ISOMETRY TYPES OF FRAME BUNDLES

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ABSTRACT. We consider the oriented orthonormal frame bundle $SO(M)$ of an oriented Riemannian manifold M. The Riemannian metric on M induces a canonical Riemannian metric on $SO(M)$. We prove that for two closed oriented Riemannian n-manifolds M and N, the frame bundles $SO(M)$ and $SO(N)$ are isometric if and only if M and N are isometric, except possibly in dimensions 3, 4, and 8. This answers a question of Benson Farb except in dimensions 3, 4, and 8.

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1. INTRODUCTION

Let M be an oriented Riemannian manifold, and let $X := SO(M)$ be the oriented orthonormal frame bundle of M. The Riemannian structure g on M induces in a canonical way a Riemannian metric g_{SO} on $SO(M)$. This construction was first carried out by O'Neill [O'N66] and independently by Mok [Mok77], and is very similar to Sasaki's construction of a metric on the unit tangent bundle of M [Sas58, Sas62], so we will henceforth refer to g_{SO} as the *Sasaki-Mok-O'Neill metric* on $SO(M)$. Let us sketch the construction of g_{SO} and refer to Section 2 for the details. Consider the natural projection $\pi : SO(M) \to M$. Each of the fibers of p is naturally equipped with a free and transitive $SO(n)$-action, so that this fiber carries an $SO(n)$-bi-invariant metric $g_{\mathcal{V}}$. The metric $g_{\mathcal{V}}$ is determined uniquely up to scaling. Further, the Levi-Civita connection on the tangent bundle $TM \to M$ induces a horizontal subbundle of TM. This in turn induces a horizontal subbundle $\mathcal{H}$ of $TSO(M)$. We can pull back the metric on M along π to get a metric $g_{\mathcal{H}}$ on $\mathcal{H}$. The Sasaki-Mok-O'Neill metric on $SO(M)$ is defined to be $g_{SO} := g_{\mathcal{V}} \oplus g_{\mathcal{H}}$.

Note that g_{SO} is determined uniquely up to scaling of $g_{\mathcal{V}}$, and hence determined uniquely after fixing a bi-invariant metric on $SO(n)$. The work of O'Neill [O'N66], Mok [Mok77], and later Takagi-Yawata [TY91, TY94] have established many natural properties of Sasaki-Mok-O'Neill metrics and connections between the geometry of M and $SO(M)$. The following natural question then arises, which was to my knowledge first posed by Benson Farb.

Question 1.1. Let M, N be Riemannian manifolds. If $\mathrm{SO}(M)$ is isometric to $\mathrm{SO}(N)$ (with respect to Sasaki-Mok-O'Neill metrics on each), is M isometric to N?

The purpose of this paper is to answer Question 1.1 except when $\dim M = 3, 4$ or 8. The question is a bit subtle, for it is not true in general that an isometry of $\mathrm{SO}(M)$ preserves the fibers of $\mathrm{SO}(M) \to M$ as shown by the following example.

Example 1.2. Let M be a constant curvature sphere S^n. Then $\mathrm{SO}(M)$ is diffeomorphic to $\mathrm{SO}(n+1)$. (To see this, identify S^n with the unit sphere in $\mathbb{R}^{n+1}$. If $p \in S^n$ and $v_1, \ldots, v_n$ is a positively oriented orthonormal frame at p, then the matrix with columns $p, v_1, \ldots, v_n$ belongs to $\mathrm{SO}(n+1)$.) There is a unique Sasaki-Mok-O'Neill metric that is isometric to the bi-invariant metric on $\mathrm{SO}(n+1)$. However, of course there are many isometries of $\mathrm{SO}(n+1)$ that do not preserve the fibers of $\mathrm{SO}(n+1) \to S^n$.

By differentiating the action of $\mathrm{SO}(n+1)$ in the above example, we obtain many Killing fields that do not preserve the fibers of $\mathrm{SO}(n+1) \to S^n$. However, by a theorem of Takagi-Yawata [TY91], manifolds with constant positive curvature are the only Riemannian manifolds whose orthonormal frame bundles admit Killing fields that do not preserve the fibers. More examples of non-fiber-preserving isometries appear if we consider isometries that are not induced by Killing fields, as the following example shows.

Example 1.3. Let M be a flat 2-torus obtained as the quotient of $\mathbb{R}^2$ by the subgroup generated by translations by $(l_1, 0)$ and $(0, l_2)$ for some $l_1, l_2 > 0$. Further fix $l_3 > 0$ and equip $\mathrm{SO}(M)$ with the Sasaki-Mok-O'Neill metric associated to the scalar l_3. It is easy to see $\mathrm{SO}(M)$ is the flat 3-torus obtained as the quotient of $\mathbb{R}^3$ by the subgroup generated by translations by $(l_1, 0, 0), (0, l_2, 0)$ and $(0, 0, l_3)$.

Now let N be the flat 2-torus obtained as the quotient of $\mathbb{R}^2$ by the subgroup generated by translations by $(l_1, 0)$ and $(0, l_3)$, and equip $\mathrm{SO}(N)$ with the Sasaki-Mok-O'Neill metric associated to the scalar l_2. Then $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ are isometric but if l_1, l_2, l_3 are distinct, M and N are not isometric.

On the other hand if $l_1 = l_3 \neq l_2$, then this construction produces an isometry $SO(M) \to SO(M)$ that is not a bundle map.

Example 1.3 produces counterexamples to Question 1.1. Note that we used different bi-invariant metrics $g_{\mathcal{V}}$ on the fibers. Therefore to give a positive answer to Question 1.1 we must normalize the volume of the fibers of $\mathrm{SO}(M) \to M$.

Our main theorem is that under the assumption of normalization Question 1.1 has the following positive answer, except possibly in dimensions $3, 4$ and 8.

Theorem A. *Let M, N be closed oriented connected Riemannian n-manifolds. Equip $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ with Sasaki-Mok-O'Neill metrics where the fibers of $\mathrm{SO}(M) \to M$ and $\mathrm{SO}(N) \to N$ have fixed volume $\nu > 0$. Assume $n \neq 3, 4, 8$. Then M, N are isometric if and only if $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ are isometric.*

We do not know if counterexamples to Question 1.1 exist in dimensions $3, 4$, and 8.

Outline of proof. If $f : M \to N$ is an isometry, then the induced map $\mathrm{SO}(f) : SO(M) \to SO(N)$ is also an isometry (see Proposition 2.5). This proves one direction of the theorem.

For the other direction, our strategy is to identify the fibers of the bundle $\mathrm{SO}(M) \to M$ using only the geometry of $\mathrm{SO}(M)$. To accomplish this, note that $X = SO(M)$ carries an action of $\mathrm{SO}(n)$ by isometries, and the orbits of this action are exactly the fibers of $\mathrm{SO}(M) \to M$. This action gives rise to an algebra of Killing fields isomorphic to $\mathfrak{o}(n)$.

The full Lie algebra $i(X)$ of Killing fields on $X = SO(M)$ has been computed by Takagi-Yawata [TY94] except in dimensions $2, 3, 4$ or 8, or when M has positive constant curvature. We show that if this computation applies, either $i(X)$ contains a unique copy

of $\mathfrak{o}(n)$ or $\mathrm{Isom}(M)$ is extremely large or M is flat. If $i(X)$ contains a unique copy of $\mathfrak{o}(n)$, then the fibers of $X = SO(M) \to M$ and $X = SO(N) \to N$ coincide, and we deduce that M and N are isometric.

We are able to resolve the flat case separately. If $\mathrm{Isom}(M)$ is large we use classifcation theorems from the theory of compact transformation groups to prove that M and N are isometric.

Finally we prove the theorem in two situations where the computation of Takagi-Yawata does not apply, namely constant positive curvature and dimension 2. In these situations it is in general impossible to identify the fibers of $SO(M) \to M$ using the geometry of $SO(M)$ alone as shown by Examples 1.2 and 1.3. However, we are still able to obtain the main result using the scarcity of manifolds with a metric of constant positive curvature, and the classification of surfaces.

Outline of the paper. In Section 3 we will review preliminaries about actions of Lie groups of G on a manifold M when $\dim G$ is large compared to $\dim M$. In Section 4 we will prove the Main Theorem A except when M and N are surfaces or have metrics of consnt positive curvature. The proof in the case that at least one of M or N has constant positive curvature will be given in Section 5. We prove Theorem A in the case that M and N are surfaces in Section 6.

2. Preliminaries

In this section we introduce the Sasaki-Mok-O'Neill metric, and we recall some basic properties. Then we discuss the classical relationship between isometries and Killing fields, and Takagi-Yawata's computations of Killing fields of Sasaki-Mok-O'Neill metrics. We end this section with a useful lemma for normalizing Sasaki-Mok-O'Neill metrics, and some general remarks about frame bundles of fiber bundles that will also be useful later.

2.1. **Definition of the Sasaki-Mok-O'Neill metric.** Our discussion here follows the construction of Mok [Mok77], where more details can be found. Let (M, g) be an oriented Riemannian manifold of dimension n, and let $X := SO(M)$ be the oriented orthonormal frame bundle of M with natural projection map $\pi : SO(M) \to M$. For $e \in SO(M)$, the *vertical subspace* at e is defined to be $\mathcal{V}_e := \ker D_e\pi$. The collection of vertical subspaces forms a subbundle $\mathcal{V} \to TM$ of $T\,SO(M) \to TM$.

Let ω be the Riemannian connection $\mathfrak{o}(n)$-valued 1-form associated to the Riemannian metric on M. Explicitly, if $p \in M$ and $e = (e_1, \ldots, e_n)$ is a frame at p, we define for $X \in T_eSO(M)$:

$$\omega_{ij}(X) := \theta_j(\nabla_X(e_i)) \qquad (1 \le i, j \le n),$$

where θ_j is the form dual to e_j with respect to the Riemannian metric g.

We set $\mathcal{H}_e := \ker \omega_e$. We call $\mathcal{H}_e$ the *horizontal subspace* at e. We have a decomposition $T_e\,SO(M) = \mathcal{V}_e \oplus \mathcal{H}_e$. Define an inner product on $T_e\,SO(M)$ via

$$g_{\mathrm{SO}}(X, Y) = \langle \omega(X), \omega(Y) \rangle + g(\pi_*X, \pi_*Y)$$

where $\langle \cdot, \cdot \rangle$ is an $O(n)$-invariant inner product on $\mathfrak{o}(n)$. Note that the choice of an $O(n)$-invariant inner product on $\mathfrak{o}(n)$ is uniquely determined up to scaling by a positive number

λ, so that we obtain a 1-parameter family of Sasaki-Mok-O'Neill metrics. Explicitly such an inner product is given by

$$\langle A, B \rangle_\lambda := -\lambda \operatorname{tr}(AB) = \lambda \sum_{i,j} A_{ij} B_{ij}$$

for $A, B \in \mathfrak{o}(n)$. We call $\langle \cdot, \cdot \rangle_1$ the *standard metric* on $\mathfrak{o}(n)$.

Remark 2.1. The oriented orthonormal frame bundle $\mathrm{SO}(M) \to M$ is an example of a $\mathrm{SO}(n)$-principal bundle of over M, and it has a natural connection form ω as defined above. For a principal G-bundle $E \to B$ with a principal connection form θ, one can construct a so-called connection metric (see e.g. [Zil, Section 1]). The Sasaki-Mok-O'Neill metric is exactly this connection metric in the case of the principal $\mathrm{SO}(n)$-bundle $\mathrm{SO}(M) \to M$ with the connection form ω.

As mentioned above, the geometry of the above-defined metric was first investigated by O'Neill and Mok. In particular they showed:

Proposition 2.2 (O'Neill [O'N66, p. 467], Mok [Mok77, Theorem 4.3])**.** *The fibers of* $\mathrm{SO}(M) \to M$ *are totally geodesic submanifolds of* $\mathrm{SO}(M)$ *with respect to any Sasaki-Mok-O'Neill metric.*

2.2. **Vector fields on frame bundles.** Let X be a vector field on $\mathrm{SO}(M)$. If $X_e \in \mathcal{V}_e$ for any $e \in \mathrm{SO}(M)$, we say X is *vertical*. If $X_e \in \mathcal{H}_e$ for any $e \in \mathrm{SO}(M)$, we say X is *horizontal*.

