Symmetry and self-similarity in geometry

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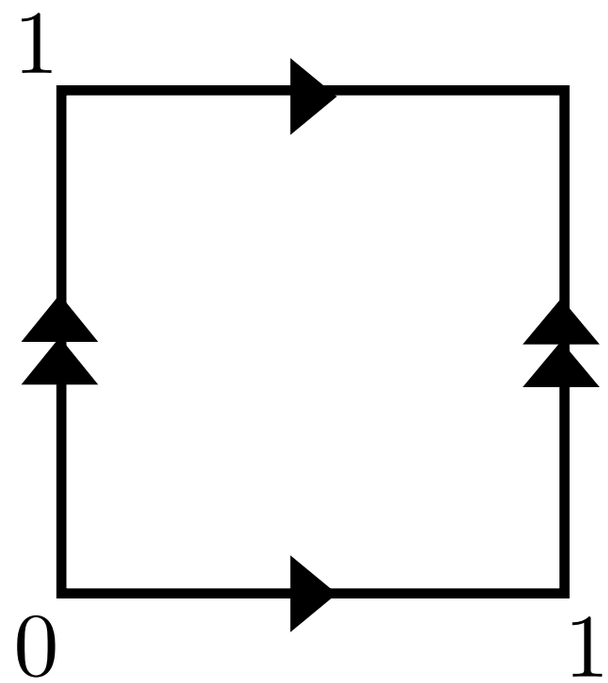
An example

$T^2 =$



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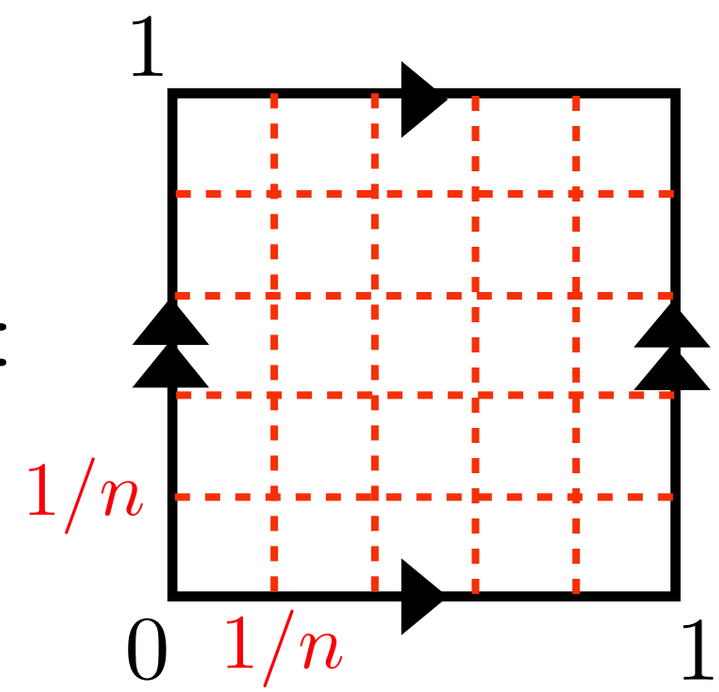

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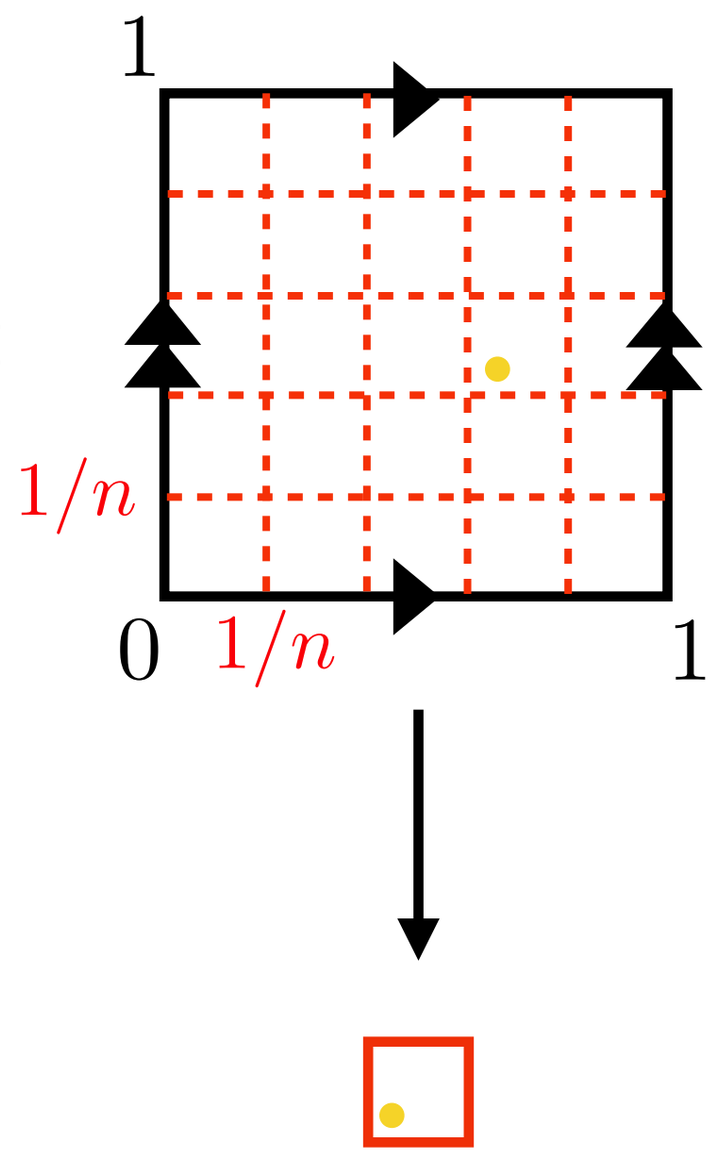

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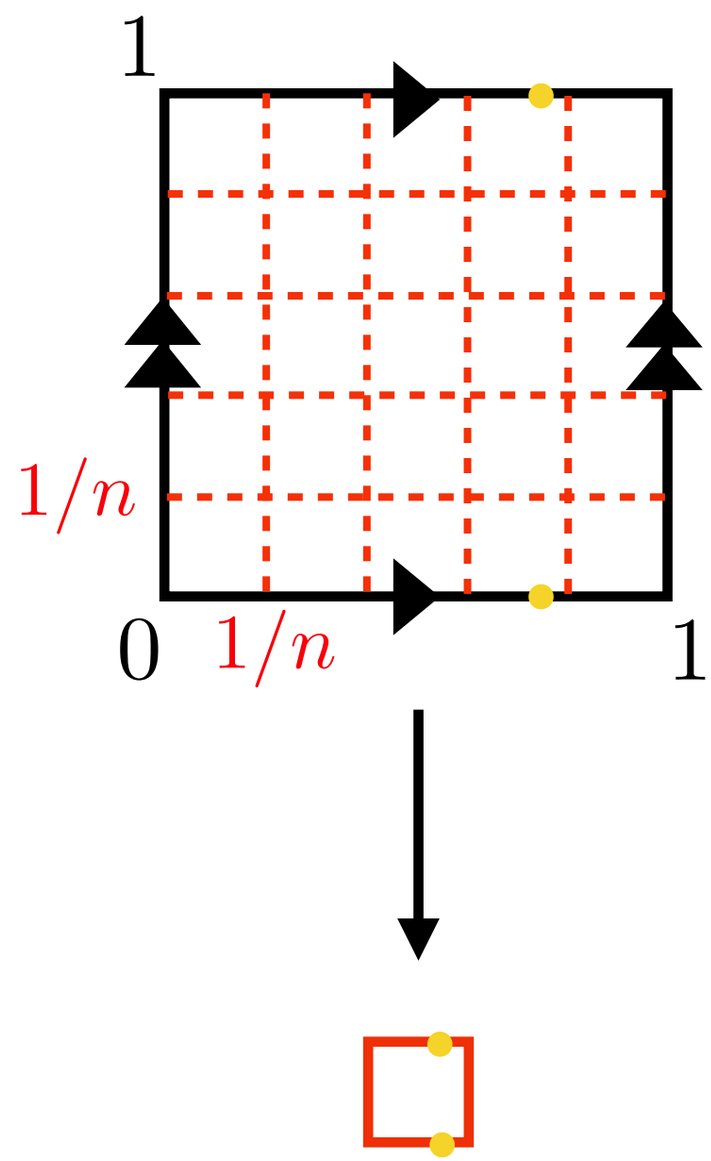

