Riemannian manifolds with nontrivial local symmetry

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The problem

- Let $M$ be a closed Riemannian manifold.
- $\text{Isom}(\tilde{M})$ contains the deck group $\pi_1(M)$.
- Generically: $[\text{Isom}(\tilde{M}) : \pi_1 M] < \infty$.

**Problem**

Classify $M$ such that $[\text{Isom}(\tilde{M}) : \pi_1 M] = \infty$. 
Example

- $M$ closed hyperbolic manifold.

**Theorem (Bochner, Yano)**

$\text{Isom}(M)$ is finite.

- But

  $\text{Isom}(\tilde{M}) = \text{Isom}(\mathbb{H}^n) = O^+(n, 1)$.

  Note: $\tilde{M}$ is homogeneous.
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Farb-Weinberger theorem

Theorem (Farb, Weinberger (2008))

Let $M$ be a closed aspherical manifold. Then either

1. $[\text{Isom}(\tilde{M}), \pi_1 M] < \infty$ or
2. $M$ is on a list.

Further, every item on the list occurs.
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Applications

1. Differential geometry
2. Complex geometry
3. etc.
The list

- A finite cover of $M$ is a ‘Riemannian orbibundle’ $F \to M' \to B$.

- The fibers $F$ are locally homogeneous.

- $[\text{Isom}(\tilde{B}) : \pi_1 B] < \infty$. 
Problems in general case

- Proof of Farb and Weinberger fails:
  \[ M \text{ aspherical} \Rightarrow \]
  \[
  \text{(geometry of } \tilde{M} \text{)} \leftrightarrow \text{(geometry of } \pi_1(M)\text{)}.
  \]

Crucial in F-W: Isom( ˜M) -orbits are of the same type.
Not true in general case.
More options for Isom( ˜M): E.g. compact factors.
So the 'list' is more complicated.
The general (nonaspherical) case

Problems in general case

- Proof of Farb and Weinberger fails: $M$ aspherical $\Rightarrow$

  $$(\text{geometry of } \tilde{M}) \leftrightarrow (\text{geometry of } \pi_1(M)).$$

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More complicated example

Example

\[ H := \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix}, \quad Z(H) = \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \]
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- Set \( N := H / \mathbb{Z} \).
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- Let \( X := (S^2 \times N) / Z(N) \).
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- Set \( N := H/\mathbb{Z} \).
- Let \( Z(N) \cong S^1 \) act on \( S^2 \) by rotations.
- Let \( X := (S^2 \times N)/Z(N) \).
- Orbits in \( X \) are of two types:
  1. \( N \) (generic)
  2. \( N/S^1 = \mathbb{R}^2 \) (north/south poles)
General fact about Lie groups

**Theorem (Levi decomposition)**

Let $G$ be a connected Lie group. Then

- There exists a solvable subgroup $G_{sol}$ and
- there exists a semisimple subgroup $G_{ss}$ such that

$$G = G_{sol}G_{ss}.$$

**Remark**

This decomposition is essentially unique.
Nonaspherical case

Theorem (VL)

Let $M$ be a closed Riemannian manifold, $G := \text{Isom}(\tilde{M})$. Then either

1. $G$ is compact
2. $G_0$ is compact and $G$ has infinitely many components
3. $M$ is on a list
Nonaspherical case

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2. $G^0_{ss}$ is compact and $G$ has infinitely many components or
3. $M$ is on a ‘list’.
Nonaspherical case: The list

1. $G_{0, ss}$ noncompact $\Rightarrow M$ 'virtually' fibers over locally symmetric space.

2. $G_{0, ss}$ is compact, $\#(\text{components of } G) < \infty \Rightarrow M$ is 'virtually' an 'iterated bundle' over tori.
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Proof.

$G^0$ is nilpotent: Outline

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**Proof.**

- $\Gamma \subseteq G^0$ lattice in nilpotent group $\mapsto$ Map $f_1 : M \to N$
- $\Gamma$ starts ‘tower of lattices’ $(\Gamma_q)_q$

![Diagram showing the tower of lattices](image)
\( G^0 \) is nilpotent: Outline

Proof.

- \( \Gamma \subseteq G^0 \) lattice in nilpotent group \( \leadsto \) Map \( f_1 : M \rightarrow N \)
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$G^0$ is nilpotent: Outline

**Proof.**

- $\Gamma \subseteq G^0$ lattice in nilpotent group $\rightsquigarrow$ Map $f_1 : M \to N$
- $\Gamma$ starts ‘tower of lattices’ $(\Gamma_q)_q \rightsquigarrow$ Map $f_q : M_q \to N_q$
- Arzelà-Ascoli $\rightsquigarrow$ Limit $\tilde{f} : \tilde{M} \to \tilde{N}$
$G^0$ is nilpotent: Outline

Proof.

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- Arzelà-Ascoli $\leadsto$ Limit $\tilde{f} : \tilde{M} \to \tilde{N}$

- Smooth $\tilde{f}$ while keeping it equivariant.
$G^0_{ss}$ noncompact: Outline

- Find a lattice $\Lambda$ in a semisimple Lie group and a map $\Gamma \rightarrow \Lambda$.

- $\sim$ homotopy class of maps $M \rightarrow N$ ($N$ locally symmetric space for $\Lambda$).

**Theorem (Eells, Sampson, Hartman, Schoen-Yau)**

$\exists!$ harmonic $f : M \rightarrow N$ *in this class.*
$G_{ss}^0$ noncompact: Outline

- Lift to $\tilde{f} : \tilde{M} \rightarrow \tilde{N}$.

**Theorem (Frankel, 1994)**

One can average $\tilde{f}$.

- $\rightsquigarrow$ the fiber bundle $M \rightarrow N$.

**Remarks**

- Frankel’s method relies heavily on symmetric space theory.
- This does not work if $G_{ss}^0$ is compact.
Open question

Question
Let $M$ be a closed Riemannian manifold. Is it true that either
1. $\text{Isom}(\tilde{M})^0$ is compact or
2. $M$ is virtually an iterated orbibundle, at each step with locally homogeneous fibers or base?

More specifically:

Problem
Describe closed Riemannian manifolds $M$ such that $G$ has infinitely many components and $G_{ss}^0$ is compact.