We will now discuss how to lift a vector field Y on M to a vector field X on $\mathrm{SO}(M)$ such that $\pi_* X = \mathrm{SO}(M)$. There are two useful constructions, called the *horizontal* and *complete* lift of Y. Both constructions start by considering the derivative of the bundle map $\pi : \mathrm{SO}(M) \to M$. For a frame $e \in \mathrm{SO}(M)$, we have a decomposition $T_e \mathrm{SO}(M) = \mathcal{V}_e \oplus \mathcal{H}_e$ as discussed above. Here $\mathcal{V}_e = \ker \pi_*$, and hence π_* restricts to an isomorphism $\mathcal{H}_e \to T_{\pi(e)} M$. Therefore for a vector field Y on M, there exists a unique horizontal vector field Y^H on M with $Y = \pi_* Y^H$. We call Y^H the *horizontal lift* of Y.

The *complete lift* Y^C of Y was first introduced by Kobayashi-Nomizu [KN63]. First observe that given a map $f : M \to M$, we can consider its induced map $\mathrm{SO}(f) : \mathrm{SO}(M) \to \mathrm{SO}(M)$ on frames. Then we can define Y^C as follows: Let φ_t be the 1-parameter family of diffeomorphims of M obtained by integrating Y, so that $Y = \frac{d}{dt}\big|_{t=0} \varphi_t$. Then we define

$$Y^C := \frac{d}{dt}\Big|_{t=0} \mathrm{SO}(\varphi_t).$$

Note that Y^C is in general neither vertical nor horizontal. Mok [Mok79, Section 3] has given a description of Y^C in terms of local coordinates.

2.3. **Killing fields and isometries.** Before considering the isometries of $\mathrm{SO}(M)$ equipped with a Sasaki-Mok-O'Neill metric g_{SO}, we will review some classical facts about the structure of the group of isometries $\mathrm{Isom}(M)$ of a Riemannian manifold M.

Myers-Steenrod [MS39] have proved that $\mathrm{Isom}(M)$ of a Riemannian manifold is a Lie group. If $(h_t)_t$ is a 1-paremeter group of isometries, then $Y := \frac{d}{dt}\big|_{t=0} h_t$ is a vector field on M. Differentiating the condition $h_t^* g = g$ gives the *Killing relation* for Y:

$$\mathcal{L}_Y g = 0, \tag{2.1}$$

where $\mathcal{L}$ is the Lie derivative. Any vector field Y satisfying Equation 2.1 is called a Killing field. Given a Killing field Y on M, the 1-parameter group $(h_t)_t$ obtained by integrating Y consists of isometries. The Killing fields on M form a Lie algebra $i(M)$ of vector fields. We have:

Theorem 2.3. *Let M be a Riemannian manifold. Then* $\mathrm{Isom}(M)$ *is a Lie group (possibly not connected), with Lie algebra $i(M)$.*

2.4. **The Takagi-Yawata theorem on Killing fields.** We will now discuss a complete description due to Takagi-Yawata [TY94] of the Killing fields on $\mathrm{SO}(M)$ in terms of the geometry of M for many manifolds M. Let us first discuss three constructions of Killing fields on $\mathrm{SO}(M)$.

For the first construction, recall that Sasaki showed (see [Sas58, Corollary 1]) that whenever $f : M \to M$ is an isometry of M, the derivative $Df : TM \to TM$ is an isometry of TM (where TM is equipped with a Sasaki metric). Therefore if Y is a Killing field on M, then the complete lift of Y is a Killing field on TM. This is also true for frame bundles:

Proposition 2.4 (Mok [Mok77, Proposition 5.3]). *If Y is a Killing field on M, then Y^C is a Killing field on $\mathrm{SO}(M)$ with respect to any Sasaki-Mok-O'Neill metric.*

In fact the following more general statement is true:

Proposition 2.5. *Let M be a Riemannian manifold and $f : M \to M$ any isometry. Then the induced map $\mathrm{SO}(f) : \mathrm{SO}(M) \to \mathrm{SO}(M)$ is an isometry of $\mathrm{SO}(M)$ with respect to any Sasaki-Mok-O'Neill metric.*

Proof. Note that since the Riemannian connection form ω is canonically associated to the metric, we have $f^*\omega = \omega$. In particular $\mathrm{SO}(f)$ preserves the horizontal subbundle $\mathcal{H} := \ker \omega$. Also note that $\mathrm{SO}(f)$ is a bundle map of $\pi : \mathrm{SO}(M) \to M$ (i.e. we have $\mathrm{SO}(f) \circ \pi = \pi \circ f$), and in particular $\mathrm{SO}(f)$ preserves the vertical subbundle $\mathcal{V} := \ker \pi_*$. Using these facts it is easy to check $\mathrm{SO}(f)$ is an isometry. $\qquad\square$

The second construction of Killing fields comes from the structure of $\mathrm{SO}(M) \to M$ as a principal $\mathrm{SO}(n)$-bundle. There is an action of $\mathrm{SO}(n)$ on the fibers of $\mathrm{SO}(M) \to M$, which is easily seen to be isometric with respect to any Sasaki-Mok-O'Neill metric. Differentiating any 1-parameter subgroup of $\mathrm{SO}(n)$ then gives a Killing field on $\mathrm{SO}(M)$. Explicitly, we can define these as follows: For $A \in \mathfrak{o}(n)$, define the vector field A^* on $\mathrm{SO}(M)$ via $\omega(A^*) = A$, where ω is the connection form as above. Then A^* is a vertical Killing field. Write i_V^M for the Killing fields thus obtained. In particular $i_V^M \cong \mathfrak{o}(n)$ as Lie algebras.

Finally, here is the third construction of a Killing field on $\mathrm{SO}(M)$. Let φ be a 2-form on M, so that it defines a skew-symmetric bilinear form on every tangent space T_pM for $p \in M$. With respect to a frame e of T_pM, the skew-symmetric form φ_p can be represented as a skew-symmetric matrix $A_e \in \mathfrak{o}(n)$. We then define a vector field X^φ on $\mathrm{SO}(M)$ via $\omega_e(X_e^\varphi) := A_e$ and $\pi_*(X_e^\varphi) = 0$. Note that the latter condition just means that we define X^φ to be a vertical vector field. An explicit computation shows that if φ is parallel, then X^φ is a Killing field (see e.g. [TY91]). Denote by $(\Lambda^2 M)_0$ the Lie algebra of parallel 2-forms on M.

Takagi-Yawata have proved that for many manifolds, the above three constructions are the only ways of producing Killing fields on $\mathrm{SO}(M)$:

Theorem 2.6 (Takagi-Yawata [TY94]). *Let M be a closed Riemannian manifold and equip $\mathrm{SO}(M)$ with the Sasaki-Mok-O'Neill metric corresponding to the standard inner product $\langle \cdot, \cdot \rangle_1$ on $\mathfrak{o}(n)$. Suppose M does not have constant curvature $\frac{1}{2}$ and $\dim M \neq 2, 3, 4, 8$. Then for any Killing field X on $\mathrm{SO}(M)$ there exist unique $Y \in i(M), A \in \mathfrak{o}(n)$, and $\varphi \in (\Lambda^2 M)_0$ such that*

$$X = Y^C + A^* + X^\varphi.$$

Remark 2.7. Of course a version of the above result holds for different Sasaki-Mok-O'Neill metrics as well: If we use the inner product $\langle \cdot, \cdot \rangle_\lambda = \lambda \langle \cdot, \cdot \rangle_1$ on $\mathfrak{o}(n)$, the same conclusion holds except that we should now require that M does not have constant curvature $\frac{1}{2\sqrt{\lambda}}$.

An explicit computation shows that if $Y \in i(M), A \in \mathfrak{o}(n)$, and $\varphi \in (\Lambda^2 M)_0$, then the vector fields Y^C, A^*, and X^φ pairwise commute. Combining this with Theorem 2.6, we obtain the following Lie algebra decomposition of Killing fields on $\mathrm{SO}(M)$:

Corollary 2.8 (Takagi-Yawata [TY94]). *Let M be a Riemannian manifold satisfying the hypotheses of Theorem 2.6. Then there is a Lie algebra decomposition*

$$i(\mathrm{SO}(M)) = i(M) \oplus i_V^M \oplus (\Lambda^2 M)_0,$$

where $i(M)$ (resp. $i_V^M, (\Lambda^2 M)_0$) corresponds to the subalgebra of Killing fields consisting of Y^C (resp. A^*, X^φ) for $Y \in i(M)$ (resp. $A \in \mathfrak{o}(n), \varphi \in (\Lambda^2 M)_0$).

2.5. **Normalizing volume.** Given a closed oriented Riemannian manifold M, we have previously obtained a 1-parameter family of Sasaki-Mok-O'Neill metrics on M. These can be parametrized by a choice of $O(n)$-invariant inner product on $\mathfrak{o}(n)$ (which is unique up to scaling), or, equivalently, by the volume of a fiber of $\mathrm{SO}(M) \to M$. The following easy lemma will be useful to us on multiple occasions in the rest of the paper.

Lemma 2.9. *Fix $\nu > 0$. Let M, N be closed orientable connected Riemannian n-manifolds and equip $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ with Sasaki-Mok-O'Neill metrics where the fibers of $\mathrm{SO}(M) \to M$ and $\mathrm{SO}(N) \to N$ have volume ν. Suppose that $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ are isometric. Then $\mathrm{vol}(M) = \mathrm{vol}(N)$.*

Proof. Set $X := SO(M) \cong SO(N)$. Since the fiber bundle $X \to M$ has fibers with volume ν, we have

$$\mathrm{vol}(X) = \frac{\mathrm{vol}(M)}{\nu}.$$

Likewise we have $\mathrm{vol}(X) = \frac{\mathrm{vol}(N)}{\nu}$. Combining these we get $\mathrm{vol}(M) = \mathrm{vol}(N)$. $\square$

2.6. **Frame bundles of fiber bundles.** Suppose we have a fiber bundle $F \to E \to B$. It is desirable to use this structure to get a fiber bundle structure for the frame bundles. We will discuss some constructions to this end.

Let us start by considering vector bundles. If $\pi : E \to B$ is a (real) vector bundle over a manifold B of rank k, the *frame bundle* X of $E \to B$ has total space consisting of points (x, e) where $x \in B$ and e is a frame of the fiber $E_x := \pi^{-1}(x)$ over x. There is a natural map $X \to B$ defined by $(x, e) \mapsto x$, which is in fact a principal $\mathrm{GL}_k(\mathbb{R})$-bundle over B.

If $\pi : E \to B$ is oriented and has the additional structure of an inner product on each fiber E_x that smoothly varies in the variable $x \in B$, then we can also define the *oriented orthonormal frame bundle* Y associated to π as follows. Points of Y are given by (x, e) where $x \in B$ and e is a positively oriented orthonormal frame of E_x. There is a natural map $Y \to B$ defined by $(x, e) \mapsto e$. Then $Y \to B$ is a principal $\mathrm{SO}(k)$-bundle over B.

Now suppose we start with a fiber bundle $F \to E \xrightarrow{\pi} B$. We define the *vertical tangent bundle* of π to be the vector bundle over E with total space $T_V E$ consisting of points (x, v), where $x \in E$ and v is a tangent vector to the fiber of π through x. The map $T_V E \to E$ is given by $(x, v) \mapsto x$. The rank of $T_V E \to E$ (as a vector bundle) is equal to $\dim F$.

We define the *vertical frame bundle* $F_V(E)$ of $E \to B$ to be the frame bundle of the vertical tangent bundle bundle $T_V E \to E$. The vertical frame bundle is a principal $\mathrm{GL}_{\dim(F)}(\mathbb{R})$-bundle over E.

If $E \to B$ is an oriented fiber bundle and each fiber is equipped with a Riemannian metric, we define the *vertical oriented orthonormal frame bundle* $\mathrm{SO}_V(E)$ of $E \to B$ to be the oriented orthonormal frame bundle of the vertical tangent bundle. That is, $\mathrm{SO}_V(E)$ consists of points (x, e) where $x \in E$ and e is a positively oriented orthonormal frame of $\ker D_x \pi$ (which is a subspace of $T_x E$). Then $\mathrm{SO}_V(E) \to E$ is a principal $\mathrm{SO}(\dim F)$-bundle over E.

In this way we have produced, given a fiber bundle $F \to E \to B$, a principal $\mathrm{SO}(\dim F)$-bundle $\mathrm{SO}_V(E) \to E$. We can actually get a bundle over B by composing $\mathrm{SO}_V(E) \to E$ with $E \to B$. In this way we get a bundle $\mathrm{SO}_V(E) \to B$. The fiber of $\mathrm{SO}_V(E) \to B$ over a point $x \in B$ is the oriented orthonormal frame bundle of E_x (and hence diffeomorphic to $\mathrm{SO}(F)$).