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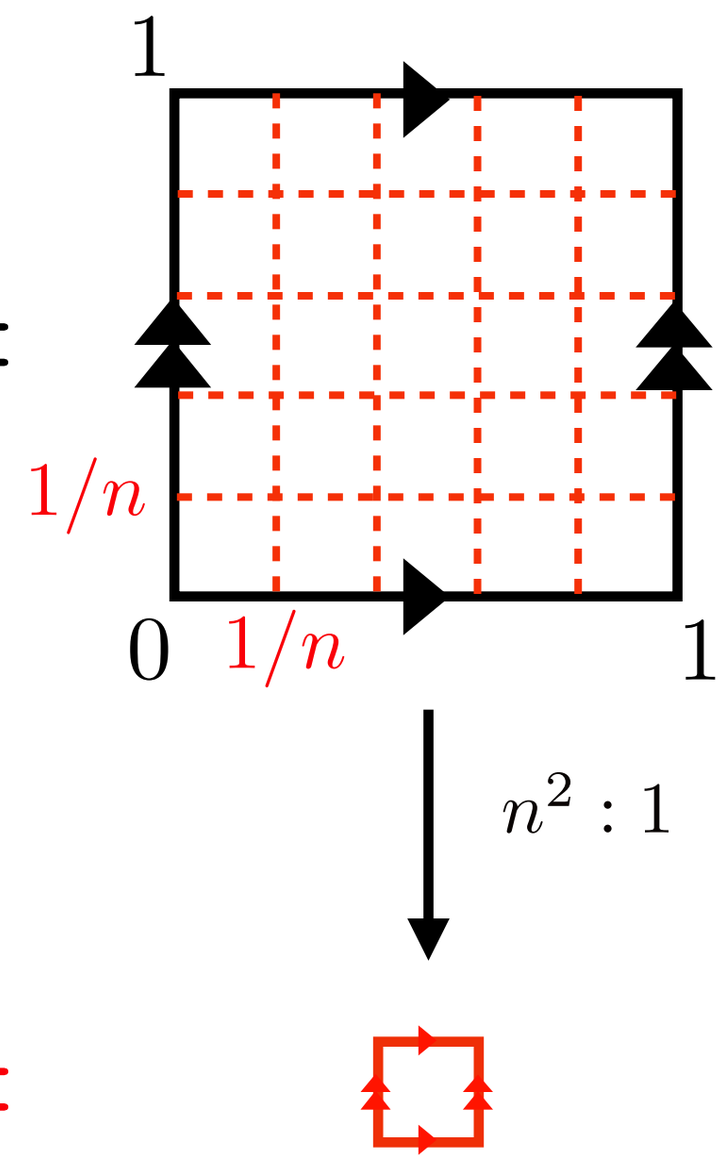

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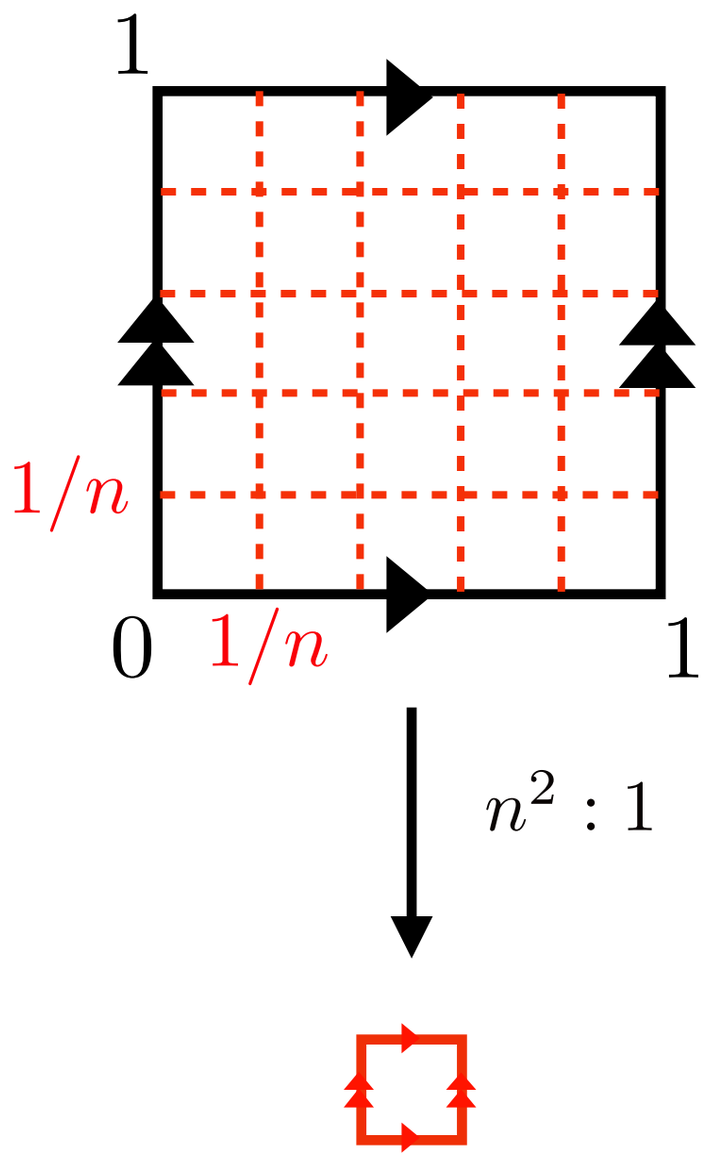

Remarks:

1. $\exists$ covers $\begin{array}{c} T^2 \\ \downarrow \\ T^2 \end{array}$ with degree > 1

2. Symmetry $\rightsquigarrow$ covers

Genus ≥ 2

1. $\forall g \geq 2 : \nexists$ covers $\Sigma_g \to \Sigma_g$

   Reason: $\chi = 2 - 2g \neq 0$

2. <u>Hurwitz's $84(g-1)$ Theorem (1893):</u>

Σ_g Riemann surface, $g \geq 2 \implies |\mathrm{Aut}^+(\Sigma_g)| \leq 84(g-1)$

Genus ≥ 2

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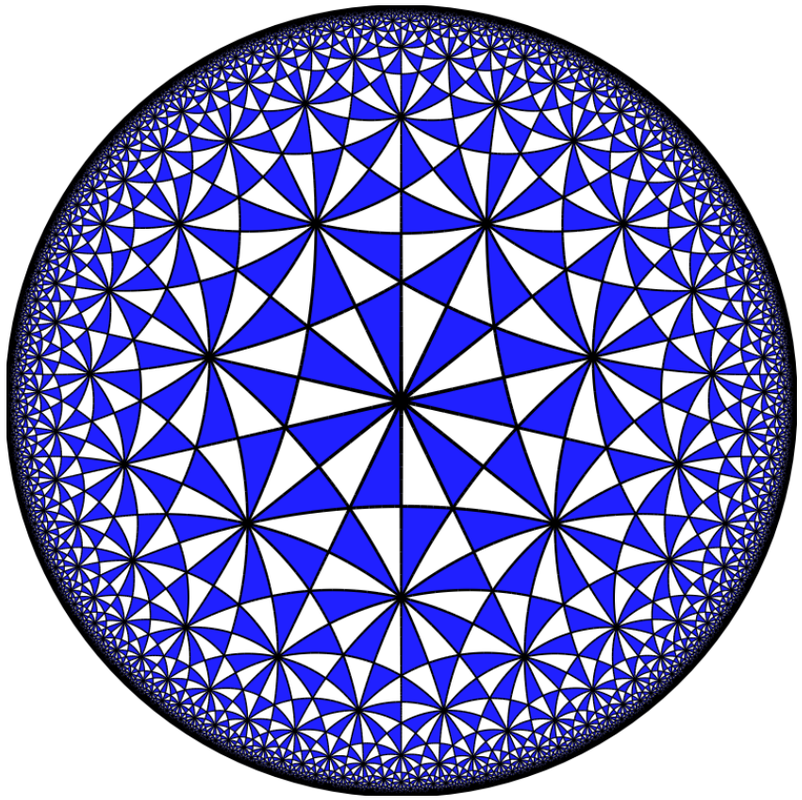
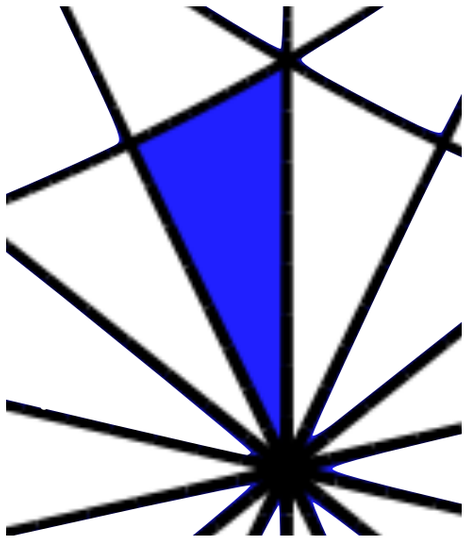


Hurwitz's $84(g-1)$ Theorem (1893):

$$|\mathrm{Aut}^+(\Sigma_g)| \leq 84(g-1)$$

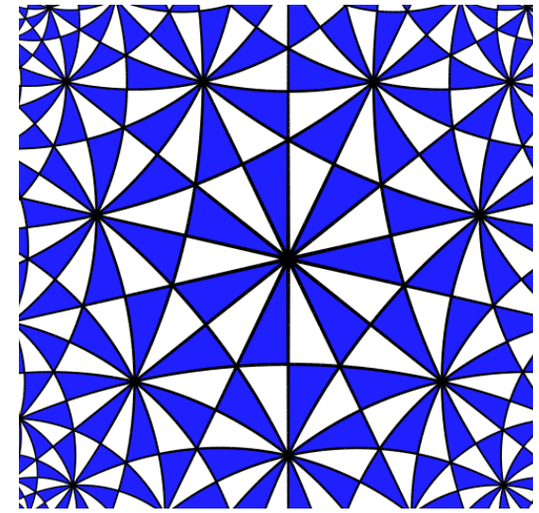
Key fact $\left[\begin{array}{c}\text{area estimate for}\\ \text{“orbifold” quotients}\end{array}\right]$:

$$\forall X = \mathbb{H}^2/\Gamma : \ \ \mathrm{Area}(X) \geq \tfrac{\pi}{21}$$



(Image: Claudio Rocchini)

