also be found in their joint paper [3, Section 3.1].

4. Constructing exotic almost-isometries

This section is devoted to the proof of Theorem 1.5. In view of Corollary 1.4, we want to produce a k-dimensional family $\mathcal{F}_k$ of equal area metrics on a higher genus surface M, which all have the same topological entropy, but whose lifts to the universal cover are not isometric to each other.

4.1. **Perturbations of metrics.**

We start with a fixed reference hyperbolic metric g_0 on M, normalized to have constant curvature -2. Pick $(k+2)$ distinct points $p_1, \ldots p_{k+2} \in M$, and choose r_2 smaller than the injectivity radius of M and satisfying $2r_2 < \inf_{i \neq j}\{d(p_i, p_j)\}$. Let U_i denote the open metric ball of radius r_2 centered at p_i – note that the U_i are all isometric to each other, and are pairwise disjoint. Now choose $r_1 < r_2$ so that the area of the ball of radius r_1 is at least $4/5$ the area of the ball of radius r_2. Denote by $V_i \subset U_i$ the ball of radius r_1 centered at each p_i.

We will vary the metric g_0 by introducing a perturbation on each of the U_i in the following manner. Let us choose a smooth bump function $\rho : [0,\infty) \to [0,1]$ with the property that $\rho|_{[0,r_1]} \equiv 1$ and $\rho|_{[r_2,\infty)} \equiv 0$. Next define $u_i : M \to [0,1]$ via $u_i(x) := \rho\,(d(x,p_i))$. Given a parameter $\vec{t} := (t_1,\dots,t_{k+2}) \in \mathbb{R}^{k+2}$, define the function $u_{\vec{t}} : M \to [0,\infty)$ by setting $u_{\vec{t}} := t_1 u_1 + \cdots + t_{k+2}u_{k+2}$. Finally, we define the metric $g_{\vec{t}} := e^{2u_{\vec{t}}} g_0$ (and note the identification $g_{\vec{0}} = g_0$). The family $\mathcal{F}_k$ will be obtained by choosing suitable values of $\vec{t}$ close to $\vec{0}$.

Since the metric $g_{\vec{t}}$ is obtained by making a conformal change on each U_i, and since the U_i are pairwise disjoint, we first analyze the behavior of such a change on an individual U_i. To simplify notation, denote by $V \subset U$ open balls of radius $r_1 < r_2$ centered at a point p in the hyperbolic plane $\mathbb{H}^2_{-2}$ of curvature -2, and set $g_t := e^{2tu} g_0$ where g_0 is the hyperbolic metric of curvature -2, and $u : \mathbb{H}^2_{-2} \to [0,\infty)$ is given by $u(x) := \rho\,(d(p,x))$. We start with the easy:

Lemma 4.1. *As $t \to 0$, we have the following estimates:*

(1) the curvatures $K(g_t)$ tend uniformly to -2.
(2) the area $Area(U; g_t)$ of the ball U tends to $Area(U; g_0)$.
(3) the area $Area(V; g_t)$ of the ball V tends to $Area(V; g_0)$.

Proof. This is straightforward from the formulas expressing how curvature and area change when one makes a conformal change of metric. We have that the new curvature $K(g_t)$ is related to the old curvature $K(g_0)$ via the formula

$$K(g_t) = (e^{-2u})^t K(g_0) - t(e^{-2u})^t \Delta u$$

where Δu denotes the Laplacian of the function u in the hyperbolic metric g_0. As t tends to zero, it is clear that the expression to the right converges to $K(g_0)$ uniformly, giving (1). Similarly, the area form dg_t for the new metric is related to the area form dg_0 for the original metric via the formula $dg_t = (e^{2u})^t dg_0$ giving us (2) and (3). $\square$

4.2. **Lifted metrics are almost-isometric.** Next we establish that, for suitable choices of the parameter $\vec{t}$, we can arrange for the lifted metrics to be almost-isometric. By Lemma 4.1, we can take the parameters $\vec{t}$ close enough to $\vec{0}$ to ensure that all the metrics we consider have sectional curvatures ≤ -1. Then from Corollary 1.4, it suffices to consider values of the parameter $\vec{t}$ for which the corresponding metrics have the same topological entropy for the geodesic flow on T^1M. Notice that varying $\vec{t}$ near $\vec{0}$ gives a C^∞ family of perturbations of the metric g_0. Work of Katok, Knieper, Pollicott and Weiss [11, Theorem 2] then implies that the topological entropy map h, when restricted to any line $l(s)$ through the origin $\vec{0}$ in the $\vec{t}$-space, is a

C^∞ map. Moreover the derivative of h along the line is given by (see [12, Theorem 3])

$$\frac{\partial}{\partial s}\Big|_{s=0} h\left(g_{l(s)}\right) = -\frac{h(g_0)}{2}\int_{T^1M}\frac{\partial}{\partial s}\Big|_{s=0} g_{l(s)}(v,v)d\mu_0$$

where T^1M denotes the unit tangent bundle of M with respect to the g_0-metric, and μ_0 denotes the Margulis measure of g_0 (the unique measure of maximal entropy for the g_0-geodesic flow on T^1M).