3. High-dimensional isometry groups of manifolds

In this section we review some known results about effective actions of a compact Lie group G on a closed n-manifold M when $\dim G$ is large compared to n. We will be especially interested in actions of $\mathrm{SO}(n)$ on an n-manifold M. First, there is the following classical upper bound for the dimension of a compact group acting smoothly on an n-manifold.

Theorem 3.1 ([Kob72, II.3.1]). *Let M be a closed n-manifold and G a compact group acting smoothly, effectively, and isometrically on M. Then $\dim G \leq \frac{n(n+1)}{2}$. Further equality holds if and only if M is isometric to either S^n or $\mathbb{R}P^n$ with a metric of constant positive curvature. In this case we have $G = PSO(n+1)$ or $\mathrm{SO}(n+1)$ or $O(n+1)$, and G acts on M in the standard way.*

Note that in the above case $G = \mathrm{Spin}(n+1)$ does not occur because there is no effective action on S^n or $\mathbb{R}P^n$. Theorem 3.1 leads us to study groups of dimension $< \frac{n(n+1)}{2}$. First, there is the following remarkable 'gap theorem' due to H.C. Wang.

Theorem 3.2 (H.C. Wang [Wan47]). *Let M be a closed n-manifold with $n \neq 4$. Then there is no compact group G acting effectively on M with*

$$\frac{n(n-1)}{2} + 1 < \dim G < \frac{n(n+1)}{2}.$$

Therefore the next case to consider is $\dim G = \frac{n(n-1)}{2} + 1$. The following characterization is independently due to Kuiper and Obata.

Theorem 3.3 (Kuiper, Obata [Kob72, II.3.3]). *Let M be a closed Riemannian n-manifold with $n > 4$ and G a connected compact group of dimension $\frac{n(n-1)}{2} + 1$ acting smoothly, effectively, and isometrically on M. Then M is isometric to $S^{n-1} \times S^1$ or $\mathbb{R}P^{n-1} \times S^1$ equipped with a product of a round metric on S^{n-1} or $\mathbb{R}P^{n-1}$ and the standard metric on S^1. Further $G = SO(n) \times S^1$ or $PSO(n) \times S^1$.*

After Theorem 3.3, the natural next case to consider is $\dim G = \frac{n(n-1)}{2}$. There is a complete classification due to Kobayashi-Nagano [KN72].

Theorem 3.4 (Kobayashi-Nagano). *Let M be a closed Riemannian n-manifold with $n > 5$ and G a connected compact group of dimension $\frac{n(n-1)}{2}$ acting smoothly, effectively, and isometrically on M. Then M must be one of the following.*

    (1) M is diffeomorphic to S^n or $\mathbb{R}P^n$ and $G = SO(n)$ or $PSO(n)$. In this case G has a fixed point on M. Every orbit is either a fixed point or has codimension 1.

Regarding S^n as the solution set of $\sum_{i=0}^n x_i^2 = 1$ in $\mathbb{R}^{n+1}$, the metric on M (or its double cover if M is diffeomorphic to $\mathbb{R}P^n$) is of the form

$$ds^2 = f(x_0) \sum_{i=0}^n dx_i^2$$

for a smooth positive function f on $[-1, 1]$.

(2) *M is diffeomorphic to $S^{n-1} \times S^1$ or $\mathbb{R}P^{n-1} \times S^1$. In this case $G = SO(n)$ or $PSO(n)$ acts on M preserving each fiber $S^{n-1} \times \{z\}$ for $z \in S^1$.*

(3) *M is a quotient $(S^{n-1} \times \mathbb{R})/\Gamma$ where Γ is generated by*

$$(v, t) \mapsto (v, t+2)$$
$$(v, t) \mapsto (-v, -t).$$

In this case $G = SO(n)$ acts on $S^{n-1} \times \mathbb{R}$ by acting orthogonally on each copy $S^{n-1} \times \{t\}$. This action commutes with the action of Γ, so that the G-action descends to M. We have $M/G = [0, 1]$. The G-orbits lying over the endpoints $0, 1$ are isometric to round projective spaces $\mathbb{R}P^{n-1}$ and the G-orbits lying over points in $(0, 1)$ are round spheres.

(4) *If $n = 6$ there is the additional case that $M \cong \mathbb{C}P^3$, equipped with the Fubini-Study metric and the standard action of $G = SO(6) \cong SU(4)/\{\pm\,\mathrm{id}\}$.*

(5) *If $n = 7$ there are the additional cases $M \cong \mathrm{Spin}(7)/G_2$ and $G = \mathrm{Spin}(7)$, or $M \cong SO(7)/G_2$ and $G = SO(7)$. In this case M is isometric to S^7 or $\mathbb{R}P^7$ with a constant curvature metric.*

Remark 3.5. Actually Kobayashi-Nagano prove a more general result that includes the possibility that M is noncompact, and there are more possibilities. Since we will not need the noncompact case, we have omitted these. In their formulation of Case (4), M is a manifold of complex dimension 3 with constant holomorphic sectional curvature, and G is the largest connected group of holomorphic isometries.

Specializing to the compact case gives an explicit description of Case (4) as follows. Hawley [Haw53] and Igusa [Igu54] independently proved that a simply-connected complex n-manifold of constant holomorphic sectional curvature is isometric to either $\mathbb{C}^n, \mathbb{B}^n$ or $\mathbb{C}P^n$ (with standard metrics). Therefore in Case (4) we obtain that M is isometric to $\mathbb{C}P^3$ (equipped with a scalar multiple of the Fubini-Study metric) and $G = SO(6) \cong SU(4)/\{\pm\,\mathrm{id}\}$.

Theorem 3.4 does not cover the case $n = 5$. In the following proposition we resolve this case for semisimple groups. We would like to thank the referee for the following statement and its proof, which improve upon those contained in an earlier version of this paper.

Proposition 3.6. *Let M be a closed oriented Riemannian 5-manifold and suppose G is a semisimple compact connected Lie group that acts on M smoothly, effectively, and isometrically, and that $\dim(G) = 10$. Then M admits a description as in Cases (1), (2) or (3) of Theorem 3.4.*

Proof. The proof of Theorem 3.4 (see [KN72, Section 3]) shows that the assumption that $n > 5$ is only used to show that no G-orbit has codimension 2. We will show under the stated assumptions there are still no codimension 2 orbits, so that the rest of the proof of Theorem 3.4 applies.

Clearly we can assume that G is connected. Note that $\dim(G) = \mathrm{rank}(G) + 2k$, where k is the number of root spaces of G. Hence the rank of G is even. Any semisimple Lie group with rank ≥ 4 has dimension > 10, so that we must have that $\mathrm{rank}(G) = 2$ and therefore G is a quotient of $\mathrm{Spin}(5)$.

Suppose now that $x \in M$ and that the orbit $G(x)$ has codimension 2 in M. Let G_x be the stabilizer of x. Note that G_x has rank either 1 or 2, and since the orbit of x is codimension 2, we must have that $\dim G_x = 7$.

If G_x has rank 1, then it must be S^1 or $\mathrm{Spin}(3)$ (possibly up to a finite quotient), but then we see that $\dim G_x < 7$, so this is impossible.

On the other hand if $\mathrm{rank}(G_x){=}2$, then the dimension of G_x is even, which is also a contradiction. $\qquad\square$

4. Geometric characterization of the fibers of $\mathrm{SO}(M) \to M$

We will now start the proof of Theorem A. In this section we aim to prove the following theorem, which proves Theorem A in all cases except for round spheres and surfaces. The remaining cases are resolved in Section 5 (round spheres) and Section 6 (surfaces).

Theorem 4.1. *Let M, N be closed oriented connected Riemannian n-manifolds. Equip $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ with Sasaki-Mok-O'Neill metrics where the fibers of $\mathrm{SO}(M) \to M$ and $\mathrm{SO}(N) \to N$ using the metric $\langle \cdot, \cdot \rangle_\lambda$ on $\mathfrak{o}(n)$. Assume that $n \neq 2, 3, 4, 8$ and that M does not have constant curvature $\frac{1}{2\sqrt{\lambda}}$. Then M, N are isometric if and only if $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ are isometric.*

Proof. Write $X := SO(M) \cong SO(N)$, and let

$$\pi_M : X \to M$$
$$\pi_N : X \to N$$

be the natural projections. The strategy of the proof is to characterize the fibers of π_M and π_N just in terms of the geometry of X, except when M is flat or $\mathrm{Isom}(M)$ has dimension at least $\frac{1}{2}n(n-1)$. It automatically follows that in all but the exceptional cases the fibers of π_M and π_N must agree, and we will use this to show that M and N are isometric. Finally we will show that in the exceptional cases M and N also have to be isometric.

Note that the assumptions of Theorem 4.1 guarantee that we can use Takagi-Yawata's computation of the Lie algebra of Killing fields on X, so we can write (see Corollary 2.8):

$$i(X) = i(M) \oplus i_V^M \oplus (\Lambda^2 M)_0.$$

Here, as before, $i(M)$ denotes the space of Killing fields on M, and i_V^M consists of the Killing fields A^* for $A \in \mathfrak{o}(n)$ (in particular $i_V^M \cong \mathfrak{o}(n)$), and $(\Lambda^2 M)_0$ denotes the space of parallel 2-forms on M. On the other hand, the natural action of $\mathrm{SO}(n)$ on the fibers of π_N induces an embedding of $\mathrm{SO}(n)$ in $\mathrm{Isom}(X)$, hence an embedding of Lie algebras

$$\mathfrak{o}(n) \cong i_V^N \hookrightarrow i(X) = i_V^M \oplus (\Lambda^2 M)_0 \oplus i(M).$$

We identify i_V^N with its image throughout. Now consider the projections of i_V^N onto each of the factors of this decomposition. We have the following cases:

(1) $i_V^N = i_V^M$, or
(2) i_V^N projects nontrivially to $(\Lambda^2 M)_0$, or
(3) i_V^N projects trivially to $(\Lambda^2 M)_0$ but nontrivially to $i(M)$.

We will show below that these cases correspond to (1) the fibers of π_M coincide with the fibers of π_N, (2) M is flat, and (3) $\dim \mathrm{Isom}(M) \geq \frac{1}{2}n(n-1)$. We will complete the proof of Theorem 4.1 in each of these cases below.

Case 1 (vertical directions agree). Assume that $i_V^N = i_V^M$. For any $x \in X$, the values of i_V^M at x, i.e. the set of vectors

$$\{Z(x) \mid Z \in i_V^M\},$$

span the tangent space to the fiber of π_M through x. On the other hand this set also spans the tangent space to the fiber of π_N through x. It follows that the fibers of π_M and π_N actually coincide. Hence we have a natural map $f : M \to N$ defined as follows: For $p \in M$, let $x \in \pi_M^{-1}(p)$ be any point in the fiber of π_M over p. Then set $f(p) := \pi_N(x)$. The fact that the fibers of π_M and π_N coincide proves that $f(p)$ does not depend on the choice of x.

We claim f is an isometry. Denote by $\mathcal{H}^M$ and $\mathcal{V}^M$ the horizontal and vertical sub-bundles with respect to $\pi_M : X \to M$. Because π_M is a Riemannian submersion, the metric on $T_x M$ coincides with the metric on the horizontal subbundle $\mathcal{H}_u$ at a point $u \in \pi_M^{-1}(x)$. We have

$$\mathcal{H}_u = \mathcal{V}_u^\perp = (\ker(\pi_M)_*)^\perp = (\ker(\pi_N)_*)^\perp.$$

Here the first identity is because by definition of the Sasaki-Mok-O'Neill metric on X, the horizontal and vertical subbundles are orthogonal. The last identity follows because we know the fibers of π_M and π_N agree. Finally, note that the space $(\ker(\pi_N)_*)^\perp$ is just the horizontal subbundle of $\pi_N : X \to N$. Since π_N is a Riemannian submersion, we conclude that the metric on $\mathcal{H}_u$ coincides with the metric on $T_{\pi_N(u)} N$. This proves the naturally induced map

$$f : M \to N$$

is a local isometry. Since f is also injective, M and N are isometric.