Genus ≥ 2



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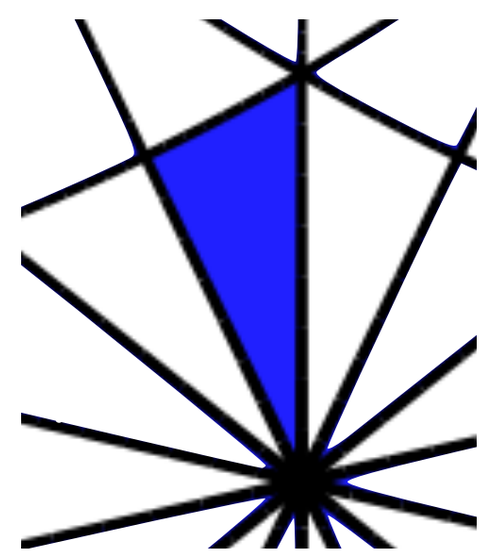
Connection:

$$\mathrm{Area}(\Sigma_g/\mathrm{Aut}^+(\Sigma_g)) = \frac{\mathrm{Area}(\Sigma_g)}{|\mathrm{Aut}^+(\Sigma_g)|}$$
$$\geq \pi/21$$



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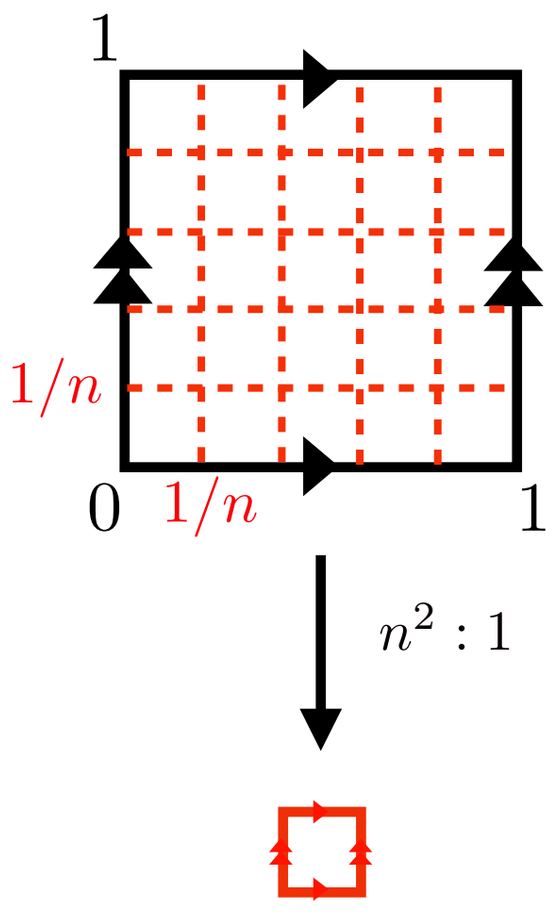
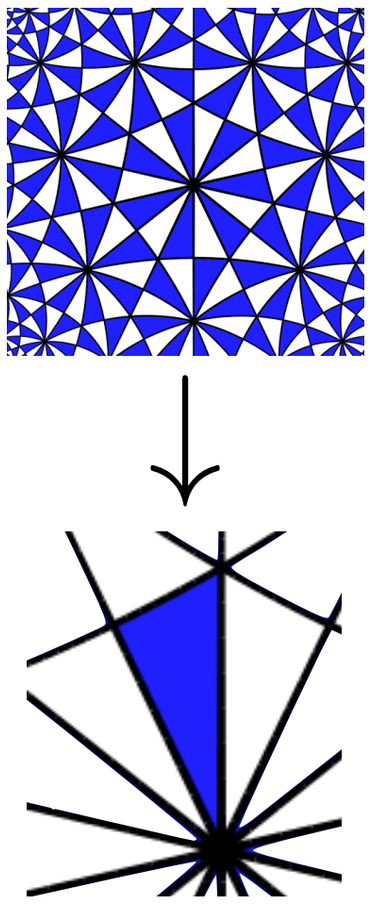
Two basic problems





1. Classify M that self-cover with $\deg > 1$.

2. Which Riem. mnfds X have "minimal quotients":

$$\exists \mu > 0 : \forall \Gamma : \mathrm{vol}(X/\Gamma) \geq \mu \ ?$$

Problem: Classify M that self-cover with $\deg > 1$.

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Low dimensions:

$$\dim = 2 : T^2, K$$

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$\dim \geq 4$ (Examples)

1. Tori $T^n = \mathbb{R}^n / \mathbb{Z}^n$

$\circlearrowright$

$A \in M_n(\mathbb{Z}), \quad \deg = |\det(A)|$

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Gromov's Expanding Maps Theorem ['81]:

$f : M \to M$ expanding self-cover $\implies$ M is nilmnfd!

(up to finite cover)

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$$\|Df(v)\| > \|v\|$$

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Q: Are these all?

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Gromov's Expanding Maps Theorem ['81]:

$f : M \to M$ expanding self-cover $\implies$ M is nilmnfd!

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Q: Are these all?

A: No! Remember: $\Sigma \times S^1$, $\begin{array}{c} T^2 \to M \\ \downarrow \\ S^1 \end{array}$

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Examples: 1) Nilmanifolds

2) $\Sigma \times S^1$, $\begin{array}{c} T^2 \to M \\ \downarrow \\ S^1 \end{array}$

3) $\begin{array}{c} [\text{nilmnfd}] \to M \\ \downarrow \\ B \end{array}$

Problem: Classify M that self-cover with $\deg > 1$.

Examples:

$$\begin{array}{ccc} [\text{nilmnfd}] \to & M \\ & \downarrow \\ & B \end{array}$$

Ambitious Conj:

Any self-cover is of this form (up to finite cover).

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Ambitious Conj:

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Agol-Teichner-vL: False!

First "exotic" examples.

[using: Baumslag–Solitar groups,
4–mnfd topology results by Hambleton–Kreck–Teichner]

New

Problem: Classify M that self-cover with $\deg > 1$.

"coming from symmetry"

(i.e. regular / Galois / map is quotient by a group action)

$$\begin{array}{c} M \circlearrowleft G \\ \Big\downarrow /G \\ M \end{array}$$

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Problem: Surprisingly mild condition.

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$$\begin{tikzcd} M \circlearrowleft G \arrow[d, "/G"] \arrow[dd, bend right=60] \\ M \circlearrowleft G \arrow[d, "/G"] \\ M \end{tikzcd}$$

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Idea: Iterate!

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$$\begin{tikzcd}
M \arrow[d] \arrow[dd, bend right] \arrow[ddd, bend left] \arrow[dddd, bend right=60] \arrow[ddddd, bend left=60] \\
M \arrow[d] \\
M \arrow[d] \\
M \arrow[d] \\
M \arrow[d] \\
M \arrow[d, no head] \\
{}
\end{tikzcd}$$

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$\iff$

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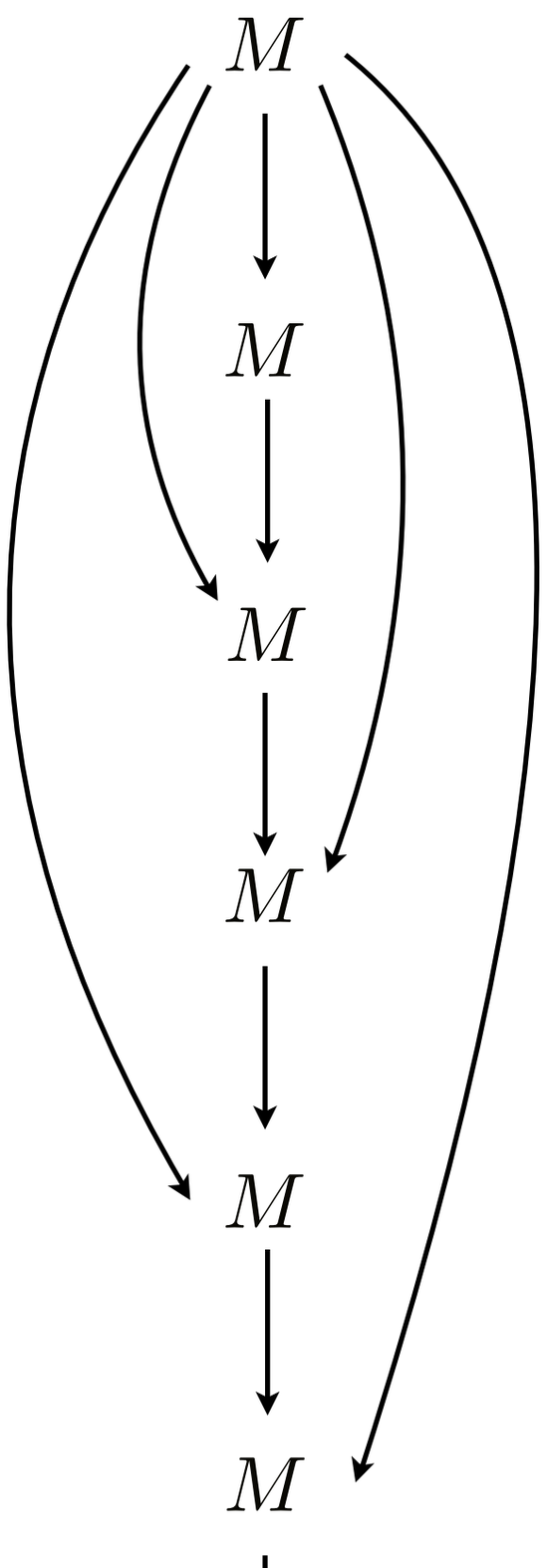

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Classify strongly reg. self-covers.

