Coarse geometry of expanders from homogeneous spaces

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Joint work with D. Fisher and T. Nguyen

Discretizing group actions (Vigolo, '16)

- Γ f.g. group
- M closed Riem. manifold
- $\Gamma \curvearrowright M$ (bi-Lipschitz)

$\rightsquigarrow$

Family of graphs
$(X_t)_{t>0}$

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Mesh $< t^{-1}$

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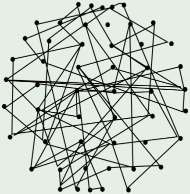
Graphs X_t



Vertices: Regions R_i
Edges: $sR_i \cap R_j \neq \emptyset$.

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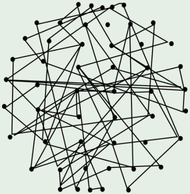

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Roe's Warped Cone

Assembles all X_t
$\rightsquigarrow \mathscr{C}(\Gamma \curvearrowright \mathrm{M})$.

Dynamics and coarse geometry

Dynamics of $\Gamma \curvearrowright M$

$\overset{?}{\leftrightsquigarrow}$

Coarse geometry of graphs $(X_t)_t$
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Spectral gap for $\Gamma \curvearrowright M$ $\quad\Longrightarrow\quad$ $(X_n)_n$ *expander.*

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Subgroups of compact Lie groups $\rightsquigarrow$ Spectral gap

Margulis, Sullivan, Drinfeld, Gamburd–Jakobson–Sarnak, Bourgain–Gamburd ($\times 2$), Benoist-De Saxcé, ...

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From now on:

- $M = G$ compact semisimple Lie
- $\Gamma \subseteq G$ dense, fin. pres.

Theorems

Coarse geometry of cones $\overset{?}{\longleftrightarrow}$ Dynamics of $\Gamma \curvearrowright M$

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Theorem (De Laat–Vigolo, Sawicki '17)

Warped cones are QI $\Longrightarrow$ *Groups are Stably QI*

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Coarse geometry of cones $\overset{?}{\leftrightsquigarrow}$ Dynamics of $\Gamma \curvearrowright M$

Theorem (De Laat–Vigolo, Sawicki, '17)

$$\textit{Warped cones are QI} \implies \textit{Groups are } {\color{red}\textit{Stably}} \textit{ QI}$$

$$\mathscr{C}(\Gamma \curvearrowright \mathrm{M}) \simeq_{QI} \mathscr{C}(\Lambda \curvearrowright \mathrm{N}) \implies \Gamma \times \mathbb{R}^{\dim M} \simeq_{QI} \Lambda \times \mathbb{R}^{\dim N}.$$

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Warped cones are QI $\implies$ *actions are commensurable*

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Similar result for graphs $\implies$

Theorem (Fisher–Nguyen–vL, '17)

There exist continua of QI disjoint expanders.