Case 2 (many parallel forms). Assume that $i_V^N \cong \mathfrak{o}(n)$ projects nontrivially to $(\Lambda^2 M)_0$. Note that the kernel of the projection of i_V^N to $(\Lambda^2 M)_0$ is an ideal in i_V^N. On the other hand $i_V^N \cong \mathfrak{o}(n)$ is simple (because $n > 4$), so the projection $i_V^N \to (\Lambda^2 M)_0$ must be an isomorphism onto its image. Therefore

$$\dim(\Lambda^2 M)_0 \geq \dim \mathfrak{o}(n) = \frac{n(n-1)}{2}. \tag{4.1}$$

We claim that we actually have equality in Equation 4.1. To see this, note that since a parallel form is invariant under parallel transport, it is determined by its values on a single tangent space, so that we have an embedding

$$(\Lambda^2 M)_0 \hookrightarrow \Lambda^2 T_x M. \tag{4.2}$$

Therefore $\dim(\Lambda^2 M)_0 \leq \frac{n(n-1)}{2}$, and equality in Equation 4.1 holds. Hence by a dimension count, the projection $i_V^N \to (\Lambda^2 M)_0$ is not only injective, but also surjective.

So we have $\mathfrak{o}(n) \cong (\Lambda^2 M)_0$, and M has the maximal amount of parallel forms it can possibly have (i.e. a space of dimension $\frac{n(n-1)}{2}$). Note that a torus is an example of such a manifold. Motivated by these examples, we claim that M is a flat manifold.

To prove that M is flat, let us first show that for any $x \in M$, the holonomy group at x is trivial. Recall that the holonomy group consists of linear maps $T_x M \to T_x M$ obtained by parallel transport along a loop in M based at x. Therefore any holonomy map will fix parallel forms pointwise. Suppose now that $T : T_x M \to T_x M$ is a holonomy map at $x \in M$. We showed above that the evaluation at x is an isomorphism $(\Lambda^2 M)_0 \hookrightarrow \Lambda^2 T_x M$ (see Equation 4.2). Since T fixes parallel forms, it is therefore clear that $\Lambda^2 T = \mathrm{id}$ (i.e. T acts trivially on oriented planes in $T_x M$). Since $\dim(M) > 2$, it follows that $T = \mathrm{id}$.

So M has trivial holonomy. Since the holonomy algebra (i.e. the Lie algebra of the holonomy group) contains the Lie algebra generated by curvature operators $R(v, w)$ where $v, w \in T_x M$ (see e.g. [Pet06, Section 8.4]), it follows that $R(v, w) = 0$ for all $v, w \in T_x M$, so M is flat.

We will use that M is flat to obtain more information about the Killing fields $i(M)$ of M. Recall that the structure of flat manifolds is described by the Bieberbach theorems. Namely, any closed flat manifold is of the form $\mathbb{R}^n/\Gamma$ for some discrete torsion-free subgroup $\Gamma \subseteq \mathrm{Isom}(\mathbb{R}^n)$, and there is a finite index normal subgroup $\Lambda \subseteq \Gamma$ that consists of translations of $\mathbb{R}^n$ (so $\mathbb{R}^n/\Lambda$ is a torus). In particular the Killing fields on $\mathbb{R}^n/\Lambda$ are just obtained by translations of $\mathbb{R}^n$, so $i(\mathbb{R}^n/\Lambda) \cong \mathbb{R}^n$ as a Lie algebra.

The Killing fields on $M = \mathbb{R}^n/\Gamma$ are exactly those Killing fields of $\mathbb{R}^n/\Lambda$ invariant under the deck group Γ/Λ of the (Galois) cover $\mathbb{R}^n/\Lambda \to M$. In particular $i(M)$ is a sub-Lie algebra of $\mathbb{R}^n$.

Therefore $i(M)$ is abelian. Recall that we have

$$i(X) \cong i_V^M \oplus (\Lambda^2 M)_0 \oplus i(M)$$

We know that $i_V^N \cong \mathfrak{o}(n)$ has no abelian quotients, so we must have $i_V^N \subseteq i_V^M \oplus (\Lambda^2 M)_0$. Since the vector fields in $i_V^M \oplus (\Lambda^2 M)_0$ are vertical with respect to π_M (see Section 2), it follows that for $x \in N$ and $\widetilde{x} \in \pi_N^{-1}(x)$, we have

$$T_{\widetilde{x}} \pi_N^{-1}(x) = i_V^N|_{\widetilde{x}} \subseteq (i_V^M \oplus (\Lambda^2 M)_0)|_{\widetilde{x}} = T_{\widetilde{x}} \pi_M^{-1}(\pi_M(\widetilde{x})).$$

Since $\pi_N^{-1}(x)$ and $\pi_M^{-1}(\pi_M(\widetilde{x}))$ are connected submanifolds with the same dimension, we must have $\pi_N^{-1}(x) = \pi_M^{-1}(\pi_M(\widetilde{x}))$. Therefore the fibers of π_M and π_N agree. We conclude that M and N are isometric in the same way as Case 1.

Case 3 (many Killing fields). Assume i_V^N projects nontrivially to $i(M)$. Again we use that $\mathfrak{o}(n)$ is a simple Lie algebra because we have $n > 4$. By assumption $i_V^N \cong \mathfrak{o}(n)$ projects nontrivially to $i(M)$, hence i_V^N projects isomorphically to $i(M)$. Let $\mathfrak{h}$ be the image of i_V^N in $i(M)$. At this point we would like to say that $i_V^N \subseteq i(M)$. We cannot in general establish this, but we have the following.

Claim 4.2. Assume that $\mathfrak{o}(n) \nsubseteq (\Lambda^2 M)_0$ and that $\mathfrak{o}(n) \nsubseteq (\Lambda^2 N)_0$. Then

    (1) $i_V^N \subseteq i(M)$, and
    (2) $i_V^M \subseteq i(N)$.

Therefore M and N have isometry groups of dimension $\geq \frac{1}{2}n(n-1)$.

Proof. Note that i_V^M and $\mathfrak{h}$ centralize each other and are isomorphic to $\mathfrak{o}(n)$. Consider the projection

$$p_1 : \mathfrak{h} \oplus i_V^M \subseteq i(X) \cong i_V^N \oplus (\Lambda^2 N)_0 \oplus i(N) \longrightarrow i_V^N.$$

Note that $\dim(\mathfrak{h} \oplus i_V^M) = 2 \dim i_V^N$, so p_1 cannot be injective. If p_1 is trivial, then we have

$$\mathfrak{h} \oplus i_V^M \subseteq (\Lambda^2 N)_0 \oplus i(N).$$

Using again that $\mathfrak{o}(n)$ is simple, and since $(\Lambda^2 N)_0$ does not contain a copy of $\mathfrak{o}(n)$ by assumption, we must have that $\mathfrak{h} \oplus i_V^M$ projects isomorphically to $i(N)$. However note that $\dim i(N) \leq \frac{n(n+1)}{2}$ by Theorem 3.1. Again by comparing dimensions we see that this is impossible. Therefore $\ker p_1$ is a proper ideal of $\mathfrak{h} \oplus i_V^M$, so $\ker p_1$ is either $\mathfrak{h}$ or i_V^M.

Now consider the projection

$$p_2 : \mathfrak{h} \oplus i_V^M \subseteq i(X) \cong i_V^N \oplus (\Lambda^2 N)_0 \oplus i(N) \to i(N).$$

As above we see that p_2 can be neither injective nor trivial. Hence we have that $\ker p_2$ is either $\mathfrak{h}$ or i_V^M.

If $\ker p_2 = i_V^M$, then we have $i_V^M = i_V^N$, but this contradicts the assumption that i_V^N projects nontrivially to $i(M)$. Therefore we must have $\ker p_1 = i_V^M$ and $\ker p_2 = \mathfrak{h}$. The latter implies $i_V^N = \mathfrak{h}$, which proves (1).

Since $\ker p_1 = i_V^M$, we have $i_V^M \subseteq (\Lambda^2 N)_0 \oplus i(N)$ and i_V^M projects trivially to $(\Lambda^2 N)_0$. Therefore we have $i_V^M \subseteq i(N)$, which proves (2). $\qquad\square$

If $\mathfrak{o}(n) \subseteq (\Lambda^2 M)_0$ or $\mathfrak{o}(n) \subseteq (\Lambda^2 N)_0$, the proof is finished in Case 2. Therefore we assume $i_V^N \subseteq i(M)$ and $i_V^M \subseteq i(N)$. Write $H_M := \exp(i_V^N)$ and $H_N := \exp(i_V^M)$.

Then H_M and H_N are commuting subgroups of $\mathrm{Isom}(X)$, so that the action of H_M on X descends to an action on $N = X/H_N$ with kernel $H_M \cap H_N$. We will write $\overline{H}_M := H_M/(H_M \cap H_N)$ for the group of isometries of N thus obtained. Similarly, H_N acts by isometries on $M = X/H_M$ with finite kernel $H_M \cap H_N$, and we will write $\overline{H}_N := H_N/(H_M \cap H_N)$ for this group of isometries.

Note that $H_M \cap H_N$ is discrete, since its Lie algebra is $i_V^M \cap i_V^N = 0$. In particular, since H_M and H_N are compact, it follows that $H_M \cap H_N$ is finite. Therefore the natural quotient map $H_M \to \overline{H}_M$ is a covering of finite degree, and $\overline{H}_M$ and H_M have the same Lie algebra. Similarly, $\overline{H}_N$ and H_N have the same Lie algebra. Therefore $\overline{H}_M$ and $\overline{H}_N$ are groups of isometries of closed n-manifolds with Lie algebras isomorphic to $\mathfrak{o}(n)$. The results of Section 3 exactly apply to such actions; these results will restrict the possibilities for M and N tremendously, as we will see below.

Motivated by the results of Section 3, we will now consider two cases: Either one of $\overline{H}_M$ or $\overline{H}_N$ acts transitively, or neither acts transitively.

Case 3(a) ($\overline{H}_M$ or $\overline{H}_N$ acts transitively). Suppose $\overline{H}_M$ acts transitively on M. We will show that this is impossible. Since $\overline{H}_M$ has Lie algebra $\mathfrak{o}(n)$ and $\dim M = n$, Theorem 3.4 and Proposition 3.6 give a classification of the possibilities for M and $\overline{H}_M$. Since in Cases (1), (2), and (3) of Theorem 3.4 the group of isometries is not transitive, but by assumption $\overline{H}_M$ acts transitively on M, we know that either

- M is isometric to $S^7 \cong \mathrm{Spin}(7)/G_2$, equipped with a constant curvature metric, and $\overline{H}_M = \mathrm{Spin}(7)$, or
- M is isometric to $\mathbb{R}P^7 \cong \mathrm{SO}(7)/G_2$, equipped with a constant curvature metric, and $\overline{H}_M = \mathrm{SO}(7)$, or
- M is isometric to $\mathbb{C}P^3$, equipped with a metric of constant holomorphic sectional curvature, and $\overline{H}_M = \mathrm{SO}(6) \cong \mathrm{SU}(4)/\{\pm\,\mathrm{id}\}$.

We will show that the first and last cases are impossible, and that in the second case M and N are isometric.

Lemma 4.3. *M is not isometric to S^7.*

Proof. Since $\overline{H}_M = H_M/(H_M \cap H_N)$, we know that $\overline{H}_M$ is a quotient of $H_M \cong \mathrm{SO}(7)$. In particular, H_M is not simply-connected. On the other hand, $\mathrm{Spin}(7)$ is simply-connected. This is a contradiction. $\qquad\square$

Lemma 4.4. *If M is isometric to $\mathbb{R}P^7$, then M and N are isometric.*

Proof. Suppose now M is isometric to $\mathbb{R}P^7$, and consider the action of H_N on N. From the classification in Theorem 3.4, and using that $\dim(N) = \dim(M) = 7$, we see that N must be diffeomorphic to one of the following:

(1) $\mathbb{R}P^7$,
(2) S^7, or
(3) a fiber bundle $L_N \to N \to S^1$ where L_N is S^6 or $\mathbb{R}P^6$.

Claim 4.5. *We must have that N is diffeomorphic to $\mathbb{R}P^7$ (and hence to M).*

Proof. We will show that we can distinguish the frame bundles of the manifolds appearing in Cases (2) and (3) from $\mathrm{SO}(\mathbb{R}P^7)$ by their fundamental group.

First, let us compute the fundamental group of $\mathrm{SO}(N) = \mathrm{SO}(\mathbb{R}P^7)$. Note that $\mathrm{SO}(S^7) \cong SO(8)$ (see Example 1.2). It easily follows that $\mathrm{SO}(\mathbb{R}P^7) \cong SO(8)/\{\pm\,\mathrm{id}\}$. In

particular, $\pi_1 \operatorname{SO}(\mathbb{R}P^7)$ is obtained as an extension

$$1 \to \pi_1 \operatorname{SO}(8) \to \pi_1 \operatorname{SO}(\mathbb{R}P^7) \to \{\pm \operatorname{id}\} \to 1.$$

So $\pi_1(\operatorname{SO}(\mathbb{R}P^7))$ has order 4. So let us now show that in each of the Cases (2) and (3), π_1 does not have order 4.