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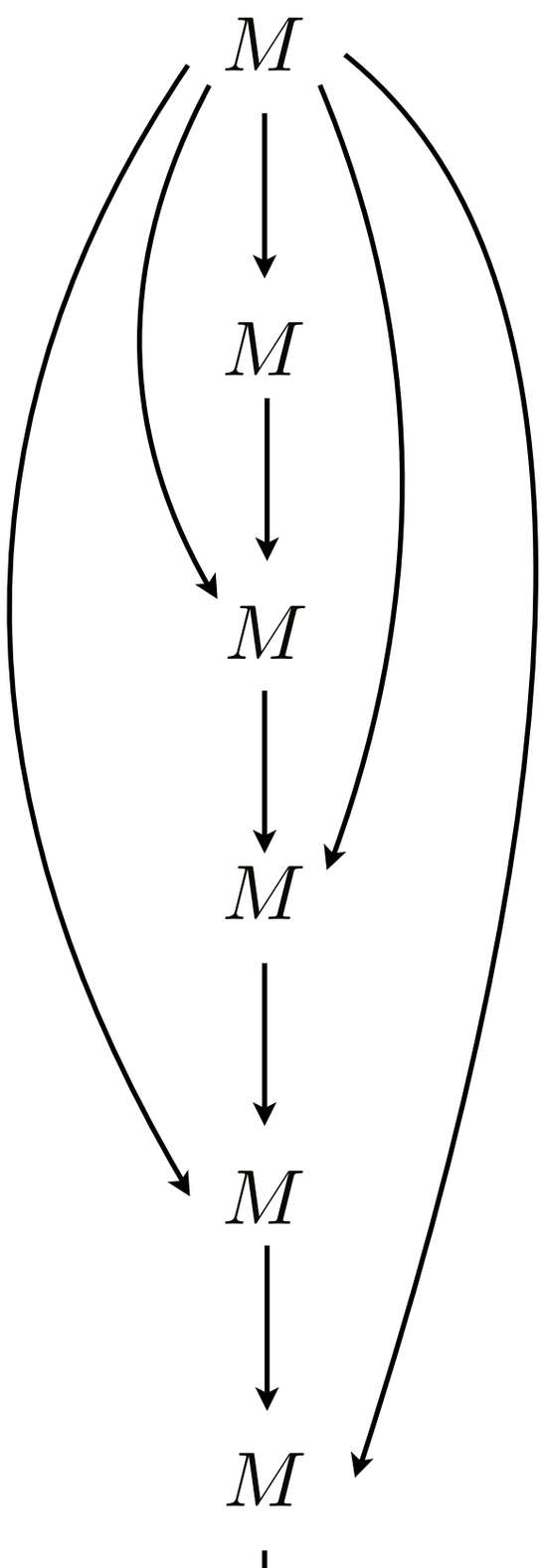

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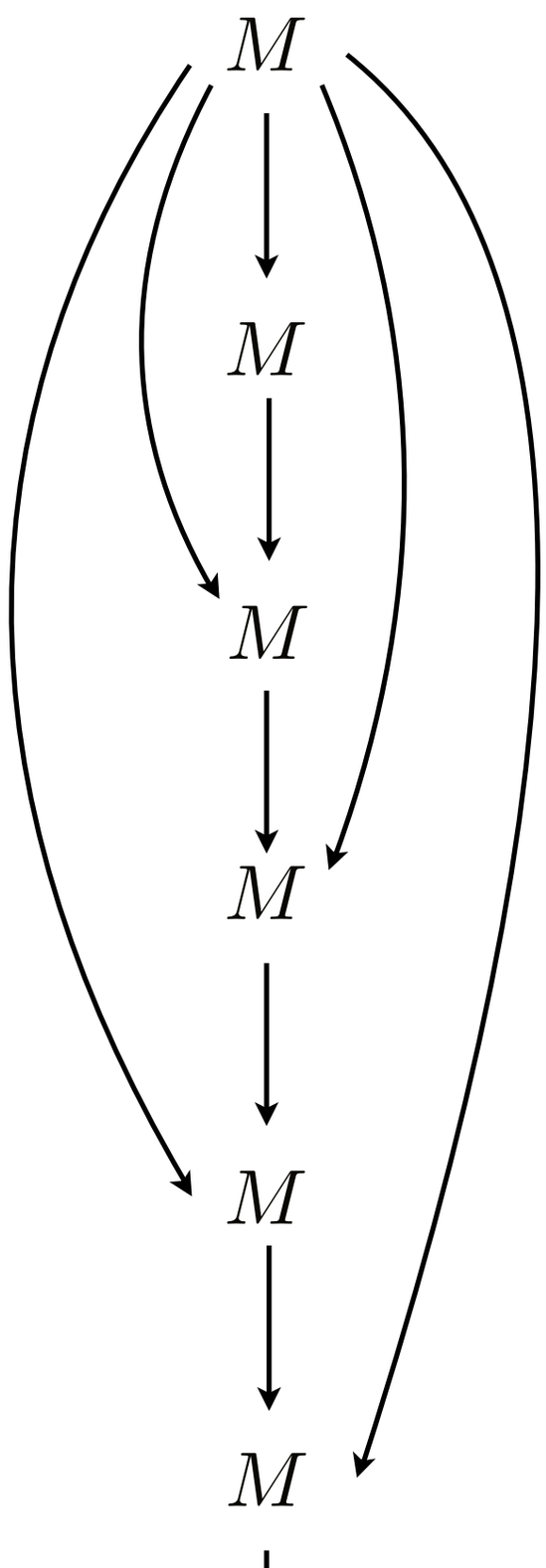

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$$\begin{tikzcd} \pi_1(M) \arrow[r, two heads, "\exists q"] \arrow[d, "p_*"] & \mathbb{Z}^k \arrow[d, "\exists A"] \\ \pi_1(M) \arrow[r, two heads] & \mathbb{Z}^k \end{tikzcd}$$

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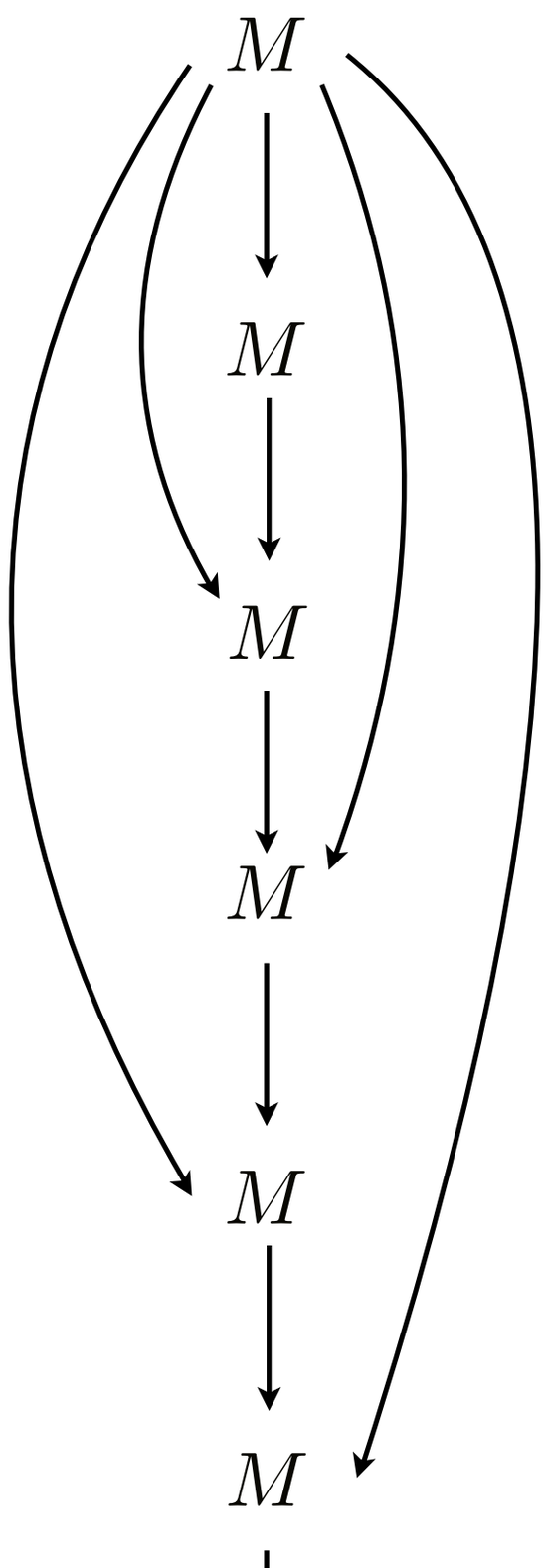