Consider the map $F : \mathbb{R}^{k+2} \to \mathbb{R}$ given by $F(t_1, \ldots, t_{k+2}) := h(g_{(t_1,\ldots,t_{k+2})})$, where h denotes the topological entropy of (the geodesic flow associated to) a metric. Let us compute the directional derivative in the direction $\frac{\partial}{\partial t_1}$:

$$\begin{aligned}\frac{\partial F}{\partial t_1}(0,\ldots,0) &= \frac{d}{dt}\Big|_{t=0} h\left(g_{(t,0,\ldots,0)}\right) = -\frac{h(g_0)}{2}\int_{T^1M}\frac{d}{dt}\Big|_{t=0} g_{(t,0,\ldots,0)}(v,v)d\mu_0\\ &= -\frac{h(g_0)}{2}\int_{T^1M}\frac{d}{dt}\Big|_{t=0} e^{2tu_1\left(\pi(v)\right)}d\mu_0\\ &= -\frac{h(g_0)}{2}\int_{T^1M} 2u_1\left(\pi(v)\right)d\mu_0\end{aligned}$$

where $\pi : T^1M \to M$ is the projection from the unit tangent bundle onto the surface M. Finally, we observe that by construction u_1 is a non-negative function, which is identically zero on the complement of U_1, and identically one on the set V_1. Hence the integral above is positive, and we obtain $\frac{\partial F}{\partial t_1}(\vec{0}) < 0$.

Now, a similar calculation applied to each of the other coordinates gives us the general formula for the directional derivative of F. The gradient of F is given by the non-vanishing vector:

$$\nabla F = -h(g_0)\int_{T^1M}\langle u_1\left(\pi(v)\right), \ldots, u_{k+2}\left(\pi(v)\right)\rangle d\mu_0.$$

In fact, since each u_i is supported solely on U_i, and each u_i is defined as $\rho\left(d(p_i, x)\right)$ on the U_i, each of the integrals in the expression for ∇F has the same value. So ∇F is just a nonzero multiple of the vector $\langle 1, \ldots, 1\rangle$.

The implicit function theorem now locally gives us an embedded codimension one submanifold $\sigma(z)$ (where $z \in \mathbb{R}^{k+1}$, $||z|| < \epsilon$) in the $(t_1, \ldots, t_{k+2})$-space, with normal vector $\langle 1, \ldots, 1\rangle$ at the point $\sigma(\vec{0}) = \vec{0}$, on which the topological entropy functional is constant. From Corollary 1.4, we see that the lifts of these metrics to the universal cover are all pairwise almost-isometric.

4.3. **Lifted metrics are not isometric.**

Lemma 4.2. *There is an $\epsilon > 0$ so that if the parameters $\vec{s} = (s_1, \ldots, s_{k+2})$ and $\vec{t} = (t_1, \ldots, t_{k+2})$ satisfy $0 < |s_i| < \epsilon$ and $0 < |t_i| < \epsilon$ and the lifted metrics $(\tilde{M}, \tilde{g}_{\vec{s}})$ and $(\tilde{M}, \tilde{g}_{\vec{t}})$ are isometric to each other, then we must have an equality of multisets $\{s_1, \ldots, s_{k+2}\} = \{t_1, \ldots, t_{k+2}\}$.*

We recall that a multi-set is a set with multiplicities associated to each element. Equality of multi-sets means not only that the underlying sets are equal, but that the corresponding multiplicities are equal.

Proof. By Lemma 4.1, it is possible to pick ϵ small enough so that, for all parameters $\vec{s}, \vec{t}$ within the ϵ-ball around $\vec{0}$, we have that

$$\text{Area}(V_i; g_{\vec{t}}) \geq \frac{3}{4} \text{Area}(U_j; g_{\vec{s}})$$

for every $1 \leq i, j \leq k+2$.

Now let us assume that there is an isometry $\Phi : (\tilde{M}, \tilde{g}_{\vec{s}}) \to (\tilde{M}, \tilde{g}_{\vec{t}})$. Observe that the lifted metrics have the following properties:

(i) on the complement of the lifts of the U_i, both metrics have curvature identically -2.
(ii) on any lift of the set V_1, the metric $\tilde{g}_{\vec{s}}$ has curvature identically $-2e^{-2s_1}$.
(iii) on any lift of the set V_i, the metric $\tilde{g}_{\vec{t}}$ has curvature identically $-2e^{-2t_i}$.