- In Case (2), note that $\pi_1 \operatorname{SO}(S^7) = \pi_1 \operatorname{SO}(8) \cong \mathbb{Z}/(2\mathbb{Z})$ has order 2.
- In Case (3), N is a fiber bundle $L_N \to N \to S^1$ with $L_N \cong S^6$ or $\mathbb{R}P^6$, it follows that $\pi_1 N$ is infinite (use the long exact sequence on homotopy groups for the fiber bundle $L_N \to N \to S^1$), and also that $\pi_1 \operatorname{SO}(N)$ is infinite (by the long exact sequence on homotopy groups for the fiber bundle $\operatorname{SO}(7) \to \operatorname{SO}(N) \to N$).

The only remaining possibility is that N is diffeomorphic to $\mathbb{R}P^7$ (and hence also to M). $\qquad\square$

So we find that N is diffeomorphic to $\mathbb{R}P^7$. Let us now determine the metric on N:

Claim 4.6. N has constant curvature.

Proof. Theorem 3.4 classifies the possible metrics on N. Namely, if $\overline{H}_N$ acts transitively on N, then N has constant curvature, as desired.

Suppose now that H_N does not act transitively on N. Identify the universal cover $\widetilde{N}$ of N (which is diffeomorphic to S^7) with the solution set of $\sum_{i=0}^{7} x_i^2 = 1$ in $\mathbb{R}^8$. Then by Theorem 3.4.(3) the metric on $\widetilde{N}$ is of the form

$$ds_{\widetilde{N}}^2 = f(x_0) \sum_{i=0}^{7} dx_i^2$$

for some smooth positive function f on $[-1, 1]$. The function $|x_0|$ descends from $\widetilde{N}$ to N, and $\overline{H}_N$ acts isometrically and transitively on each level set

$$\left\{ [x_0, \ldots, x_7] \in \mathbb{R}P^7 \ \middle| \ \sum_{i=1}^{7} x_i^2 = 1 - c^2 \right\}$$

for $0 \le c \le 1$. For $c = 0$ this level set is a copy of $\mathbb{R}P^6$ (the image of the equator $S^6 \subseteq S^7 \cong \widetilde{N}$ in $\mathbb{R}P^7 \cong N$) and for $c = 1$ the level set consists of a single point (the image of the north and south pole). For $0 < c < 1$, the level set is a copy of S^6.

Let $x \in N$ be any point with $0 < x_0 < 1$, so that the $\overline{H}_N$-orbit of x is a copy of S^6. Since the metric on $\overline{H}_N x$ is given by $f(x_0) \sum_i dx_i^2$, we have

$$\operatorname{vol}(\overline{H}_N x) = (f(x_0))^{\frac{n}{2}} \operatorname{vol}(S^6)$$

where on the right-hand side $\operatorname{vol}(S^6)$ is computed with respect to the standard metric $\sum_i dx_i^2$. Now consider the fiber bundle $\pi_N : \operatorname{SO}(N) \to N$. Recall that each fiber in $\operatorname{SO}(N)$ has a fixed volume $\nu > 0$, and is an H_M-orbit. Therefore for $e \in \pi_N^{-1}(x)$, we have

$$\operatorname{vol}(H_M H_N e) = \nu \operatorname{vol}(\overline{H}_N x) = \nu (f(x_0))^{\frac{6}{2}} \operatorname{vol}(S^6). \tag{4.3}$$

On the other hand, e is a frame at some point $y \in M$. Since the fibers of $\operatorname{SO}(M) \to M$ also have volume ν, it follows that

$$\operatorname{vol}(H_M H_N e) = \nu \operatorname{vol}(\overline{H}_M y).$$

Since $\overline{H}_M$ acts transitively on M, the right-hand side is just equal to $\nu \operatorname{vol}(M)$. In particular, the left-hand side does not depend on e. Using Equation 4.3, we see that $f(x_0)$ does not depend on the point x chosen. Since the only requirements for x were

that $-1 < x_0 < 1$ and $x_0 \neq 0$, we see that f is constant on $(-1, 1)\backslash\{0\}$. Since f is also continuous, it is in fact constant on $[-1, 1]$, so the metric on $\widetilde{N}$ is given by

$$ds_{\widetilde{N}}^2 = c \sum_{i=0}^{7} dx_i^2$$

for some $c > 0$. Therefore the metric is some multiple of the standard round metric, so N has constant curvature. $\qquad\square$

So we have shown that both M and N are diffeomorphic to $\mathbb{R}P^7$ with constant curvature metrics. Since by Lemma 2.9, we also have that $\text{vol}(M) = \text{vol}(N)$, it follows that M and N have the same curvature, so that they are isometric, as desired. $\qquad\square$

Lemma 4.7. *If M is isometric to $\mathbb{C}P^3$, equipped with a metric of constant holomorphic sectional curvature, then M and N are isometric.*

Proof. Again consider the action of $\overline{H}_N$ on N. From the classification in Theorem 3.4, and using that $\dim(N) = \dim(M) = 6$, we see that N must be one of the following:

(1) diffeomorphic to S^6 or $\mathbb{R}P^6$,
(2) a fiber bundle $L_N \to N \to S^1$ where L_N is S^5 or $\mathbb{R}P^5$, or
(3) isometric to $\mathbb{C}P^3$ with a metric of constant holomorphic sectional curvature.

We can rule out Cases (1) and (2) by computations of π_2. Namely, let us first compute $\pi_2(\text{SO}(\mathbb{C}P^3))$. The long exact sequence on homotopy groups of the fibration $\text{SO}(6) \to \text{SO}(\mathbb{C}P^3) \to \mathbb{C}P^3$ gives

$$1 = \pi_2 SO(6) \to \pi_2(\text{SO}(\mathbb{C}P^3)) \to \pi_2(\mathbb{C}P^3) \to \pi_1(SO(6)) = \mathbb{Z}/(2\mathbb{Z}).$$

Since $\pi_2(\mathbb{C}P^3) \cong \mathbb{Z}$ it follows that $\pi_2(\text{SO}(\mathbb{C}P^3)) \cong \mathbb{Z}$. On the other hand, in Cases (1) and (2), we have $\pi_2(\text{SO}(S^6)) = \pi_2(SO(7)) = 1$ and similarly $\pi_2(\text{SO}(\mathbb{R}P^6)) = 1$. Therefore in Cases (1) and (2) we cannot have $\text{SO}(N) \cong \text{SO}(\mathbb{C}P^3)$, so we conclude that M and N are both isometric to $\mathbb{C}P^3$ with a metric of constant holomorphic sectional curvature.

A metric of constant holomorphic sectional curvature on $\mathbb{C}P^3$ is determined by a bi-invariant metric on $SU(4)$, which is then induced on the quotient $SU(4)/U(3) \cong \mathbb{C}P^3$. Hence the metrics on M and N differ only by scaling, so M and N are isometric if and only if $\text{vol}(M) = \text{vol}(N)$. By Lemma 2.9 we indeed have $\text{vol}(M) = \text{vol}(N)$ so M and N are isometric. $\qquad\square$

Case 3(b) (H_M and H_N do not act transitively). Theorem 3.4 and Proposition 3.6 imply that M and N are of one of the following types:

(1) diffeomorphic to S^n or $\mathbb{R}P^n$,
(2) a fiber bundle $F \to E \to S^1$ where each fiber is isometric to a round sphere or projective space, or
(3) $(S^{n-1} \times \mathbb{R})/\Gamma$ where $\Gamma \cong D_\infty$ is generated by $(v, t) \mapsto (v, t + 2)$ and $(v, t) \mapsto (-v, -t)$.

Claim 4.8. *M and N belong to the same types in the above classification.*

Proof. Again we will show that the different types can be distinguished by the fundamental group of the frame bundle. Since $\text{SO}(M) = \text{SO}(N)$, it must then follow that M and N belong to the same type.

The fundamental group of $X = \text{SO}(M)$ can be computed using the long exact sequence on homotopy groups for the fiber bundle $X \to M$ (or $X \to N$). Namely, we have

$$\pi_2(M) \to \pi_1(SO(n)) \to \pi_1(X) \to \pi_1(M) \to 1$$

and likewise for N. Since $\pi_2(M) = \pi_2(N) = 1$ for all of the above types, we have a short exact sequence

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(X) \to \pi_1(M) \to 1$$

and likewise for N. We see that $\pi_1(X) \cong \mathbb{Z}/2\mathbb{Z}$ precisely when M is diffeomorphic to S^n, and $\pi_1(X)$ has order 4 precisely when M is diffeomorphic to $\mathbb{R}P^n$. If $\pi_1(X)$ is infinite then M is of type (2) or (3). If the maximal finite subgroup of $\pi_1(X)$ has order 2 then M is of type (2), and if the maximal finite subgroup of $\pi_1(X)$ has order 4 then M is of type (3). Therefore we can distinguish all the possible cases by considering $\pi_1(X)$, so M and N are of the same type. □

We will now show that in each of these cases, M and N are isometric.

Case A (M and N are diffeomorphic to S^n or $\mathbb{R}P^n$). Identify S^n with the solution set of $\sum_{i=0}^{n} x_i^2 = 1$ in $\mathbb{R}^{n+1}$. By Theorem 3.4.(1), the metric on M (or its double cover if M is diffeomorphic to $\mathbb{R}P^n$) is of the form

$$ds_M^2 = f_M(x_0) \sum_{i=0}^{n} dx_i^2. \tag{4.4}$$

Similarly the metric on N (or its double cover) can be written as

$$ds_N^2 = f_N(x_0) \sum_{i=0}^{n} dx_i^2 \tag{4.5}$$

We will now show that $f_M(x) = f_N(x)$ for all x. We will just do this in case M and N are diffeomorphic to S^n, since the proof for $\mathbb{R}P^n$ is similar (note that it is not possible that one of M and N is diffeomorphic to S^n, and the other to $\mathbb{R}P^n$, since $SO(S^n)$ and $SO(\mathbb{R}P^n)$ are not diffeomorphic). Theorem 3.4.(1) also describes the action of $\overline{H}_N$ on N. Namely, $\overline{H}_N$ leaves the coordinate x_0 invariant and acts transitively on each level set of x_0. This yields an identification

$$N/\overline{H}_N \cong [-1, 1].$$

The $\overline{H}_N$-orbits lying over the points in $(-1, 1)$ are copies of S^{n-1}, and the orbits lying over ± 1 are fixed points (corresponding to the north and south pole). Similarly we can identify $M/\overline{H}_M$ with $[-1, 1]$. Of course we can also write $M = X/H_M$, and this yields an identification

$$X/(H_M H_N) = M/\overline{H}_M.$$

Let $-1 < x < 1$ and choose a lift $y_M \in M$ of x. Equation 4.4 shows that $\mathrm{vol}(H_M y_M) = f_M(x) \mathrm{vol}(S^{n-1})$ where S^{n-1} is equipped with the metric $\sum_{i=1}^{n} dx_i^2$. Similarly if y_N is a lift of x to N we have $\mathrm{vol}(H_N y_N) = f_N(x) \mathrm{vol}(S^{n-1})$. Now choose a common lift $\widetilde{y}$ of y_M and y_N to X, i.e. $\widetilde{y}$ is an oriented orthonormal frame at the point $y_M \in M$ and at the point $y_N \in N$. Recall that the volume of a fiber of $X \to M$ is a fixed constant $\nu > 0$. Hence we have

$$\mathrm{vol}(H_M H_N \widetilde{y}) = \nu \, \mathrm{vol}(\overline{H}_M y_M) = \nu f_M(x) \mathrm{vol}(S^{n-1}).$$

Since the volume of a fiber of $X \to N$ is also equal to ν, we also have

$$\mathrm{vol}(H_M H_N \widetilde{y}) = \nu \, \mathrm{vol}(H_N y_N) = \nu f_N(x) \mathrm{vol}(S^{n-1}).$$

It follows that $f_M(x) = f_N(x)$. Hence M and N are isometric.