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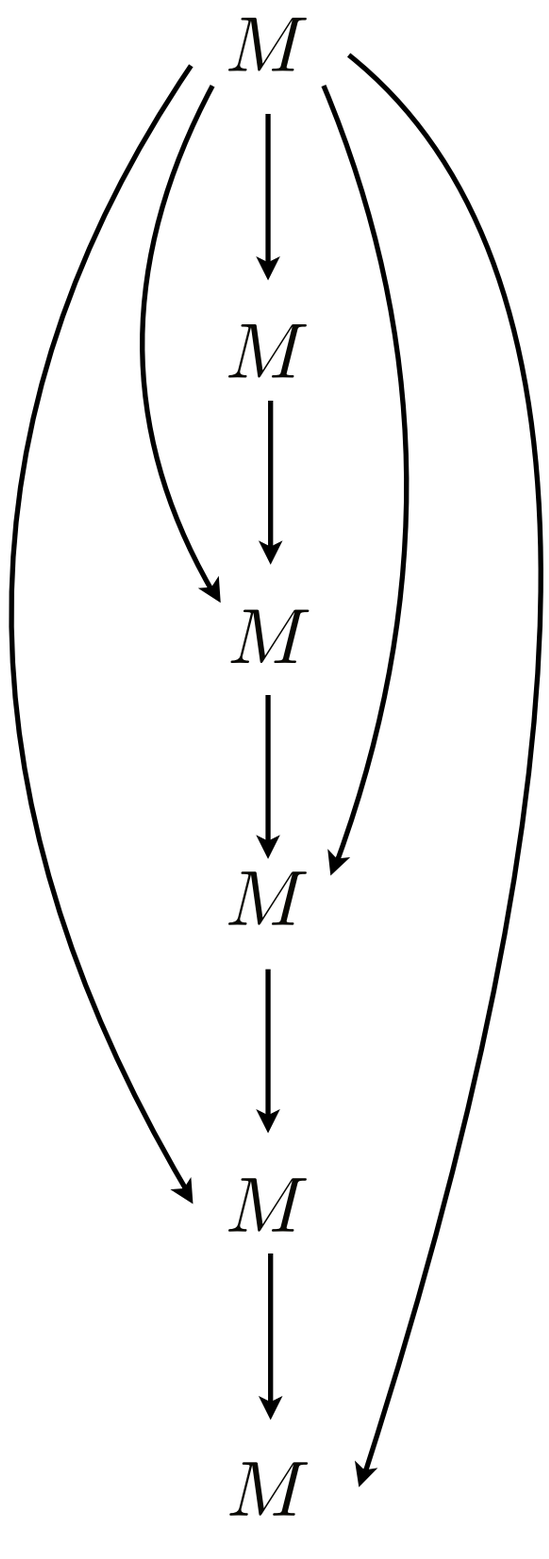

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Thm 2 [vL]: M Kähler,

$p : M \to M$ hol. strongly reg.

$\implies M \cong N \times T$ (up to finite cover).

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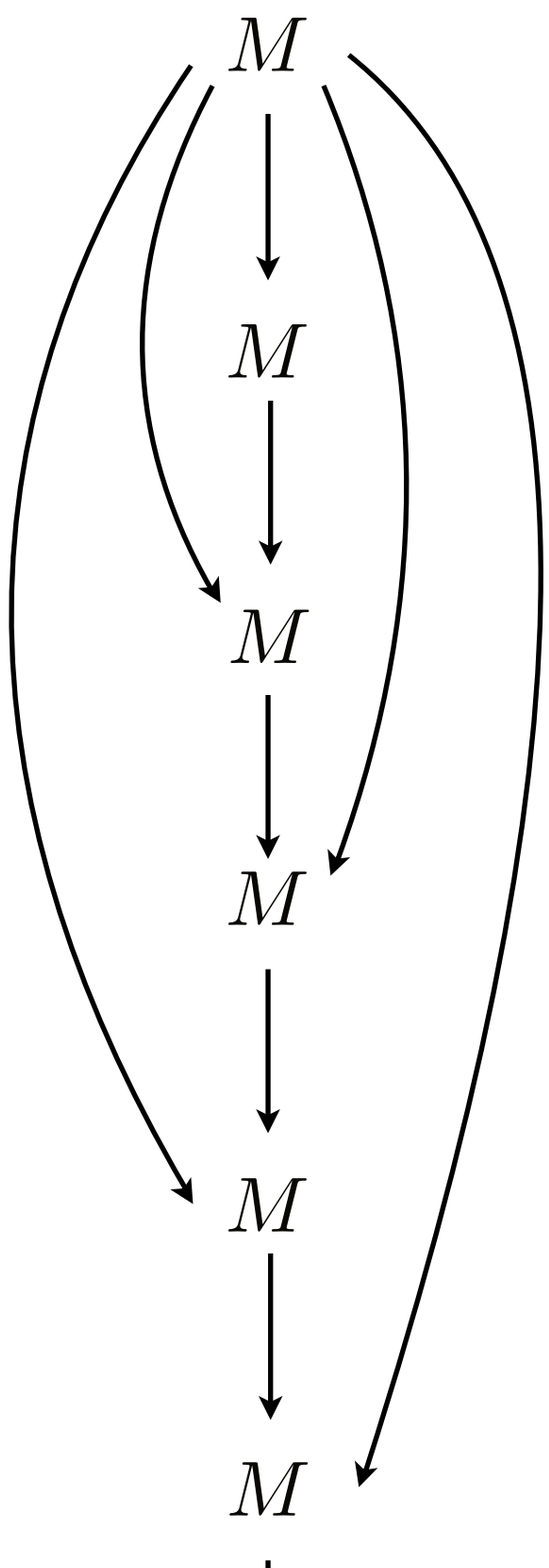
$$\Longrightarrow \quad \pi_1(M) \xrightarrow{\exists q} \mathbb{Z}^k$$

Proof idea:



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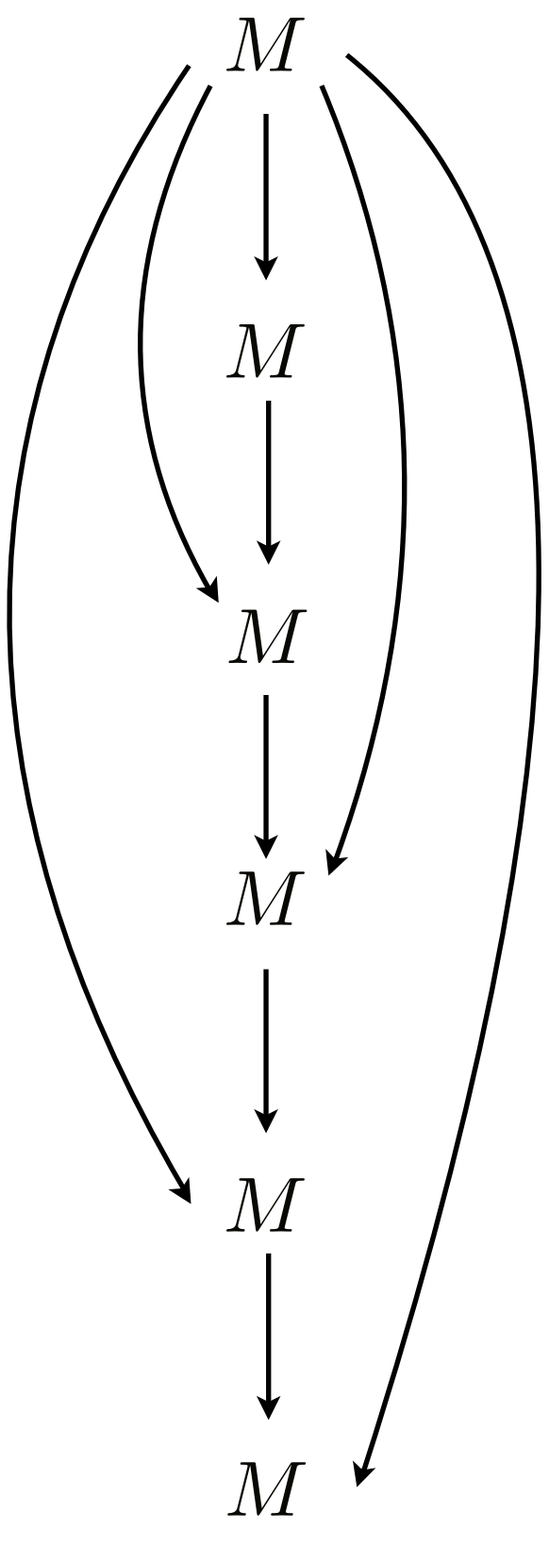

Proof idea:

Step 1: Change perspective.

$$\begin{tikzcd}
 & \\
M \arrow[d] & \Gamma/\varphi^5(\Gamma) \arrow[d] \\
M \arrow[d] & \Gamma/\varphi^4(\Gamma) \arrow[d] \\
M \arrow[d] & \Gamma/\varphi^3(\Gamma) \arrow[d] \\
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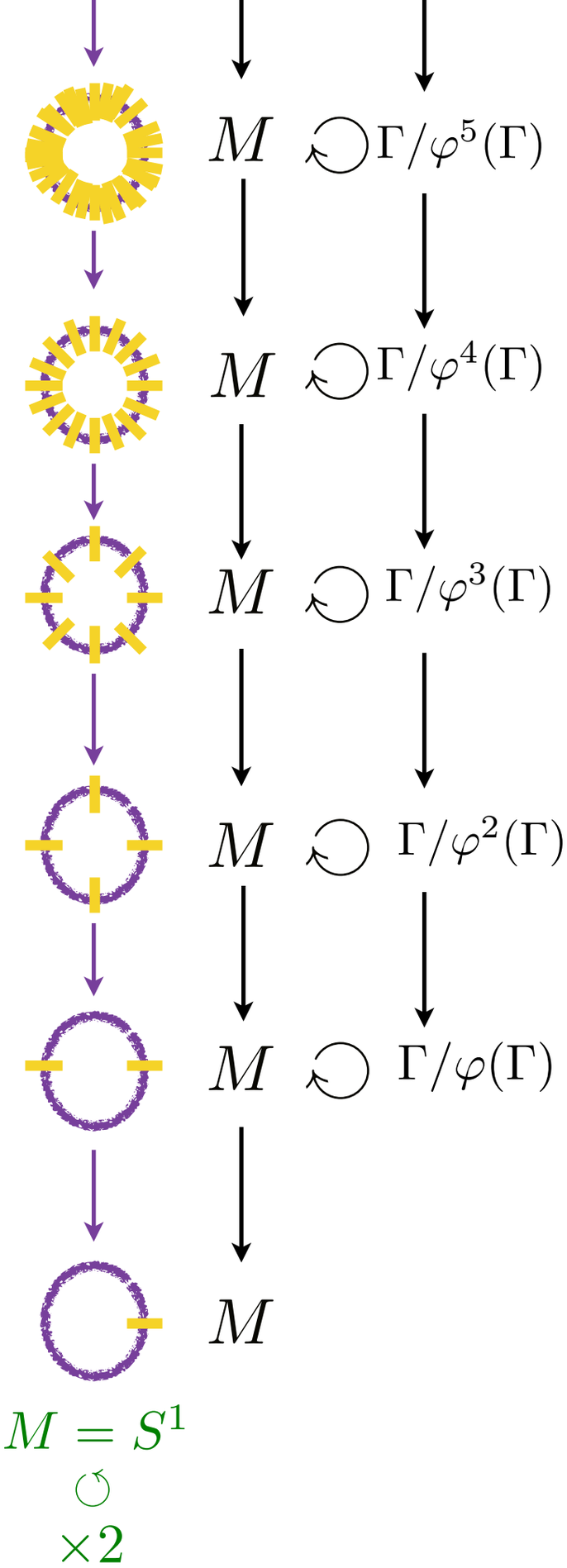