Take a lift $\tilde{V}_1$ of V_1 in the source, and consider its image under Φ. The metric in the source has curvature identically $-2e^{-2s_1}$ on this lift $\tilde{V}_1$, and since Φ is an isometry, the image set $\Phi(\tilde{V}_1)$ must have the same curvature. From property (i), we see that $\Phi(\tilde{V}_1)$ must lie, as a set, inside the union of lifts of the U_i. Since $\Phi(\tilde{V}_1)$ is path-connected, it must lie inside a single connected lift $\tilde{U}_i$ of one of the U_i. But from the area estimate, we see that for the $\tilde{V}_i \subset \tilde{U}_i$ inside the lift, one has that the intersection $\Phi(\tilde{V}_1) \cap \tilde{V}_i$ is non-empty. Looking at the curvature of a point in the intersection, we see that

$$-2e^{-2s_1} = -2e^{-2t_i}$$

and hence that $s_1 = t_i$ for some i. Applying the same argument to each of s_i, t_i completes the proof. □

Now pick a vector $\vec{v} = \langle v_1, \dots, v_{k+2} \rangle$ with the property that $v_1 + \dots + v_{k+2} = 0$, and such that $v_i \neq v_j$ for each $i \neq j$. Notice that the first constraint just means that $\vec{v} \cdot \nabla F = 0$, and hence that $\vec{v}$ is tangent to the $(k+1)$-dimensional submanifold σ. So there exists a curve $\gamma \subset \sigma$ satisfying $\gamma(0) = \vec{0}$, and $\gamma'(0) = \vec{v}$. Notice that, from our second condition, when $t \approx 0$ we have $\gamma(t) \approx (v_1 t, \dots, v_{k+2} t)$, and hence the point $\gamma(t)$ has *all coordinates distinct*. It follows from Lemma 4.2 that, for any $t \approx 0$ $(t \neq 0)$, one can find a small enough connected neighborhood W_t of $\gamma(t)$ with the property that all the metrics in that neighborhood have lifts to the universal cover that are pairwise non-isometric.

4.4. **Metrics with equal area.** Now consider the smooth function

$$A : \sigma \to \mathbb{R}$$

defined by $A(z) := \text{Area}(g_{\sigma(z)})$ for each $z \in \sigma$. The change of area formula for a conformal change of metric (see the proof of Lemma 4.1) implies that A is nonconstant on W_t. By Sard's theorem, there is a regular value r

of A in the interval $A(W_t)$. Then $\tau := A^{-1}(r)$ is a smooth k-dimensional submanifold of the $(k+1)$-dimensional manifold σ consisting of parameters for area r metrics. A connected component $\mathcal{F}_k$ of $W_t \cap A^{-1}(r)$ satisfies all of the constraints of Theorem 1.5.

5. Concluding remarks

As the reader undoubtedly noticed, our results rely heavily on the surprising result of Falconer and Marsh. As such, it is very specific to the case of circle boundaries – which essentially restricts us to surface groups (see Gabai [8]). In higher dimensions, we would not expect the bi-Lipschitz class of a self-similar metric on a sphere to be classified by its Hausdorff dimension. Thus, the following problem seems substantially more difficult.

Conjecture. Let M be a smooth closed manifold of dimension ≥ 3, and assume that M supports a negatively curved Riemannian metric. Then M supports a pair of equal volume Riemannian metrics g_1, g_2 with curvatures ≤ -1, and having the property that the Riemannian universal covers $(\tilde{M}, \tilde{g}_i)$ are almost-isometric, but are **not** isometric.

In another direction, if one were to *drop* the dimension, then there are many examples of 0-dimensional spaces having analogous self-similarity properties (i.e. properties (ii), (iii) in Definition 2.6). The metrics on these boundaries turn them into Cantor sets – and the classification of (metric) Cantor sets up to bi-Lipschitz equivalence seems much more complex than in the circle case (for some foundational results on this problem, see for instance Falconer and Marsh [7] and Cooper and Pignatoro [4]). Of course, from the viewpoint of boundaries, such spaces would typically arise as the boundary at infinity of a metric tree T. This suggests the following:

Problem. Study periodic metrics on trees up to the relation of almost-isometry.

In particular, invariance of the metric under a cocompact group action translates to additional constraints on the canonical visual metric on ∂T, e.g. the existence of a large (convergence) group action via conformal automorphisms (compare with the main theorem in [4]). It would be interesting to see if this makes the bi-Lipschitz classification problem any easier.

Finally, given a pair of quasi-isometric spaces, we can consider the collection of *all* quasi-isometries between them, and try to find the quasi-isometry which has smallest multiplicative constant. More precisely, given a pair of quasi-isometric metric spaces X_1, X_2, define the real number $\mu(X_1, X_2)$ to be the infimum of the real numbers C with the property that there exists some (C, K)-quasi-isometry from X_1 to X_2. We can now formulate the:

Problem. Given a pair of quasi-isometric metric spaces X_1, X_2, can one estimate $\mu(X_1, X_2)$? Can one find a (C, K)-quasi-isometry from X_1 to X_2, where $C = \mu(X_1, X_2)$? In particular, can one find a pair of quasi-isometric spaces X_1, X_2 which are **not** almost-isometric, but which nevertheless satisfy $\mu(X_1, X_2) = 1$?

Our Corollary 1.6 gives a complete answer in the case where the X_i are universal covers of locally CAT(-1) metrics on surfaces – the real number $\mu(X_1, X_2)$ is exactly the ratio of the Hausdorff dimensions of the canonical visual metrics on the boundary, and one can always find a quasi-isometry with multiplicative constant $\mu(X_1, X_2)$. It is unclear what to expect in the more general setting.

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