Riemannian manifolds with nontrivial local symmetry

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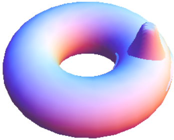
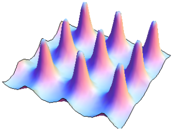
The problem

- Let M be a closed Riemannian manifold.
- $\mathrm{Isom}(\tilde{M})$ contains the deck group $\pi_1(M)$.
- Generically: $[\mathrm{Isom}(\tilde{M}) : \pi_1 M] < \infty$.



Problem

Classify M such that $[\mathrm{Isom}(\tilde{M}) : \pi_1 M] = \infty$.



Example

- M closed hyperbolic manifold.

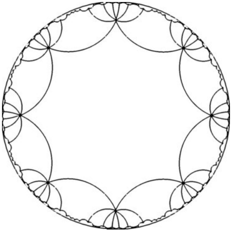
Theorem (Bochner, Yano)

Isom(M) is finite.

- But

$$\mathrm{Isom}(\tilde{M}) = \mathrm{Isom}(\mathbb{H}^n) = O^+(n,1).$$

Note: $\tilde{M}$ is homogeneous.



Farb-Weinberger theorem

Theorem (Farb, Weinberger (2008))

Let M be a closed aspherical manifold. Then either

1. $[Isom(\tilde{M}), \pi_1 M] < \infty$ *or*
2. *M is on a list.*

Further, every item on the list occurs.

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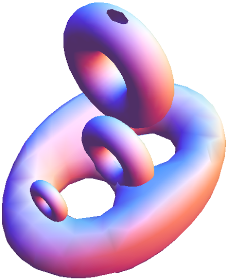
Applications

1. Differential geometry
2. Complex geometry
3. etc.

The list

- A finite cover of M is a 'Riemannian orbibundle' $F \to M' \to B$.
- The fibers F are locally homogeneous.
- $[\mathrm{Isom}(\tilde{B}) : \pi_1 B] < \infty$.



The general (nonaspherical) case

Problems in general case

- Proof of Farb and Weinberger fails:
  M aspherical $\Rightarrow$

$$(\text{geometry of } \tilde{M}) \leftrightarrow (\text{geometry of } \pi_1(M)).$$

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- More options for $\mathrm{Isom}(\tilde{M})$: E.g. compact factors.

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  Not true in general case.
- More options for $\text{Isom}(\tilde{M})$: E.g. compact factors.
- So the 'list' is more complicated.

More complicated example

Example

- $$H := \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, Z(H) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

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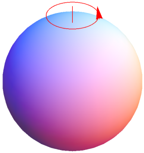
- Let $Z(N) \cong S^1$ act on S^2 by rotations.



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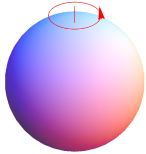

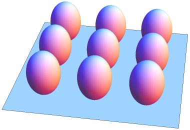

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- Set $N := H/\mathbb{Z}$.
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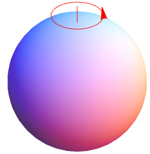
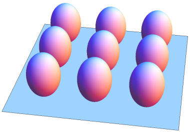
- Orbits in X are of two types:
  1. N (generic)
  2. $N/S^1 = \mathbb{R}^2$ (north/south poles)





General fact about Lie groups

Theorem (Levi decomposition)

Let G be a connected Lie group. Then

- *There exists a solvable subgroup G_{sol} and*
- *there exists a semisimple subgroup G_{ss} such that*

$$G = G_{sol}G_{ss}.$$

Remark

This decomposition is essentially unique.

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Theorem (VL)

Let M be a closed Riemannian manifold, $G := Isom(\tilde{M})$. Then either

1. G^0 *is compact or*
2. G^0_{ss} *is compact and* G *has infinitely many components or*
3. M *is on a 'list'.*

Nonaspherical case: The list

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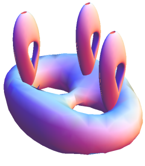
1. G^0_{ss} noncompact $\Rightarrow$

   M ‘virtually’ fibers over locally symmetric space.



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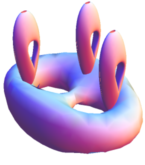
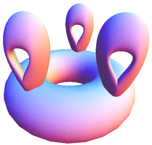


2. G_{ss}^0 is compact, $\#(\text{components of } G) < \infty \Rightarrow$

   M is 'virtually' an 'iterated bundle' over tori.



G^0 is nilpotent: Outline

Proof.

□

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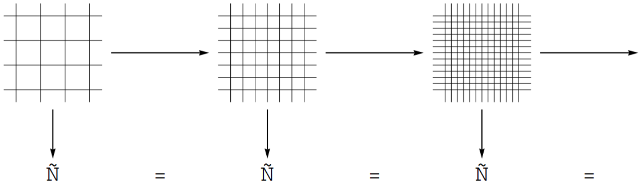

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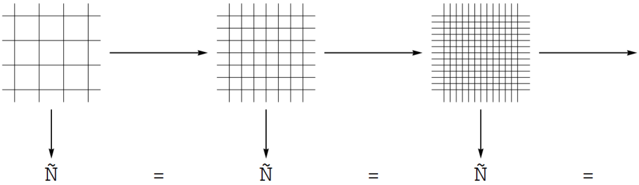

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- Arzelà-Ascoli $\rightsquigarrow$ Limit $\tilde{f} : \tilde{M} \to \tilde{N}$

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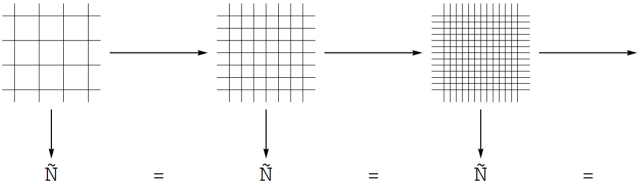

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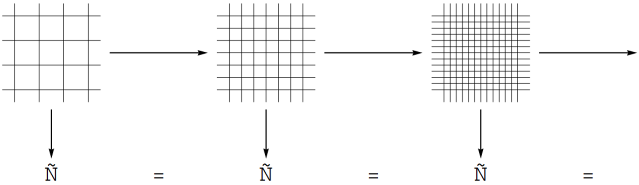
- Arzelà-Ascoli $\rightsquigarrow$ Limit $\tilde{f} : \tilde{M} \to \tilde{N}$
- Smooth $\tilde{f}$ while keeping it equivariant.

□



G_{ss}^0 noncompact: Outline

- Find a lattice Λ in a semisimple Lie group and a map $\Gamma \to \Lambda$.
- $\rightsquigarrow$ homotopy class of maps $M \to N$ (N locally symmetric space for Λ).

Theorem (Eells, Sampson, Hartman, Schoen-Yau)

$\exists!$ *harmonic* $f : M \to N$ *in this class.*

G_{ss}^0 noncompact: Outline

- Lift to $\tilde{f} : \tilde{M} \to \tilde{N}$.

Theorem (Frankel, 1994)

One can average $\tilde{f}$.

- $\rightsquigarrow$ the fiber bundle $M \to N$.

Remarks

- Frankel's method relies heavily on symmetric space theory.
- This does not work if G_{ss}^0 is compact.

Open question

Question

Let M be a closed Riemannian manifold. Is it true that either

1. $\mathrm{Isom}(\tilde{M})^0$ is compact or
2. M is virtually an iterated orbibundle, at each step with locally homogeneous fibers or base?

More specifically:

Problem

Describe closed Riemannian manifolds M such that G has infinitely many components and G_{ss}^0 is compact.