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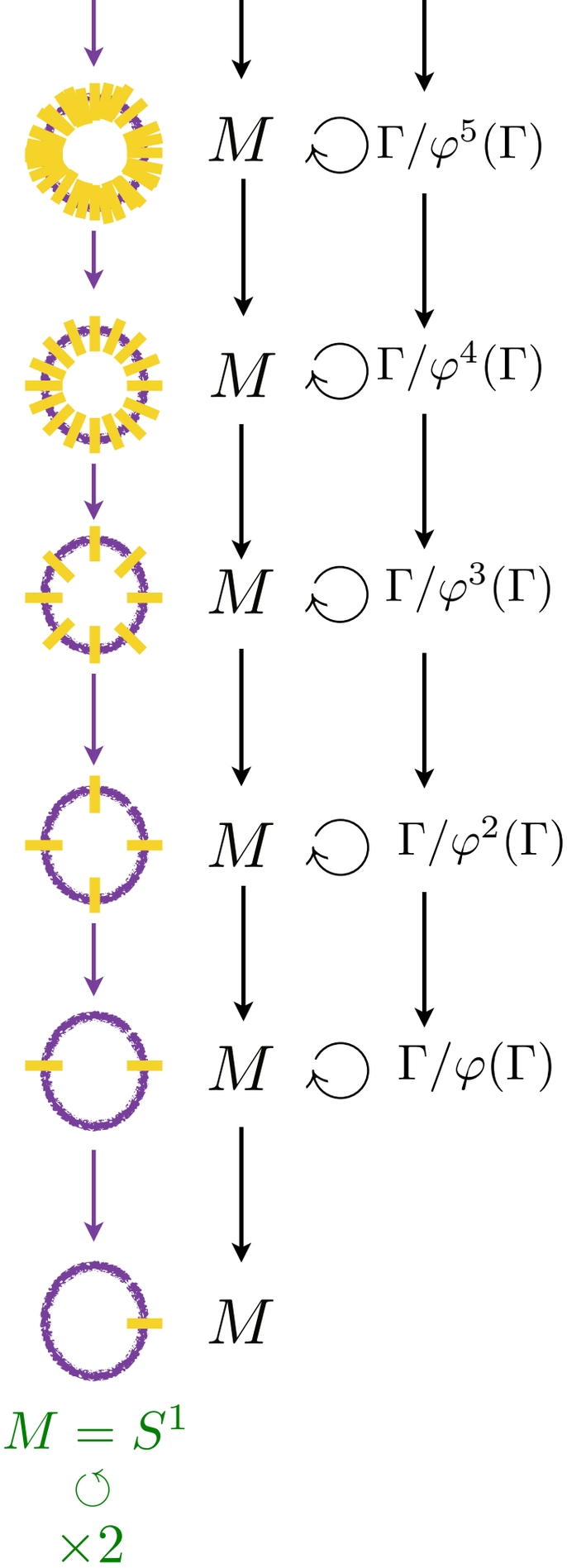

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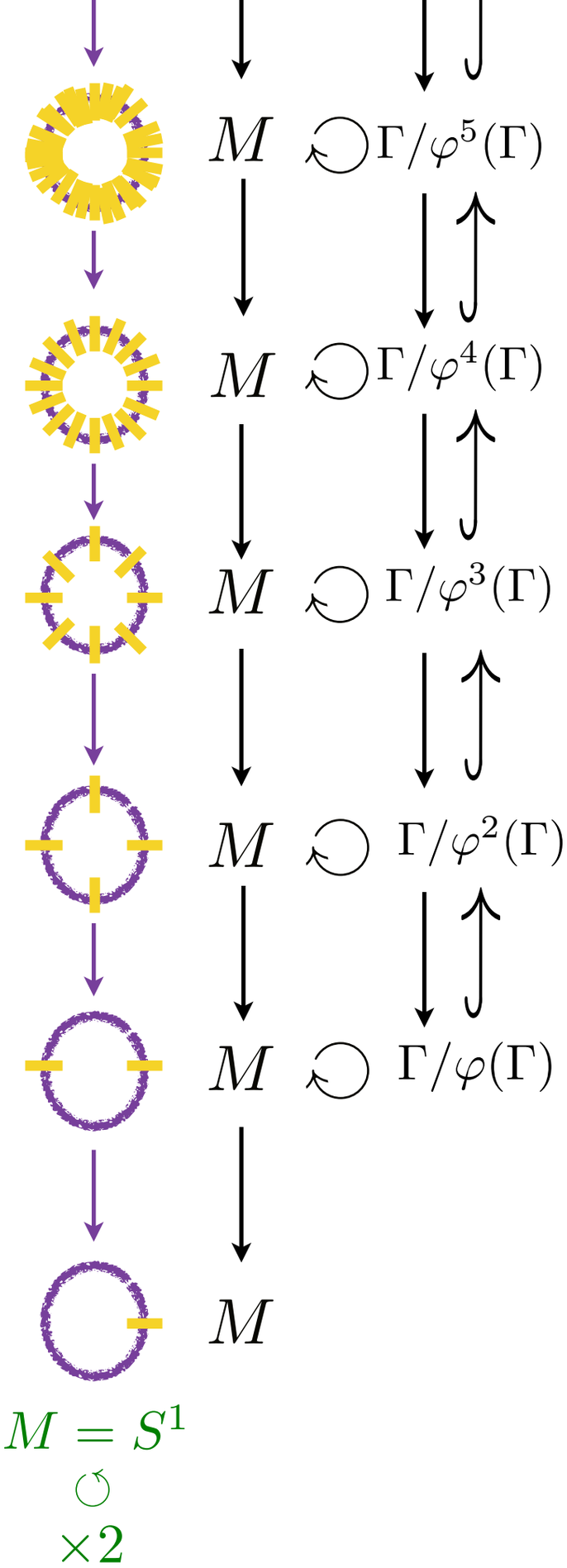

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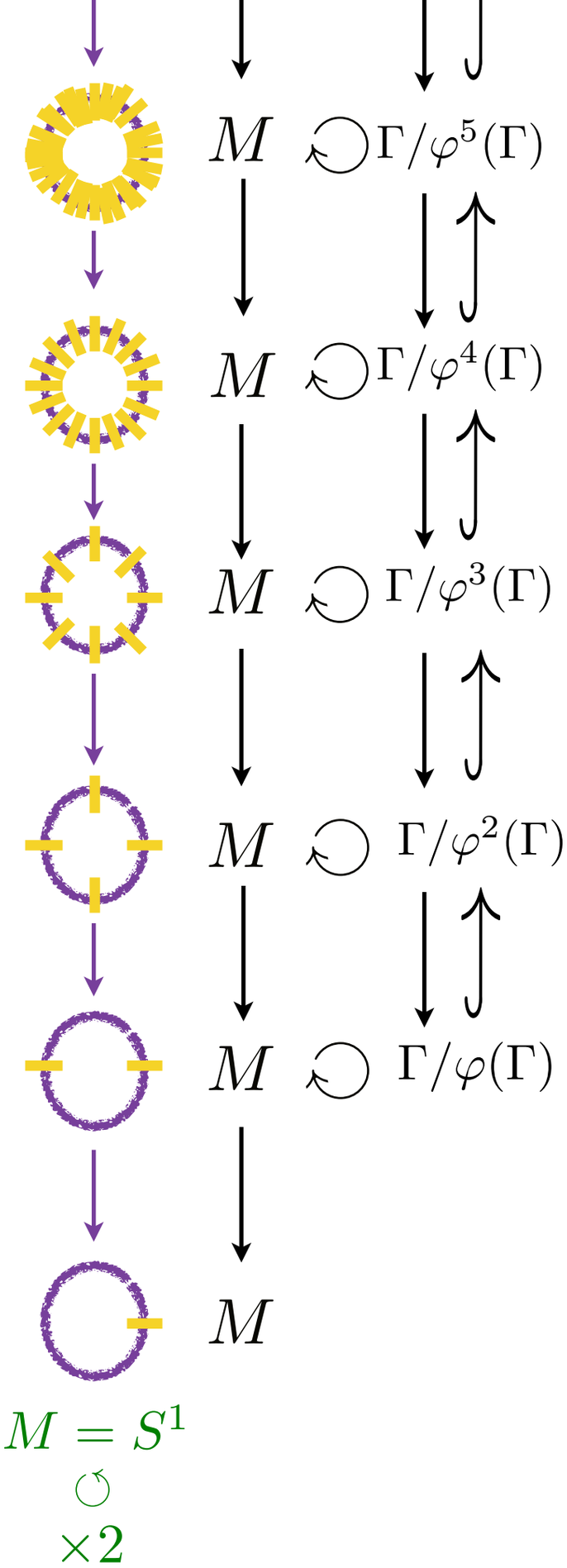
$$\rightsquigarrow \text{“}\Gamma/\varphi^{\overbrace{\infty}^{\text{(direct!)}}}\text{”} \qquad \left(:= \varinjlim \Gamma/\varphi^n(\Gamma)\right)$$

(i) Acts on M,

(ii) Self-similar algebr. struct.



