Symmetry and self-similarity in geometry

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$T^2 = \begin{array}{c} 1 \\ 0 \\ 1 \end{array}$
An example

$T^2 = \begin{array}{c}
1 \\
1/n \\
0 \\
1/n \\
1
\end{array}$
An example

\[ T^2 = \]

\[
\begin{array}{|c|c|}
\hline
1 & 1/n \\
\hline
1/n & 0 \\
\hline
\end{array}
\]

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An example

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\[ 1/n \]
An example

\[ T^2 = \begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array} \]

\[ T^2 = \]

\[ n^2 : 1 \]
An example

Remarks:
1. \( \exists \) covers \( T^2 \) with degree \( > 1 \)
2. Symmetry \( \sim \) covers
Genus $\geq 2$

1. $\forall g \geq 2 : \emptyset$ covers $\Sigma_g \rightarrow \Sigma_g$

   Reason: $\chi = 2 - 2g \neq 0$

2. **Hurwitz’s $84(g - 1)$ Theorem (1893):**

   $\Sigma_g$ Riemann surface, $g \geq 2 \implies |\text{Aut}^+(\Sigma_g)| \leq 84(g - 1)$
Genus $\geq 2$

Hurwitz’s $84(g - 1)$ Theorem (1893):

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Genus $\geq 2$

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Key fact

$$\forall X = \mathbb{H}^2/\Gamma : \text{Area}(X) \geq \frac{\pi}{21}$$
Genus $\geq 2$

Hurwitz’s $84(g - 1)$ Theorem (1893):

$$|\text{Aut}^+(\Sigma_g)| \leq 84(g - 1)$$

Key fact

\[
\forall X = \mathbb{H}^2/\Gamma : \text{Area}(X) \geq \frac{\pi}{21}
\]

Connection:

\[
\text{Area}(\Sigma_g/\text{Aut}^+(\Sigma_g)) = \frac{\text{Area}(\Sigma_g)}{|\text{Aut}^+(\Sigma_g)|} \\
\geq \frac{\pi}{21}
\]
Two basic problems

1. Classify $M$ that self-cover with $\text{deg} > 1$.

2. Which Riem. mnfds $X$ have “minimal quotients”:

$$\exists \mu > 0 : \forall \Gamma : \text{vol}(X/\Gamma) \geq \mu ?$$
Problem: Classify $M$ that self-cover with deg $> 1$. 
**Problem:** Classify \( M \) that self-cover with \( \text{deg} > 1 \).

**Low dimensions:**

\[
\text{dim} = 2 : T^2, K
\]
**Problem:** Classify $M$ that self-cover with $\text{deg} > 1$.

**Low dimensions:**

$\dim = 2 : T^2, K$

$\dim = 3$ [Yu-Wang ’99]: $\Sigma \times S^1$, $T^2 \rightarrow M$

\[ \downarrow \]

\[ S^1 \]
**Problem:** Classify $M$ that self-cover with $\text{deg} > 1$.

**Low dimensions:**

- $\text{dim} = 2 : T^2, K$
  \[ T^2 \to M \]
- $\text{dim} = 3$ [Yu-Wang ’99]: $\Sigma \times S^1$,
  \[ \downarrow \]
  \[ S^1 \]
  Kähler and $\text{dim}_\mathbb{C} = 2, 3$ [Höring-Peternell, ’11]
**Problem:** Classify $M$ that self-cover with $\text{deg} > 1$.

**Low dimensions:**

$$\text{dim} = 2 : T^2, K$$

$$\text{dim} = 3 \text{[Yu-Wang '99]} : \Sigma \times S^1,$$

$$\text{dim} \geq 4 \text{(Examples)}$$

1. Tori $T^n = \mathbb{R}^n / \mathbb{Z}^n$

   $$A \in M_n(\mathbb{Z}), \quad \text{deg} = |\det(A)|$$
**Problem:** Classify $M$ that self-cover with $\deg > 1$.