Case B (M and N are of type (2)). In this case there is a fiber bundle $q : M \to S^1$ where the fibers are isometric to round spheres or projective spaces. The group $\overline{H}_N$ acts orthogonally on each fiber. The main observation is that in this case we can obtain a larger group of isometries as follows:

Lemma 4.9. *Let Z be a unit length vector field on M that is orthogonal to the fibers of q. Then Z is a Killing vector field.*

Proof. Write M as a fiber bundle

$$L_M \to M \xrightarrow{q} S^1$$

where all fibers L_M are isometric to round spheres or projective spaces. $\overline{H}_N$ is isomorphic to either $\mathrm{SO}(n)$ or $\mathrm{PSO}(n)$, and acts orthogonally on each fiber of q.

It will be useful to use this structure to obtain an H_N-invariant submanifold of X. In order to do this, let us recall the following construction from Section 2. There we considered an oriented fiber bundle $\pi : E \to B$, whose fibers are diffeomorphic to some manifold F and are equipped with smoothly varying Riemannian metrics, with fibers F. Then we constructed a bundle $\mathrm{SO}_V(E) \to B$ with fibers diffeomorphic to $\mathrm{SO}(F)$. Points of $\mathrm{SO}_V(E)$ are given by (x, e) where $x \in E$ and e is a positively oriented orthonormal frame of the tangent space to the fiber through x. The map $\mathrm{SO}_V(E) \to B$ is just the composition

$$\mathrm{SO}_V(E) \to E \to B$$
$$(x, e) \mapsto x \mapsto q(x).$$

We apply this construction to the fiber bundle $q : M \to S^1$ with fibers L_M. Fix an orientation on S^1. This induces a transverse orientation of the fibers of q, and since M is already oriented, we obtain an orientation on each fiber of q. Further note that we have smoothly varying Riemannian metrics on the fibers of q obtained by restricting the Riemannian metric of M.

Hence the above construction yields a bundle $q_V : \mathrm{SO}_V(M) \to S^1$ with fibers diffeomorphic to $\mathrm{SO}(L_M)$. Since the fibers of q are either round spheres or round projective spaces, we have that the fibers of q_V are diffeomorphic to either $\mathrm{SO}(n)$ or $\mathrm{PSO}(n)$.

Now let us relate the action of $\overline{H}_N$ on M to this bundle. Since $\overline{H}_N$ acts isometrically on M leaving each fiber of q invariant, it follows that $\overline{H}_N$ acts on $\mathrm{SO}_V(M)$ and preserves each of the fibers of q_V. Explicitly, note that the action of $\overline{H}_N$ on a fiber F of q is just the standard action of $\mathrm{SO}(n)$ on S^{n-1} (or the standard action of $\mathrm{PSO}(n)$ on $\mathbb{R}P^{n-1}$). Then the action of $\overline{H}_N$ on a fiber $\mathrm{SO}(F)$ of q_V is just the action of $\mathrm{SO}(n)$ (or $\mathrm{PSO}(n)$) on itself by left-translations.
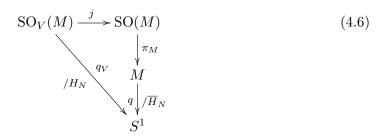
Now the fact that the fibers of q are codimension 1 in M can be used to get an inclusion

$$j : \mathrm{SO}_V(M) \hookrightarrow \mathrm{SO}(M)$$

defined in the following way. A point $(x, e) \in \mathrm{SO}_V(M)$ consists of an oriented orthonormal frame e of the vertical tangent space (with respect to q) at x. Since the foliation by fibers of $q : M \to S^1$ are transversely oriented, e can be extended to a frame for M at x by adding to e the unique unit vector $v \in T_x M$ such that (e, v) is a positively oriented orthonormal frame for M.

Using that $\overline{H}_N$ preserves each fiber of q, it is easy to see that $j(\mathrm{SO}_V(M))$ is an H_N-invariant submanifold of $\mathrm{SO}(M)$. All maps so far are summarized in the following

commutative diagram.

$$\begin{array}{ccc}
\mathrm{SO}_V(M) & \xrightarrow{\;\;j\;\;} & \mathrm{SO}(M) \\
& & \Big\downarrow{\scriptstyle \pi_M} \\
& {\scriptstyle q_V} & M \\
{\scriptstyle /H_N} & & \Big\downarrow{\scriptstyle q\;/\overline{H}_N} \\
& & S^1
\end{array} \tag{4.6}$$

Here "$/H_N$" means the projection modulo the H_N-action.

We equip $\mathrm{SO}_V(M)$ with the Riemannian metric on $j(\mathrm{SO}_V(M))$ induced from $\mathrm{SO}(M)$. Since the H_N-orbits in $\mathrm{SO}(M)$ are the fibers of the map $\pi_N : X \to N$, the H_N-orbits are totally geodesic in $\mathrm{SO}_V(M)$ (see Proposition 2.2). We conclude that the foliation $\mathcal{F}$ of $\mathrm{SO}_V(M)$ by H_N-orbits is a totally geodesic codimension 1 foliation of $\mathrm{SO}_V(M)$. Note that $\mathcal{F}$ is also the foliation of $\mathrm{SO}_V(M)$ by fibers of q_V. Consider the horizontal foliation $\mathcal{F}^\perp$ of $\mathrm{SO}_V(M)$. Since $\mathcal{F}^\perp$ is 1-dimensional, it is integrable.

Johnson-Whitt proved that if the horizontal distribution associated to a totally geodesic foliation is integrable, then the horizontal distribution is also totally geodesic [JW80, Theorem 1.6]. Further they showed that a manifold with two orthogonal totally geodesic foliations is locally a Riemannian product [JW80, Proposition 1.3]. Therefore $\mathrm{SO}_V(M)$ is locally a Riemannian product $F \times U$ where F (resp. U) is an open neighborhood in a leaf of $\mathcal{F}$ (resp. $\mathcal{F}^\perp$).

Our goal is now to show that M is also a local Riemannian product. For this, recall (see Section 2) that we also have the *vertical oriented orthonormal frame bundle p* : $\mathrm{SO}_V(M) \to M$ associated to $q : M \to S^1$. Recall that p is a principal $\mathrm{SO}(\dim L_M)$-bundle over M. The relation of p to the maps above is just that p is the composition

$$\mathrm{SO}_V(M) \xrightarrow{\;j\;} \mathrm{SO}(M) \xrightarrow{\;\pi_M\;} M.$$

Since j is an isometric embedding (by definition of the metric on $\mathrm{SO}_V(M)$), and π_M is a Riemannian submersion, p is also a Riemannian submersion.

Since the diagram given in Equation 4.6 commutes, we have

$$\begin{aligned}
q_V &= q \circ \pi_M \circ j \\
&= q \circ p.
\end{aligned}$$

Since $q_V : \mathrm{SO}_V(M) \to S^1$ and $q : M \to S^1$ are bundles, we know that any $x_0 \in S^1$ has an open neighborhood U such that

$$q_V^{-1}(U) \cong q_V^{-1}(x_0) \times U$$

and

$$q^{-1}(U) \cong q^{-1}(x_0) \times U,$$

where in local coordinates q_V and q are given by projection on the second factor. Since $q_V = q \circ p$, the map p restricts to a map $q_V^{-1}(U) \to q^{-1}(U)$ taking the fiber $q_V^{-1}(x)$ to $q^{-1}(x)$ for any $x \in U$. Therefore in the above coordinates, this is a map

$$q_V^{-1}(x_0) \times U \to q^{-1}(x_0) \times U \tag{4.7}$$

$$(y, x) \mapsto (p_x(y), x) \tag{4.8}$$

for some maps p_x.

We have shown above that the metric on the left-hand side of Equation 4.8 (i.e. on $\mathrm{SO}_V(M)$) is locally a product, and further that p is a Riemannian submersion. It follows that the metric on the right-hand side of Equation 4.8 (i.e. on M) is also locally a product.

Now let Z be a unit length vector field orthogonal to the fibers of q. Then Z is everywhere tangent to $p_* \mathcal{F}^\perp$. In the coordinates above, this means that Z is everywhere tangent to U. Since U is 1-dimensional and Z has constant length, Z is then a Killing field on U. Since the metric on M is locally a product, it follows that Z is a Killing field on M. $\qquad \square$

Hence we have the following lower bound on the dimension of the Lie algebra $i(M)$ of Killing fields on M:

$$\dim i(M) \geq \dim \overline{H}_M + 1 = \frac{1}{2} n(n-1) + 1.$$

By Theorem 3.3 we know that M is isometric to a product $F_M \times S^1$, where F_M is either a round sphere or projective space. Of course, the same proof applies to N, and shows that N is also isometric to a product $F_N \times S^1$. Further the metrics on the constant curvature spheres or projective spaces F_M and F_N only depend on their curvatures.

Claim 4.10. F_M and F_N have the same curvature.

Proof. Recall that we normalized the Sasaki-Mok-O'Neill metrics on $\mathrm{SO}(M) \cong \mathrm{SO}(N)$ so that the fibers of $\mathrm{SO}(M) \to M$ and $\mathrm{SO}(N) \to N$ have volume ν. These fibers are exactly H_M and H_N-orbits in $\mathrm{SO}(M)$, and by definition of the Sasaki-Mok-O'Neill metric, the metric restricted to an H_M or H_N-orbit is bi-invariant. On the other hand the H_M-orbit of a point $x \in \mathrm{SO}(M)$ is exactly the bundle of oriented orthonormal frames of the fiber $F_{\pi_M(x)}$ of q containing the point $\pi_M(x)$.

So π_M restricts to a bundle

$$\pi_M H_M x \to \overline{H}_M \pi_M(x) \cong F_{\pi_M(x)}. \tag{4.9}$$

Here $H_M x$ is diffeomorphic to $\mathrm{SO}(n)$ (if the fibers of q are spheres) or $\mathrm{PSO}(n)$ (if the fibers of q are projective spaces), and the fiber of the bundle in Equation 4.9 is diffeomorphic to $\mathrm{SO}(n-1)$.

Since the metric on $H_M x$ (viewed as a submanifold of $\mathrm{SO}(M)$) is a bi-invariant metric, the above bundle is isometric to a standard bundle

$$\mathrm{SO}(n) \to S^{n-1}(r_M) \qquad \text{if } F_{\pi_M(x)} \cong S^{n-1}$$

or

$$\mathrm{PSO}(n) \to \mathbb{R}P^{n-1} \qquad \text{if } F_{\pi_M(x)} \cong \mathbb{R}P^{n-1}$$

where the base is a round sphere or projective space of some radius r_M. It follows that the volume of $H_M x$ only depends on r_M. Likewise the volume of $H_N x$ will only depend on the radius r_N of F_N. On the other hand we know that $\mathrm{vol}(H_M x) = \mathrm{vol}(H_N x) = \nu$, so we must have that $r_M = r_N$, as desired. $\qquad \square$

At this point we know that there are $r > 0$, $\ell_M > 0$ and $\ell_N > 0$ such that M is isometric to $S^n(r) \times S^1(\ell_M)$ (or $\mathbb{R}P^n(r) \times S^1(\ell_M)$) and N is isometric to $S^n(r) \times S^1(\ell_N)$ (or $\mathbb{R}P^n(r) \times S^1(\ell_N)$). It only remains to show that $\ell_M = \ell_N$.

To see this, we need only recall that by normalization of the Sasaki-Mok-O'Neill metrics, we have $\mathrm{vol}(M) = \mathrm{vol}(N)$ (see Lemma 2.9).

Case C (M and N are of type (3)). The unique torsion-free, index 2 subgroups of $\pi_1(M)$ and $\pi_1(N)$ give double covers M' and N'. We claim that the frame bundles $\mathrm{SO}(M')$ and $\mathrm{SO}(N')$ are also isometric. The fiber bundle $\mathrm{SO}(n) \to X \to M$ gives

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \pi_1(X) \to D_\infty \to 1.$$

Now $\pi_1(\mathrm{SO}(M'))$ and $\pi_1(\mathrm{SO}(N'))$ are both index 2 subgroups of $\pi_1(X)$. Since M' and N' are diffeomorphic to $S^{n-1} \times S^1$ we see that $\pi_1(\mathrm{SO}(M')) \cong (\mathbb{Z}/2\mathbb{Z}) \times \mathbb{Z}$ and likewise for $\pi_1(\mathrm{SO}(N'))$. Therefore $\pi_1(\mathrm{SO}(M'))$ and $\pi_1(\mathrm{SO}(N'))$ correspond to the same index 2 subgroup of $\pi_1(X)$. It follows that $\mathrm{SO}(M')$ and $\mathrm{SO}(N')$ are also isometric.

Since M' and N' are diffeomorphic to $S^{n-1} \times S^1$ and H_M acts on S^{n-1} orthogonally, the argument from Case A applies and yields that M' and N' are isometric to the same product $S^{n-1} \times S^1$. Then M and N are obtained as the quotient of $S^{n-1} \times S^1$ by the map $(v, z) \mapsto (-v, z^{-1})$. Hence M and N are isometric.