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Proof idea: $\rightsquigarrow$ "Γ/φ^∞"

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$$
\begin{tikzcd}
 & \\
M \arrow[d] & \Gamma/\varphi^5(\Gamma) \arrow[d, shift right] \\
M \arrow[d] & \Gamma/\varphi^4(\Gamma) \arrow[d, shift right] \arrow[u, hook, shift right] \\
M \arrow[d] & \Gamma/\varphi^3(\Gamma) \arrow[d, shift right] \arrow[u, hook, shift right] \\
M \arrow[d] & \Gamma/\varphi^2(\Gamma) \arrow[d, shift right] \arrow[u, hook, shift right] \\
M & \Gamma/\varphi(\Gamma) \arrow[u, hook, shift right] \\
M &
\end{tikzcd}
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Step 3: $+$

Loc. fin. gps $+$ Fin. Gp. Actions

$$\Longrightarrow$$

F is Artinian

$$
\begin{tikzcd}
M \arrow[d] \arrow[r, phantom, "\circlearrowleft"] & \Gamma/\varphi^5(\Gamma) \arrow[d, shift right] \\
M \arrow[d] \arrow[r, phantom, "\circlearrowleft"] & \Gamma/\varphi^4(\Gamma) \arrow[d, shift right] \arrow[u, hook, shift right] \\
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Proof idea:

$$F \text{ is Artinian} \overset{\left[\substack{\text{Šunkov} \\ \text{Kegel-Wehrfritz}}\right]}{\Longrightarrow} F \text{ is virt. abelian}$$

$\square$

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$\overline{F} \subseteq \mathrm{Hol}(M)$ is a torus T

(using Kähler geometry)

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Difficult pt: $T \curvearrowright M$ freely

Lift to $\widetilde{T} \curvearrowright \widetilde{M}$

$\circlearrowright$

conj by $\widetilde{p}$

Geom. linear map

Thm 2 [vL]: M Kähler,

$p : M \to M$ hol. strongly reg.

$\Longrightarrow \; M \cong N \times T$ (up to finite cover)

Proof idea:

$\overline{F} \subseteq \mathrm{Hol}(M)$ is a torus T

(using Kähler geometry)

Difficult pt: $T \curvearrowright M$ freely

Lift to $\widetilde{T} \curvearrowright \widetilde{M}$

$\circlearrowright$

conj by $\widetilde{p}$

Geom. linear map

Thm 1 $\rightsquigarrow$

$$\begin{tikzcd}
\pi_1(M) \arrow[r, two heads, "\exists q"] \arrow[d, "p_*"] & \mathbb{Z}^k \arrow[d, "\exists A"] \\
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KEY!

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Problem: Which Riemannian manifolds X have "minimal quotients":

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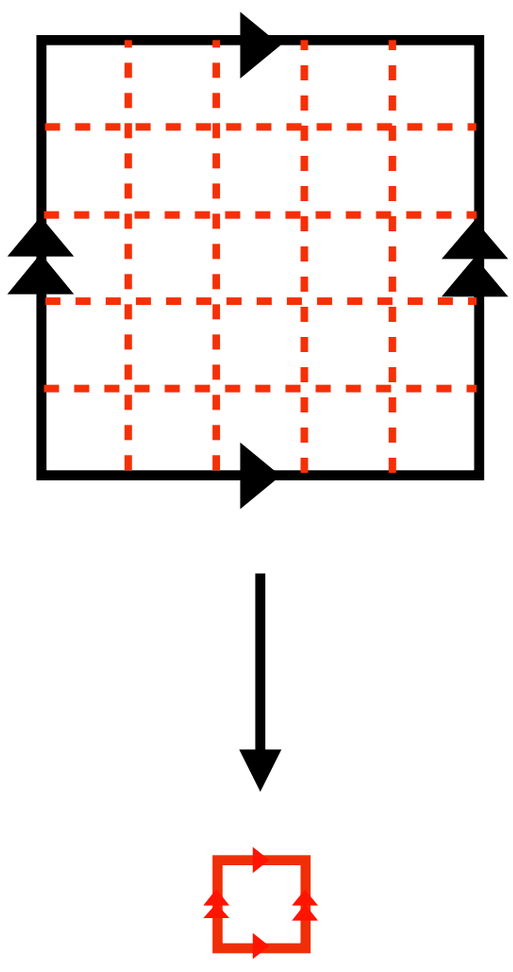
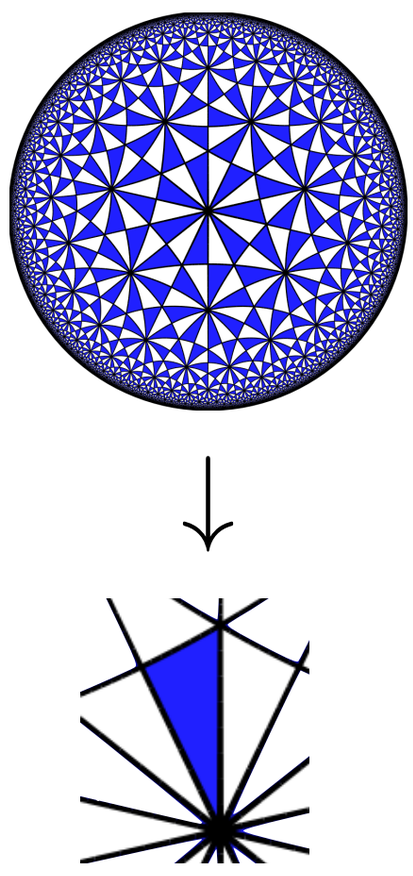
Earlier: $\mathbb{R}^2$ NO , $\mathbb{H}^2$ YES





(Image: Claudio Rocchini)

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Remark:

Thm 3 [vL] $\Longrightarrow$ **Thm (Kazhdan-Margulis, 1968)**

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Geom. bounds $\implies$ $g_n \to g$ (Cheeger–Anderson)