**Low dimensions:**

- $\dim = 2 : T^2, K$
  
  \[ T^2 \to M \]

- $\dim = 3$ [Yu-Wang ’99]: $\Sigma \times S^1$

  Kähler and $\dim_C = 2, 3$ [Höring-Peternell, ’11]

**$\dim \geq 4$ (Examples)**

1. Tori $T^n = \mathbb{R}^n / \mathbb{Z}^n$

\[ A \in M_n(\mathbb{Z}), \; \deg = |\det(A)| \]

2. Nilmanifolds
Problem: Classify $M$ that self-cover with $\text{deg} > 1$.

Examples: Nilmanifolds
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**Examples:** Nilmanifolds

**Gromov’s Expanding Maps Theorem [’81]:**

\[ f : M \to M \text{ expanding self-cover} \implies M \text{ is nilmnfd!} \]

(up to finite cover)
**Problem:** Classify $M$ that self-cover with $\deg > 1$.

**Examples:** Nilmanifolds

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$$\|Df(v)\| > \|v\|$$
**Problem:** Classify $M$ that self-cover with deg $> 1$.

**Examples:** Nilmanifolds

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**Examples:** Nilmanifolds

**Gromov’s Expanding Maps Theorem [’81]:**

$$f : M \to M \text{ expanding self-cover } \implies M \text{ is nilmnfd!}$$

(up to finite cover)

**Q:** Are these all?
**Problem:** Classify $M$ that self-cover with $\deg > 1$.

**Examples:** Nilmanifolds

**Gromov’s Expanding Maps Theorem [’81]:**

$f : M \to M$ expanding self-cover $\implies M$ is nilmanifold!

(up to finite cover)

**Q:** Are these all?

**A:** No! Remember: $\Sigma \times S^1$, $T^2 \to M$ \hspace{1cm} $\downarrow$

$\hspace{1cm} S^1$
**Problem:** Classify $M$ that self-cover with deg $> 1$.

**Examples:**

1) Nilmanifolds

2) $\Sigma \times S^1$

3) $[\text{nilmnnfd}] \to M$
Problem: Classify $M$ that self-cover with $\text{deg} > 1$.

Examples: $\text{[nilmnfd]} \rightarrow M$ \\
$\downarrow$ \\
$B$

Ambitious Conj:

Any self-cover is of this form (up to finite cover).
**Problem:** Classify $M$ that self-cover with deg $> 1$.

**Examples:** $[\text{nilmnfd}] \to M$

\[ \downarrow \]

$B$

---

**Ambitious Conj:**

Any self-cover is of this form (up to finite cover).

**Agol-Teichner-vL:** False!

First "exotic" examples.

- using: Baumslag–Solitar groups,
- 4-mnfd topology results by Hambleton–Kreck–Teichner
New Problem: Classify $M$ that self-cover with $\deg > 1$. “coming from symmetry” (i.e. regular / Galois / map is quotient by a group action)
**New Problem:** Classify \( M \) that self-cover with \( \text{deg} > 1 \).

"coming from symmetry"

(regular)

\[
M \acts \bowtie G \\
\downarrow \text{regular} \\
M
\]
**New Problem:** Classify $M$ that self-cover with deg $> 1$. “coming from symmetry”

(regular)

\[ M \bowtie G \]

**Problem:** Surprisingly mild condition.

**Idea:** Iterate!
New Problem: Classify $M$ that self-cover with $\text{deg} > 1$.

“coming from symmetry” (regular)

Problem: Surprisingly mild condition.

Idea: Iterate!

Define:

- strongly regular
  \[ \iff \]
  - all iterates are regular
Define:

**strongly regular** $\iff$ all iterates are regular

New **Problem:**

Classify strongly reg. self-covers.
Define:

**strongly regular** $\iff$ all iterates are regular

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Classify strongly reg. self-covers.

**Thm 1 [vL]:** On level of $\pi_1$, strongly reg. covers come from torus endo’s:
Define:

**strongly regular** ⇐⇒ all iterates are regular

**New Problem:**

Classify strongly reg. self-covers.