5. Proof for M with positive constant curvature

In the previous section we have proved Theorem A in all cases except when M has constant curvature $\frac{1}{2\sqrt{\lambda}}$ or M is a surface. We will resolve the latter case in the next section. In this section we will prove:

Theorem 5.1. *Let M, N be closed oriented connected Riemannian n-manifolds and assume M has constant curvature $\frac{1}{2\sqrt{\lambda}}$ for some $\lambda > 0$. Equip $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ with Sasaki-Mok-O'Neill metrics using the invariant inner product $\langle \cdot, \cdot \rangle_\lambda$ on $\mathfrak{o}(n)$. Assume $n \neq 2, 3, 4, 8$. Then M, N are isometric if and only if $\mathrm{SO}(M)$ and $\mathrm{SO}(N)$ are isometric.*

Proof. By simultaneously rescaling the metrics on M and N we can assume that the universal cover of M is a round sphere of radius 1. (Note that in the rescaling, we should also rescale the inner product on $\mathfrak{o}(n)$ that is used in the definition of the Sasaki-Mok-O'Neill metric.)

Since M has positive constant curvature, M is a Riemannian quotient of S^n by a finite group of isometries. Since the isometry group of S^n is $\mathrm{SO}(n+1)$, can write $M = S^n / \pi_1(M)$ for some (finite) group $\pi_1(M) \subseteq \mathrm{SO}(n+1)$.

Further we can write $S^n = \mathrm{SO}(n) \backslash \mathrm{SO}(n+1)$ where the quotient is on the left by the standard copy $\mathrm{SO}(n) \subseteq \mathrm{SO}(n+1)$. The action of $\mathrm{SO}(n+1)$ on S^n by isometries is then just the action of $\mathrm{SO}(n+1)$ by right-translations on $\mathrm{SO}(n) \backslash \mathrm{SO}(n+1)$, so that we have

$$M \cong \mathrm{SO}(n) \backslash \mathrm{SO}(n+1) / \pi_1(M).$$

Passing to the frame bundle, we obtain $X \cong SO(n+1)/\pi_1(M)$, where the cover $\mathrm{SO}(n+1)$ is equipped with a bi-invariant metric. Further N is a quotient of X by a group $L \cong \mathrm{SO}(n)$ acting effectively and isometrically on X.

Consider now the cover $\mathrm{SO}(n+1) \to X$. The (effective) action of L on X induces an effective action of a unique connected cover $\overline{L}$ of L on $\mathrm{SO}(n+1)$. In fact in this case we can show $\overline{L}$ is isomorphic to L. Namely, to compute $\overline{L}$ it suffices to identify the subgroup $\pi_1 \overline{L} \subseteq \pi_1 L$. This can be done as follows. Let $x \in X$. The orbit map

$$L \to X$$
$$g \mapsto gx$$

induces a homomorphism $\alpha : \pi_1(L) \to \pi_1(X)$. The cover $\mathrm{SO}(n+1) \to X$ corresponds to a subgroup H of $\pi_1(X)$. Then $\pi_1(\overline{L})$ is given by $\alpha^{-1}(H)$. In our case we have that the bundle $\mathrm{SO}(n) \to X \to N$ induces a short exact sequence

$$1 \to \pi_1 \mathrm{SO}(n) \to \pi_1(X) \to \pi_1(N) \to 1,$$

and the subgroup H corresponding to the cover $\mathrm{SO}(n+1) \to X$ is exactly the kernel of $\pi_1(X) \to \pi_1(N)$. Since the fibers of $X \to N$ are exactly orbits of L, any loop in L maps to a constant path in N. It follows that $\pi_1(L)$ is contained in H, hence we have that $\pi_1(\overline{L}) = \pi_1(L)$, and thus $\overline{L} = L$.

We now study the (effective and isometric) action of L on $\mathrm{SO}(n+1)$ (with respect to a bi-invariant metric on the latter). We claim that L has to act either by left or right-translations on $\mathrm{SO}(n+1)$. To see this, we first consider the full isometry group of $\mathrm{SO}(n+1)$ (with respect to a bi-invariant metric), which has been computed by d'Atri-Ziller [DZ79]. Namely, they show that the isometry group of a simple compact Lie group G equipped with a bi-invariant metric is

$$\mathrm{Isom}(G) \cong G \rtimes \mathrm{Aut}(G)$$

where the copy of G acts by left-translations on G. We apply this to the group $G = \mathrm{SO}(n+1)$. Since L is connected, it follows that the image of $L \hookrightarrow \mathrm{Isom}(G)$ is contained in the connected component $\mathrm{Isom}(G)^0$ of $\mathrm{Isom}(G)$ containing the identity. We can explicitly compute $\mathrm{Isom}(G)^0$. Namely, since $\mathrm{Out}(G)$ is discrete, $\mathrm{Isom}(G)^0$ is isomorphic to

$$G \rtimes \mathrm{Inn}(G) \cong (G \times G)/Z(G)$$

where $Z(G)$ is the center of G, and $Z(G) \hookrightarrow G \times G$ is the diagonal embedding. The two copies of G act by left- and right-translations on G.

It will be convenient to work with the product $G \times G$, rather than $(G \times G)/Z(G)$. Note that the preimage of L under the natural projection

$$\mathrm{SO}(n+1) \times \mathrm{SO}(n+1) \to (\mathrm{SO}(n+1) \times \mathrm{SO}(n+1))/Z(\mathrm{SO}(n+1))$$

is a (possibly disconnected) cover of L. Let $\widetilde{L}$ denote the connected component containing the identity (so $\widetilde{L}$ is a connected cover of L).

Now L acts by left or right-translations on $\mathrm{SO}(n+1)$ if and only if L is contained in the image of one of the factors of $G \times G$ in $(G \times G)/Z(G)$, which is equivalent to:

Claim 5.2. $\overline{L}$ is contained in one of the factors of $G \times G$.

Proof. Assume the contrary. Since $\widetilde{L}$ is a simple connected Lie group projecting nontrivially under the projections $G \times G \to G$ to a factor, $\widetilde{L}$ in fact projects with finite kernel onto each factor. We will first show that one of these kernels has to be trivial, so that $\widetilde{L}$ projects isomorphically to one of the factors.

The kernel of either of these projections is a discrete normal subgroup of $\widetilde{L}$, and hence central. Since $L \cong \mathrm{SO}(n)$ and $\widetilde{L}$ is a connected cover of L, the only possibilities for $\widetilde{L}$ are $\mathrm{SO}(n)$ and $\mathrm{Spin}(n)$. The center $Z(\widetilde{L})$ of $\widetilde{L}$ is then

$$Z(\widetilde{L}) \cong \begin{cases} 1 & \text{if } L \cong \mathrm{SO}(n), n \text{ is odd,} \\ \mathbb{Z}/(2\mathbb{Z}) & \text{if } L \cong \mathrm{SO}(n), n \text{ is even,} \\ \mathbb{Z}/(2\mathbb{Z}) & \text{if } L \cong \mathrm{Spin}(n), n \text{ is odd,} \\ \mathbb{Z}/(4\mathbb{Z}) & \text{if } L \cong \mathrm{Spin}(n), n \text{ is even.} \end{cases}$$

Let $K_i, i = 1, 2$ be the kernels of the projections of $\widetilde{L}$ onto the ith factor. As discussed above, we have $K_i \subseteq Z(\widetilde{L})$ for each i. Further, since no nontrivial element of $\widetilde{L}$ projects trivially to both factors, we must have $K_1 \cap K_2 = 1$. On the other hand, none of the possibilities for $Z(\widetilde{L})$ have two nontrivial subgroups that intersect trivially, so we conclude that K_1 or K_2 is trivial. Without loss of generality, we assume that $K_1 = 1$.

Then $\widetilde{L}$ is the graph of an injective homomorphism

$$\varphi : p_1(\widetilde{L}) \hookrightarrow \mathrm{SO}(n+1).$$

Here $p_1(\widetilde{L})$ is the image of $\widetilde{L}$ under projection to the first factor of $G \times G$, which is isomorphic to $\widetilde{L}$. Recall that $\widetilde{L}$ is isomorphic to either $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$. Fortunately, the possible maps $\mathrm{SO}(n) \to \mathrm{SO}(n+1)$ and $\mathrm{Spin}(n) \to \mathrm{SO}(n+1)$ are extremely limited by the following result.

Lemma 5.3 ([Kob72, Lemma 1 in II.3]). *Let H be a closed connected subgroup of $\mathrm{SO}(n+1)$ of dimension $\frac{n(n-1)}{2}$ with $n+1 \neq 4$. Then H is isomorphic to $\mathrm{SO}(n)$ or $\mathrm{Spin}(n)$, and*

(1) if $n+1 \neq 8$, then H fixes a line in $\mathbb{R}^{n+1}$,
(2) if $n+1 = 8$, then either H fixes a line in $\mathbb{R}^{n+1}$ or $H \cong \mathrm{Spin}(7)$ is embedded in $\mathrm{SO}(8)$ via the spin representation.

Also note that in the first case $H = \mathrm{Spin}(n)$ is impossible, so we have $H \cong \mathrm{SO}(7)$. Hence we have the following possibilities for $\widetilde{L}$:

(1) $\widetilde{L} \cong SO(n)$. Then p_1 and p_2 restrict to isomorphisms on $\widetilde{L}$. The groups $p_1(\widetilde{L})$ and $p_2(\widetilde{L})$ are subgroups of $SO(n+1)$ fixing a line.

(2) $\widetilde{L} \cong \mathrm{Spin}(7)$. Then $p_1(\widetilde{L}) \subseteq SO(8)$ is the image of the spin representation. $p_2(\widetilde{L})$ is either isomorphic to $\mathrm{Spin}(7)$ embedded via the spin representation, or $p_2(\widetilde{L})$ is isomorphic to $SO(7)$ and fixes a line.

Consider the second case. We claim that it is not possible that $p_2(\widetilde{L}) \cong SO(7)$. For if this were the case, we have $(-\mathrm{id}, -\mathrm{id}) \notin \widetilde{L}$, so $\widetilde{L}$ projects isomorphically to $(G \times G)/Z(G)$. However, we know that the image of $\widetilde{L}$ is L, but $\widetilde{L} \cong \mathrm{Spin}(7)$ and $L \cong SO(7)$. Hence this is impossible.

For us, the important consequence of the above dichotomy is that $p_1(\widetilde{L})$ and $p_2(\widetilde{L})$ are conjugate subgroups of $SO(n+1)$. So there exists $h \in SO(n+1)$ such that

$$\widetilde{L} = \{(g, hgh^{-1}) \mid g \in p_1(\widetilde{L})\}.$$

Then L fixes the point $h^{-1} \in SO(n+1)$:

$$(g, hgh^{-1}) \cdot h^{-1} = gh^{-1}(hgh^{-1})^{-1} = h^{-1}.$$

But since L acts freely on the frame bundle X of N, we know L does not have a fixed point. This is a contradiction. $\qquad\square$

Since L acts by left or right-translations on $SO(n+1)$, we will identify L with a subgroup of $SO(n+1)$. Then we can conjugate L to a standard copy of $SO(n)$ by an element of $SO(n+1)$. Therefore without loss of generality we have $N \cong SO(n)\backslash SO(n+1)/\pi_1(N)$, and we have an isometry

$$f : SO(n+1)/\pi_1(M) \cong SO(M) \to SO(N) \cong SO(n+1)/\pi_1(N).$$

By composing with a left-translation of $SO(n+1)$, we can assume $f(e\pi_1(M)) = e\pi_1(N)$.

Claim 5.4. f lifts to an isometry $SO(n+1) \to SO(n+1)$.

Proof. Note that f induces an isomorphism

$$f_* : \pi_1(SO(M)) \to \pi_1(SO(N)).$$

The bundle $SO(M) \to M$ gives rise to a short exact sequence

$$1 \to \pi_1 SO(n) \to \pi_1 SO(M) \to \pi_1 M \to 1,$$

and likewise the bundle $SO(N) \to N$ gives a short exact sequence for $\pi_1 SO(N)$. The cover $SO(n+1) \to SO(M)$ corresponds to the subgroup $\pi_1 SO(n)$ of $\pi_1(SO(M))$, and likewise for the cover $SO(n+1) \to SO(N)$. Therefore to show that f lifts to a map $SO(n+1) \to SO(n+1)$, it suffices to show that $f_*\pi_1 SO(n) = \pi_1(SO(n))$.