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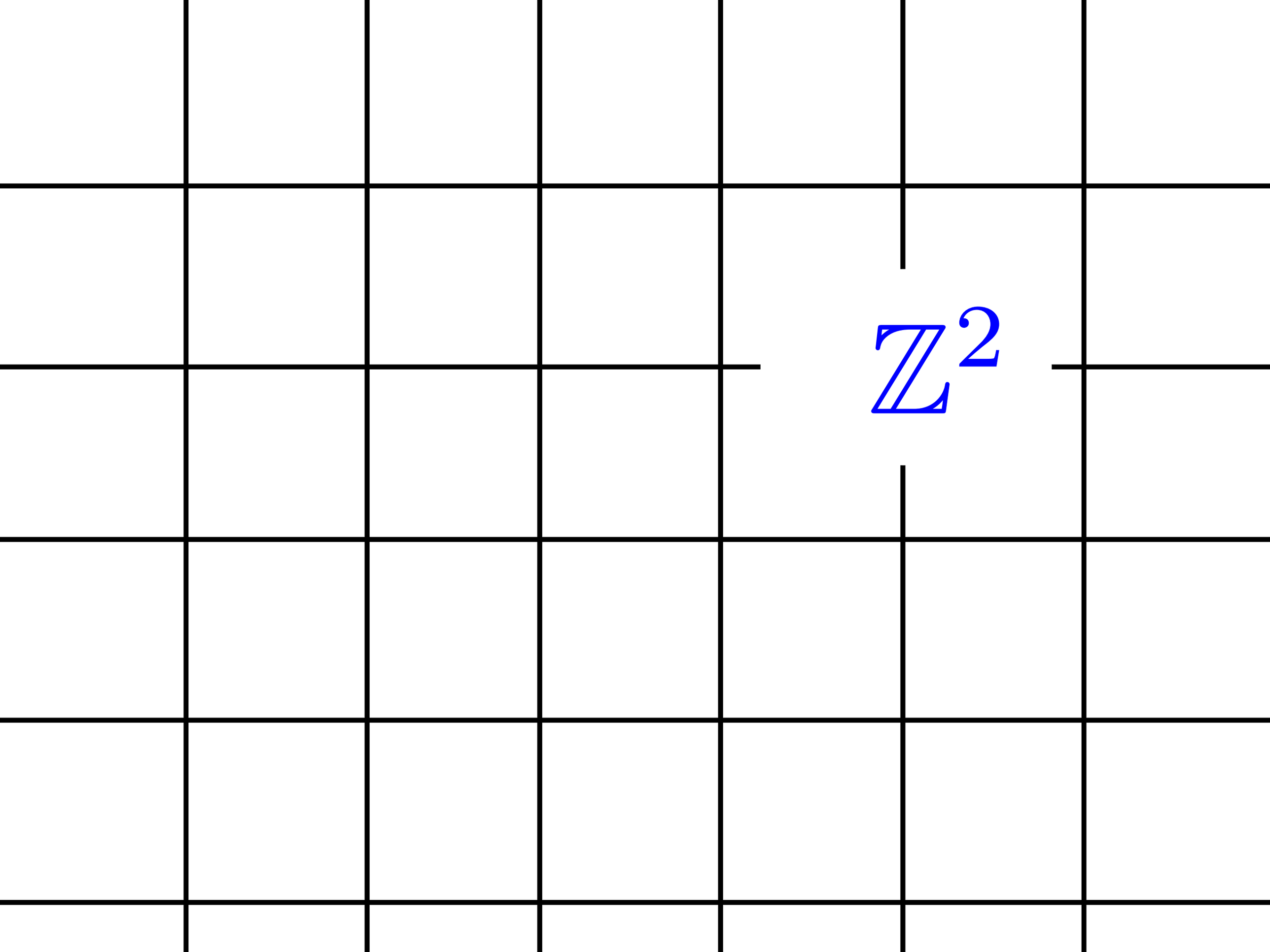
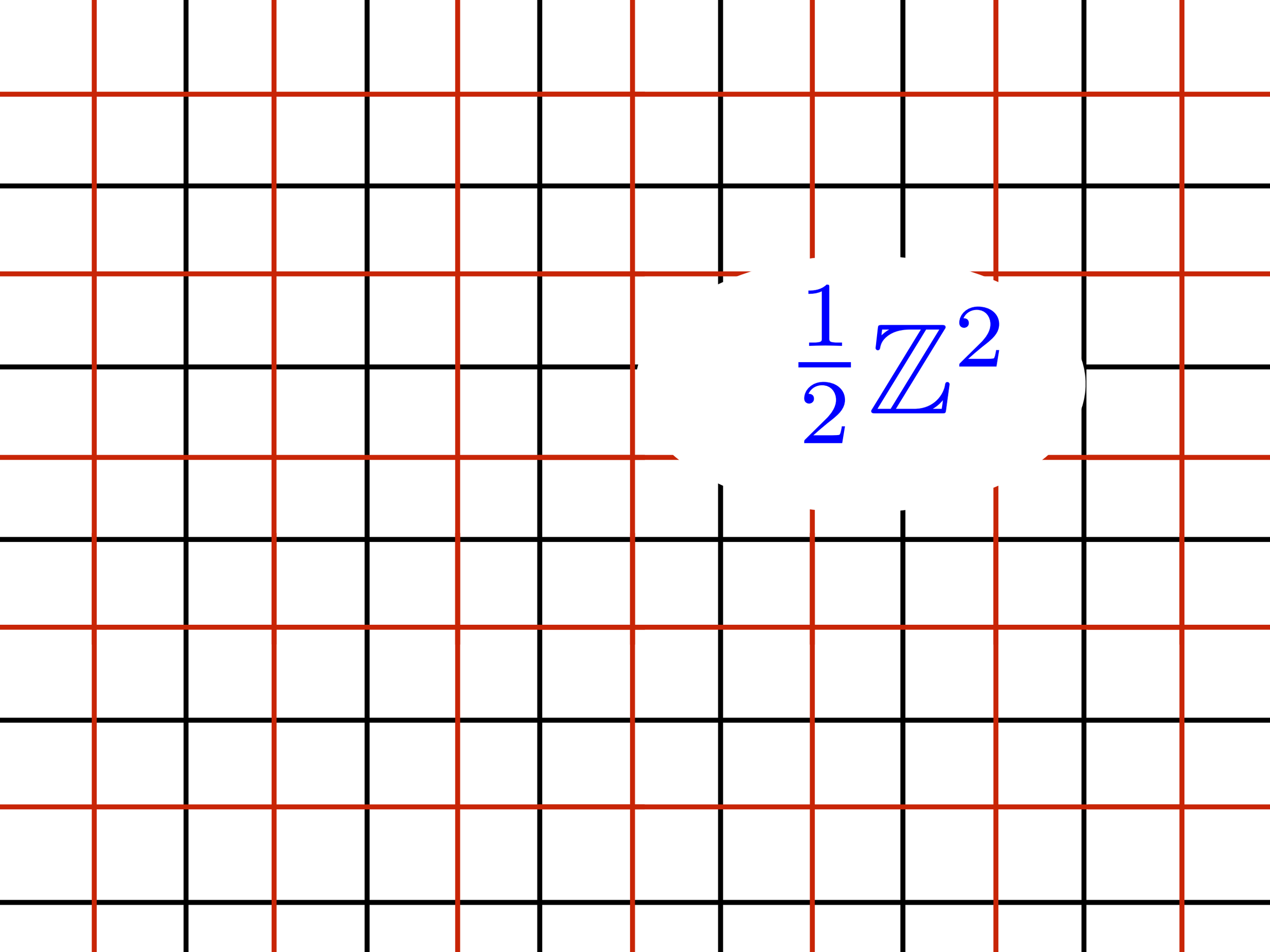
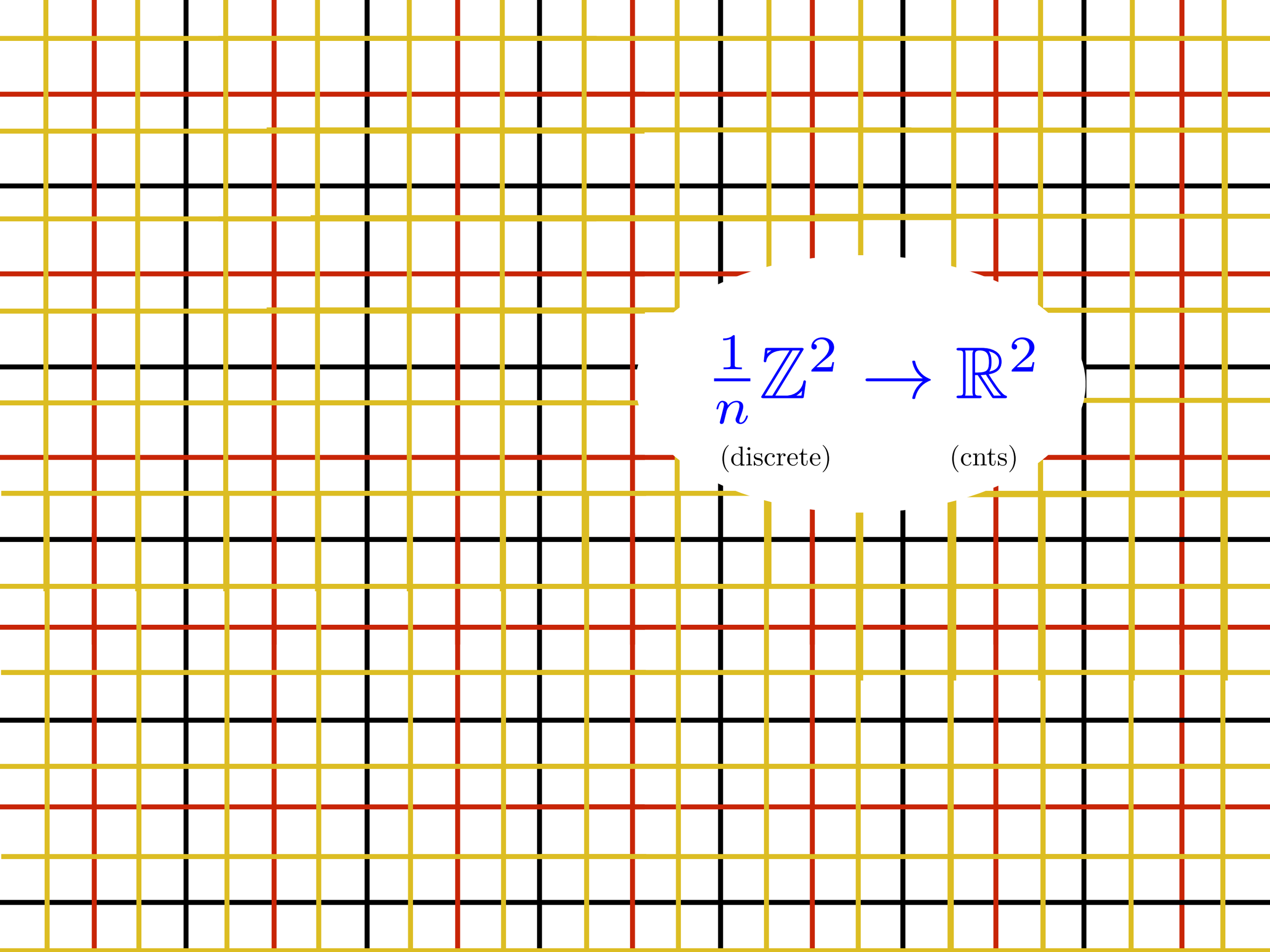

$\mathbb{Z}^2$


$\frac{1}{2}\mathbb{Z}^2$


$\frac{1}{n}\mathbb{Z}^2 \to \mathbb{R}^2$
(discrete)
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Show: G is semisimple Lie. $\square$

Common framework:

$$M \circlearrowleft \mathrm{Isom}(M)$$ "symmetry"

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"Hidden Symmetries"

"symmetry"

$\rightsquigarrow$ Lie group with natural lattice acting on a manifold

$\pi_1(M) \subseteq \mathrm{Isom}(\widetilde{M}) \circlearrowright \widetilde{M}$