**Thm 1 [vL]:** On level of $\pi_1$, strongly reg. covers come from torus endo’s:

\[
\begin{align*}
\pi_1(M) &\xrightarrow{\exists q} \mathbb{Z}^k \\
p_* &\downarrow \pi_1(M) \\
\mathbb{Z}^k &\xrightarrow{\exists A} \mathbb{Z}^k
\end{align*}
\]
**Define:**

**strongly regular** \iff all iterates are regular

**New Problem:**

Classify strongly reg. self-covers.

**Thm 1 [vL]:** On level of \( \pi_1 \), strongly reg. covers come from torus endo’s:

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\begin{align*}
\ker(q) & \to \pi_1(M) \\
\cong & \downarrow p_* \\
\ker(q) & \to \pi_1(M) \\
\end{align*}
\]

\[
\exists q \to \mathbb{Z}^k
\]

\[
\exists A
\]
Define:

**strongly regular** ⇔ all iterates are regular

**Thm 1 [vL]:** On level of $\pi_1$, strongly reg. covers come from torus endo's:

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\ker(q) &\to \pi_1(M) & \exists q &\to \mathbb{Z}^k \\
\cong &\downarrow & p_* &\downarrow & \exists A \\
\ker(q) &\to \pi_1(M) & \to &\mathbb{Z}^k
\end{align*}
\]

**Thm 2 [vL]:** $M$ Kähler, $p : M \to M$ hol. strongly reg.

\[
\implies M \cong N \times T \quad \text{(up to finite cover)}.
\]
Thm 1 [vL]: \( \rho : M \to M \) strongly reg.

\[
\begin{align*}
\pi_1(M) & \xrightarrow{\exists q} \mathbb{Z}^k
\end{align*}
\]

Proof idea:
Thm 1 [vL]: $\rho : M \rightarrow M$ strongly reg.

Proof idea:
Step 1: Change perspective.
Thm 1 [vL]: \( \rho : M \to M \) strongly reg.

\[ \pi_1(M) \xrightarrow{\exists q} \mathbb{Z}^k \]

Proof idea:

Step 1: Change perspective.

Notation: \( \Gamma := \pi_1(M) \),

\( \varphi := p_* : \Gamma \hookrightarrow \Gamma \)
Thm 1 [vL]: $\rho : M \to M$ strongly reg.

$$\Rightarrow \quad \pi_1(M) \xrightarrow{\exists q} \mathbb{Z}^k$$

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Step 2: Take limit of groups.

\( M = S^1 \quad \times 2 \)
**Thm 1 [vL]:** \( \rho : M \to M \) strongly reg.

\[
\iff \pi_1(M) \overset{\exists q}{\to} \mathbb{Z}^k
\]

**Proof idea:**

**Step 1:** Change perspective.

**Notation:** \( \Gamma := \pi_1(M), \)

\( \varphi := p_* : \Gamma \rightarrow \Gamma \)

**Step 2:** Take limit of groups.

\[
\sim \sim \sim \sim \Gamma/\varphi^\infty \quad (:= \lim_{\to} \Gamma/\varphi^n(\Gamma))
\]

(i) Acts on \( M, \)

(ii) Self-similar algebr. struct.
Thm 1 [vL]: $\rho : M \to M$ strongly reg.

Proof idea: $\rightsquigarrow \Gamma/\varphi^\infty$

(i) Acts on $M$,
(ii) Self-similar algebraic structure.
**Thm 1 [vL]:** \( \rho : M \to M \) strongly reg.

\[ \exists q \quad \pi_1(M) \twoheadrightarrow \mathbb{Z}^k \]

**Proof idea:** \( \rightsquigarrow \Gamma/\varphi^\infty \)

(i) Acts on \( M \),

(ii) Self-similar algebraic structure.

**Step 3:**

\[ \text{Loc. fin. gps} + \text{Fin. Gp. Actions} \]

\( F \) is Artinian
Thm 1 [vL]: \( \rho : M \to M \) strongly reg.

\[ \pi_1(M) \xrightarrow{\exists q} \mathbb{Z}^k \]

Proof idea:

\( F \) is Artinian \( \iff \) \( F \) is virt. abelian
Thm 2 [vL]: \( M \) Kähler,

\( p : M \to M \) hol. strongly reg.

\[ \implies M \cong N \times T \] (up to finite cover)

Proof idea:
Thm 2 [vL]: \( M \) Kähler,

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Proof idea:

\( F \subseteq \text{Hol}(M) \)
Thm 2 [vL]: $M$ Kähler,

$p : M \to M$ hol. strongly reg.