To see this, we lift f to an isometry

$$\widetilde{f} : \mathrm{Spin}(n+1) \to \mathrm{Spin}(n+1)$$

which is possible because $\mathrm{Spin}(n+1)$ is the universal cover of $SO(M)$ and $SO(N)$. We can assume that $\widetilde{f}(e) = e$ by choosing an appropriate lift. Hence by the computation of $\mathrm{Isom}(\mathrm{Spin}(n+1))$ by d'Atri-Ziller, $\widetilde{f}$ is an automorphism of $\mathrm{Spin}(n+1)$.

Because $\widetilde{f}$ is a lift of $f : SO(M) \to SO(N)$, we also know that $\widetilde{f}$ restricts to the isomorphism $f_* : \pi_1(SO(M)) \to \pi_1(SO(N))$. Therefore to prove the claim, it suffices to show that $\widetilde{f}(\pi_1 SO(n)) = \pi_1 SO(n)$.

To see this, we view $\pi_1 SO(M)$ and $\pi_1 SO(N)$ as subgroups of $\mathrm{Spin}(n+1)$. Then $\pi_1 SO(n) \subseteq \mathrm{Spin}(n+1)$ is a central subgroup of order 2. Since $\widetilde{f}$ is an automorphism of $\mathrm{Spin}(n+1)$, it follows that $\widetilde{f}(\pi_1(SO(n)))$ is also a central subgroup of order 2 of $\mathrm{Spin}(n+1)$. The center of $\mathrm{Spin}(n+1)$ is isomorphic to either $\mathbb{Z}/(2\mathbb{Z})$ (if $n+1$ is odd),

or $\mathbb{Z}/(4\mathbb{Z})$ (if $n+1$ is even), and in particular has a unique order 2 subgroup. Therefore we must have $\widetilde{f}\pi_1 \, \mathrm{SO}(n) = \pi_1 \, \mathrm{SO}(n)$. $\square$

Let
$$\widetilde{f} : \mathrm{SO}(n+1) \to \mathrm{SO}(n+1)$$
denote a lift of f. By choosing an appropriate lift, we can assume that $\widetilde{f}(e) = e$, and hence that $\widetilde{f}$ is an automorphism of $\mathrm{SO}(n+1)$ (here we again used the computation of d'Atri-Ziller of the isometry group of $\mathrm{SO}(n+1)$). Because $\widetilde{f}$ is a lift of f, we know that $\widetilde{f}$ restricts to an isomorphism $\pi_1 M \to \pi_1 N$.

By composing $\widetilde{f}$ with conjugation by an element in $\mathrm{SO}(n+1)$, we can assume that $\widetilde{f}(\mathrm{SO}(n)) = \mathrm{SO}(n)$, and hence $\widetilde{f}$ descends to an isometry
$$\overline{f} : S^n \to S^n,$$
where we identified S^n with $\mathrm{SO}(n)\backslash \mathrm{SO}(n+1)$. Since $\widetilde{f}$ conjugates $\pi_1(M)$ to $\pi_1(N)$, we further know that $\overline{f}$ descends to an isometry $M \to N$. $\square$

This finishes the proof of Theorem 5.1. $\square$

6. Proof of the main theorem for surfaces

In this section we prove Theorem A for surfaces. We cannot use the Takagi-Yawata theorem (Theorem 2.6) that computes $i(X)$ in this situation, but instead we use the classification of surfaces and Lie groups in low dimensions.

Let M and N be closed oriented surfaces with $\mathrm{SO}(M) \cong SO(N)$. Therefore M and N are each diffeomorphic to one of S^2, T^2 or Σ_g with $g \geq 2$. We know that

- $\mathrm{SO}(S^2)$ is diffeomorphic to $\mathrm{SO}(3)$,
- $\mathrm{SO}(T^2)$ is diffeomorphic to T^3, and
- $\mathrm{SO}(\Sigma_g)$ is diffeomorphic to $T^1\Sigma_g = \mathrm{PSL}_2 \, \mathbb{R}/\Gamma$ for a cocompact torsion-free lattice $\Gamma \subseteq \mathrm{PSL}_2 \, \mathbb{R}$.

In particular the diffeomorphism type of the frame bundle of a surface determines the diffeomorphim type of the surface. It follows that M and N are diffeomorphic.

Consider the Lie algebra of Killing fields $i(X)$ of X. Then $i(X)$ contains the (1-dimensional) subalgebras i_V^M and i_V^N. If $i_V^M = i_V^N$, then we proceed as in Case 1 in Section 4, and we find that M and N are isometric. Therefore we will assume that $i_V^M \neq i_V^N$. In particular we must have $\dim i(X) \geq 2$.

Let I_V^M (resp. I_V^N) be the subgroup of $\mathrm{Isom}(X)$ obtained by exponentiating the Lie algebra i_V^M (resp. i_V^N). Then I_V^M and I_V^N are closed subgroups of $\mathrm{Isom}(X)$ isomorphic to S^1.

We will now consider each of the possibilities of the diffeomorphism types of M and N, and prove that M and N have to be isometric.

Case 1 (M and N are diffeomorphic to Σ_g, $g \geq 2$). Then $X = T^1\Sigma_g$ is a closed aspherical manifold. Conner and Raymond proved [CR70] that if a compact connected Lie group G acts effectively on a closed aspherical manifold L, then G is a torus and $\dim G \leq \mathrm{rk}_{\mathbb{Z}} \, Z(\pi_1 L)$, where $Z(\pi_1 L)$ is the center of $\pi_1(L)$. In particular we find that $\dim i(X) \leq \mathrm{rk}_{\mathbb{Z}} \, Z(\pi_1 T^1\Sigma_g) = 1$. This contradicts our assumption that $\dim i(X) \geq 2$.

Case 2 (M and N are diffeomorphic to S^2). Let G be the connected component of $\mathrm{Isom}(X)$ containing the identity. Then G is a compact connected Lie group acting effectively and isometrically on $X = \mathrm{SO}(3)$, and G contains I_V^M and I_V^N.

If $\dim G = 2$, then G is a 2-torus. In particular I_V^M and I_V^N centralize each other. Therefore I_V^M acts on $X/I_V^N = N$ and similarly I_V^N acts on M. The kernel of either of these actions is $I_V^M \cap I_V^N$, which is a finite subgroup of both I_V^M and I_V^N.

Since an S^1-action on S^2 has at least one fixed point (because $\chi(S^2) \neq 0$), we see that $N/I_V^M \cong [-1,1] \cong M/I_V^N$. It is then straightforward to see that the metric on M (resp. N) is of the form

$$ds_M^2 = f_M(x_0)(dx_0^2 + dx_1^2 + dx_2^2)$$

(resp. $ds_N^2 = f_N(x_0)(dx_0^2 + dx_1^2 + dx_2^2)$) as in Theorem 3.4.(1). We can apply the reasoning from Case C of the proof of Case 3(b) in Section 4 to show M and N are isometric.

Therefore we will assume $\dim G \geq 3$. In addition we know that $\dim G \leq 6$ by Theorem 3.1. Finally, we must have $\text{rank}(G) \leq 2$: Namely let T be a maximal torus in G containing I_V^M. Since T centralizes I_V^M, the group T/I_V^M acts effectively on M. However, a torus of dimension ≥ 2 does not act effectively on S^2. (To see this, note that any 1-parameter subgroup H has a fixed point on S^2 because the Killing field generated by H has a zero on S^2. We can take H to be dense, so that the entire torus fixes a point p. The isotropy action on T_pM is a faithful 2-dimensional representation of the torus, which is impossible unless the torus is 1-dimensional.)

Therefore the only possibilities for the Lie algebra $\mathfrak{g}$ of G are

(a) $\mathfrak{g} \cong \mathfrak{o}(3)$,
(b) $\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{o}(3)$, and
(c) $\mathfrak{g} \cong \mathfrak{o}(3) \oplus \mathfrak{o}(3)$.

We will now consider each of these cases separately.

Case 2(a) ($\mathfrak{g} \cong \mathfrak{o}(3)$). Since G has rank 1, I_V^M and I_V^N are both maximal tori of G. Since all maximal tori are conjugate, there is some element $g \in G$ so that $gI_V^N g^{-1} = I_V^M$. Then g induces an obvious isometry $M \to N$.

Case 2(b) ($\mathfrak{g} \cong \mathbb{R} \oplus \mathfrak{o}(3)$). We can conjugate I_V^M by an element $g \in G$ so that $gI_V^M g^{-1}$ and I_V^N centralize each other. Then either $gI_V^M g^{-1} = I_V^N$, in which case g induces an isometry $M \to N$, or $gI_V^M g^{-1}$ and I_V^N generate a 2-torus. In the latter case the argument above in case $\dim G = 2$ shows that the metrics on M and N are of the form

$$ds^2 = f(x_0)(dx_0^2 + dx_1^2 + dx_2^2)$$

for some function f on $[-1,1]$. Then the argument of Case A of Case 3(b) in Section 4 shows that M and N are isometric.

Case 2(c) ($\mathfrak{g} \cong \mathfrak{o}(3) \oplus \mathfrak{o}(3)$). In this case $\dim \text{Isom}(X) = 6$ is maximal. By Theorem 3.1 the metric on X has positive constant curvature. Therefore the metrics on M and N have positive constant curvature. Further by Lemma 2.9 we have $\text{vol}(M) = \text{vol}(N)$. It follows that M and N are isometric.

Case 3 (M and N are diffeomorphic to T^2). In this case X is diffeomorphic to T^3. Again by the theorem of Conner-Raymond [CR70] on actions of compact Lie groups on aspherical manifolds, we know that a connected compact Lie group acting effectively on a torus is a torus. Therefore I_V^M and I_V^N centralize each other, so I_V^M and I_V^N generate a 2-torus. Further I_V^N acts on $M = X/I_V^M$ with finite kernel $I_V^M \cap I_V^N$. Again by [CR70], the action of $I_V^N/(I_V^M \cap I_V^N)$ on M is free, so that the map

$$M \to M/I_V^N \cong S^1$$

is a fiber bundle (with S^1 fibers). The argument of Case C in Case 3(b) of the proof of Theorem A constructs a (unit length) Killing field X_M on M that is orthogonal to the fibers of $M \to M/I_V^N$. It follows that M is a 2-torus equipped with a translation-invariant metric. Any such metric is automatically flat: Namely, because the torus is abelian, the metric is automatically bi-invariant. Then we use the following general fact:

On a Lie group H with a bi-invariant metric, the Lie structure and sectional curvature are tied by the identity (see e.g. [Pet06, Proposition 3.4.12])

$$K(X, Y) = -\frac{1}{2}||[X, Y]||$$

where X, Y belong to the Lie algebra $\mathfrak{h}$ (which is identified with $T_e H$ in the usual way), and the bracket is the Lie bracket. Since T^2 is abelian, it follows that the sectional curvatures with respect to any invariant metric vanish.

We conclude that M is flat. By carrying out the same construction for N, we obtain a Killing field X_N on N that is orthogonal to the I_V^M-orbits, and we conclude that N is flat.

To show that M and N are isometric, recall that the isometry type of a flat 2-torus is specified by the length of two orthogonal curves that generate its fundamental group. For M we can consider the curves given by an I_V^N-orbit on M and an integral curve of X_M. Similarly for N we can consider an I_V^N-orbit on N and an integral curve of X_N.

For $x \in M$ and $\widetilde{x} \in X$ lying over x, we have a covering

$$I_V^N \widetilde{x} \to I_V^N x$$

of degree $|I_V^M \cap I_V^N|$. Recall that the I_V^N-orbits in X have a fixed volume ν, since we normalized the Sasaki-Mok-O'Neill metric on X in this way. Therefore

$$\ell(I_V^N x) = \frac{1}{|I_V^M \cap I_V^N|}\ell(I_V^N \widetilde{x}) = \frac{\nu}{|I_V^M \cap I_V^N|}.$$

Combining this with a similar computation for the length of an I_V^M-orbit on N gives $\ell(I_V^N x) = \ell(I_V^M y)$ for every $x \in M$ and $y \in N$. Therefore we see that the length of an integral curve of X_M (resp. X_N) is $\frac{\text{vol}(M)}{\ell(I_V^N \cdot x)}$ for $x \in M$ (resp. $\frac{\text{vol}(N)}{\ell(I_V^M \cdot y)}$ for $y \in N$). Since $\text{vol}(M) = \text{vol}(N)$ by Lemma 2.9, it follows that M and N are isometric.

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