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Proof idea:

$\bar{F} \subseteq \text{Hol}(M)$ is a torus $T$

(using Kähler geometry)
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**Proof idea:**

\( \overline{F} \subseteq \text{Hol}(M) \) is a torus \( T \)  
(uses Kähler geometry)

**Difficult pt:** \( T \bowtie M \) freely
Thm 2 [vL]: \( M \) Kähler,

\[ p : M \to M \text{ hol. strongly reg.} \]

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**Proof idea:**

\( \overline{F} \subseteq \text{Hol}(M) \) is a torus \( T \)

(using Kähler geometry)

**Difficult pt:** \( T \bowtie M \) freely

Lift to \( \widetilde{T} \bowtie \widetilde{M} \)

\( \overset{\circlearrowleft}{\text{conj by } \tilde{p}} \quad \text{Geom. linear map} \)
**Thm 2 [vL]:** \( M \) Kähler,
\[
p : M \to M \text{ hol. strongly reg.}
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**Geom. linear map**  

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**Difficult pt:**  \( T \curvearrowright M \) freely

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conj by \( \widetilde{p} \)

Geom. linear map

Alg. linear map

\[ \mathbb{Z}^k \]

\[ \exists A \]

\[ \mathbb{Z}^k \]
Thm 2 [vL]: \( M \) Kähler,
p : \( M \to M \) hol. strongly reg.
\[ \implies M \cong N \times T \] (up to finite cover)

Proof idea:

\( \overline{F} \subseteq \text{Hol}(M) \) is a torus \( T \)
(using Kähler geometry)

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Lift to \( \tilde{T} \cong \tilde{M} \)

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Geom. linear map \( \implies \) Alg. linear map

\[ \mathbb{Z}^k \]
\[ \exists A \]
\[ \mathbb{Z}^k \]
Problem: Which Riemannian manifolds $X$ have “minimal quotients”:

$$\exists \mu > 0 : \forall \Gamma : \text{vol}(X/\Gamma) \geq \mu$$

Earlier: $\mathbb{R}^2$ NO, $\mathbb{H}^2$ YES
**Problem:** Which Riemannian manifolds $X$ have “minimal quotients”:

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**Earlier:** $\mathbb{R}^2$ NO, $\mathbb{H}^2$ YES

**Thm (Kazhdan-Margulis, 1968):**

$G$ semisimple (e.g. $\text{SL}(n, \mathbb{R})$)

$\implies G$ and $G/K$ have min’l quot’s
**Problem:** Which Riemannian manifolds $X$ have
$$\exists \mu > 0 : \forall \Gamma : \text{vol}(X/\Gamma) \geq \mu$$

**Thm (Kazhdan-Margulis, 1968):**

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**Conj (Margulis, 1974 ICM):**

$X$: $-1 \leq K \leq 0$, no Eucl factors $\implies$ min’l quot’s

$(\text{vol} \geq \mu(\text{dim } X))$
**Problem:** Which Riemannian manifolds $X$ have

$$\exists \mu > 0 : \forall \Gamma : \text{vol}(X/\Gamma) \geq \mu ?$$

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**Gromov [’78]:** $-1 \leq K < 0$
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Gromov [’78]: $-1 \leq K < 0$

(vol $\geq \mu(\text{dim } X))$

Thm 3 [vL] $X$ contractible with some cmpt quot $M$

$\pi_1(M)$ no normal abelian subgps

Then $\forall$ metric, $\Gamma$: $\text{vol}(X/\Gamma) \geq \mu \left[ \begin{array}{c} \text{dim} \\ \text{Ric} \\ \text{injrad} \\ \text{diam} \end{array} \right]$
**Thm (Kazhdan-Margulis, 1968)**

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$X$: $-1 \leq K \leq 0$, no Eucl factors $\implies$ min’l quot’s

**Gromov ’78:** $-1 \leq K < 0$ $(\text{vol} \geq \mu(\text{dim } X))$

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e.g. $K \leq 0$  
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Gromov ['78]: $-1 \leq K < 0$

$\mu \geq \mu(\dim X)$

Thm 3 [vL] $X$ contractible with some cmpt quot $M$

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Topology!

Then $\forall$ metric, $\Gamma$: $\text{vol}(X/\Gamma) \geq \mu \left[ \begin{array}{c} \dim \\ \text{Ric} \\ \text{injrad} \\ \text{diam} \end{array} \right]$ e.g. $K \leq 0$ w/o Eucl. factors

Remark:

Thm 3 [vL] $\iff$ Thm (Kazhdan-Margulis, 1968)
Thm 3 [vL] \( X \) contractible with some cmpt quot \( M \)
\( \pi_1(M) \) no normal abelian subgps

Then \( \forall \) metric, \( \Gamma \):
\[
\text{vol}(X/\Gamma) \geq \mu \left[ \begin{array}{c}
\dim \\
\text{Ric} \\
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\]

Proof idea:
**Thm 3 [vL]**  $X$ contractible with some cmpt quot $M$

$\pi_1(M)$ no normal abelian subgps

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**Proof idea:** Suppose $\text{vol}(X/\Gamma_n) \to 0$.

Geom. bounds $\implies g_n \to g$ (Cheeger–Anderson)
Thm 3 [vL]  \( X \) contractible with some cmpt quot \( M \)  
\( \pi_1(M) \) no normal abelian subgps  

Then \( \forall \) metric, \( \Gamma \):  
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\end{array} \right]
\]

Proof idea: Suppose \( \text{vol}(X/\Gamma_n) \to 0 \).  
Geom. bounds \( \implies g_n \to g \) (Cheeger–Anderson)  

Hard:  
\( \Gamma_n \to G \)  
(discrete) (cnts)
\( \mathbb{Z}^2 \)
\[ \frac{1}{n} \mathbb{Z}^2 \rightarrow \mathbb{R}^2 \]

(discrete) \quad (cnts)
Thm 3 [vL]  \( X \) contractible with some cmpt quot \( M \)
\[ \pi_1(M) \text{ no normal abelian subgps} \]

Then \( \forall \) metric, \( \Gamma \): \( \text{vol}(X/\Gamma) \geq \mu \left[ \begin{array}{c} \text{dim} \\ \text{Ric} \\ \text{injrad} \\ \text{diam} \end{array} \right] \)

**Proof idea:**  Suppose \( \text{vol}(X/\Gamma_n) \to 0 \).
Geom. bounds \( \implies g_n \to g \) (Cheeger–Anderson)

**Hard:** \( \Gamma_n \to G \)
\[ \begin{array}{c} \text{(discrete)} \\ \text{(cnts)} \end{array} \]

**Show:** \( G \) is semisimple Lie.  \( \square \)
Common framework:

\[ M \circlearrowleft \text{Isom}(M) \quad \text{“symmetry”} \]
Common framework:

\[ M \overset{\text{symmetry}}{\longrightarrow} \text{Isom}(M) \]

\[ \text{Isom}(M_1) \overset{\text{symmetry}}{\longrightarrow} M_1 \overset{\text{Isom}}{\longrightarrow} M_2 \overset{\text{symmetry}}{\longrightarrow} \text{Isom}(M_2) \]

\[ M \overset{\text{symmetry}}{\longrightarrow} \text{Isom}(M) \]
Common framework:

\[ \tilde{M} \circlearrowleft \text{Isom}(\tilde{M}) \]

\[ \text{Isom}(M_1) \circlearrowleft M_1 \longrightarrow M_2 \circlearrowright \text{Isom}(M_2) \]

\[ M \circlearrowleft \text{Isom}(M) \]

“Hidden Symmetries”

“symmetry”

\[ \leadsto \text{Lie group with natural lattice} \]

acting on a manifold

\[ \pi_1(M) \subseteq \text{Isom}(\tilde{M}) \circlearrowright \tilde{M} \]