Symmetry gaps in Riemannian geometry and minimal orbifolds

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A geometric dichotomy

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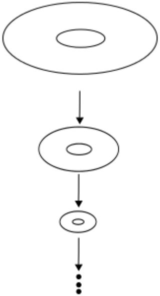

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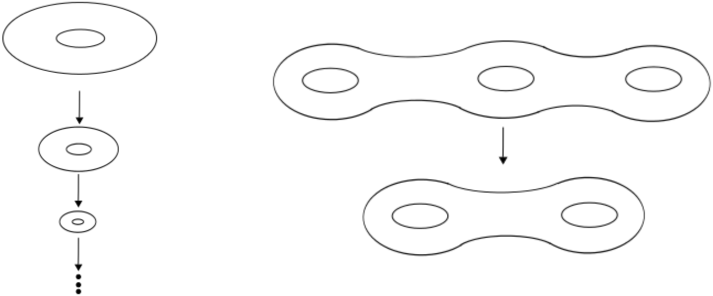

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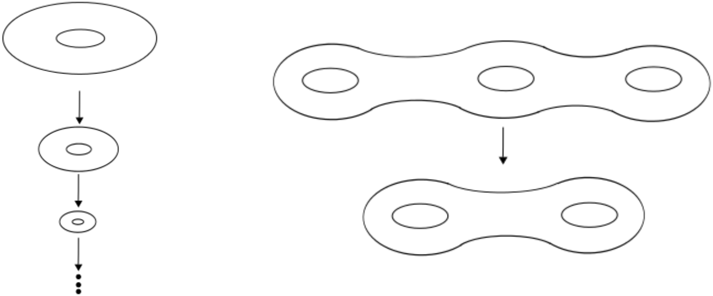

Question

Given (M, g), can we bound $|\mathrm{Isom}(M, g)|$?

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Dai-Shen-Wei	$\mathrm{Ric} < 0$	Dimension Ric Injectivity radius Diameter

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Theorem (vL, 2014)

Let M^n be a closed Riemannian manifold, such that

- $|Ric(M)| \leq \Lambda$,
- $injrad(M) \geq \varepsilon$,
- $diam(M) \leq D$,
- *M does not admit an S^1-action.*

Then $|Isom(M)| \leq C(n, \Lambda, \varepsilon, D)$.

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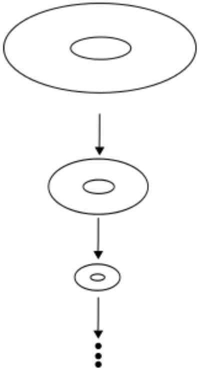

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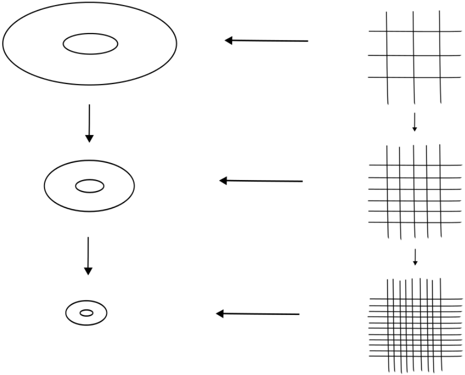

Question

Given (M, g), can we bound $[\mathrm{Isom}(\widetilde{M}, \widetilde{g}) : \pi_1(M)]$?

Higher genus surface

Let $M = \Sigma_g$, $g \geq 2$.

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$[\mathrm{Isom}(\mathbb{H}^2) : \pi_1(\Sigma_g)] = \infty$.

$\implies \mathrm{Ric}(M) < 0$ does not yield a bound!

Second theorem

Theorem (vL, 2014)

Let M^n be a closed Riemannian manifold, such that

- $|Ric(M)| \leq \Lambda$,
- $injrad(M) \geq \varepsilon$,
- $diam(M) \leq D$.
- $\widetilde{M}$ *does not admit a proper action by a nondiscrete Lie group G such that $\pi_1(M) \subseteq G$.*

Then $|Isom(M)| \leq C(n, \Lambda, \varepsilon, D)$.

Local symmetry $\implies$ locally symmetric

Theorem (Farb-Weinberger, 2008)

Let M be

- *a closed, aspherical manifold, and not virtually a product,*
- *$\pi_1(M)$ has no nontrivial normal abelian subgroups.*

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Conjecture (Farb-Weinberger, 2008)

For the conclusion above, it suffices that $[\mathrm{Isom}(\widetilde{M}, \widetilde{g}) : \pi_1(M)] \geq C$ for some C only depending on M.

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Theorem (Farb-Weinberger, 2008)

True if M is diffeomorphic to a locally symmetric space.

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Theorem (vL, 2014)

There exists $C(n, \Lambda, \varepsilon, D)$ *such that if* M^n *is as in the conjecture, and*

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- $|Ric(M)| \leq \Lambda$,
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then either

- $[Isom(\widetilde{M}, \widetilde{g}) : \pi_1(M)] \leq C$, *or*
- (M, g) *is isometric to a locally symmetric space.*

Proof

- Suppose there is no bound on $[\mathrm{Isom}(\widetilde{M}, \widetilde{g}) : \pi_1(M)]$.

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- Choose g_n such that $[\underbrace{\mathrm{Isom}(\widetilde{M},\widetilde{g_n})}_{G_n}:\underbrace{\pi_1(M)}_{\Gamma}]\to\infty$.
- $\exists g$: $g_n \xrightarrow{C^1} g$. Set $G := \mathrm{Isom}(\widetilde{M},\widetilde{g})$.
  Easy facts: G is a Lie group, possibly with infinitely many components. $\Gamma\subseteq G$ is a cocompact lattice.

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  $\implies G^0 \neq 1$ where G^0 is the connected component of the identity.
- Show: Γ contains no nontrivial normal abelian subgroups
  $\implies G^0$ is semisimple.

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- We have G_n with $[G_n : \Gamma] \to \infty$, and G_n 'converge' to G such that G^0 is semisimple.

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- Rough idea:
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- Should be impossible:
  A semisimple Lie group does not admit arbitrarily large lattices (Kazhdan-Margulis).

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- Set $\Gamma_0 := \Gamma \cap G^0$. Show:
  1. $\Gamma_0 \subseteq G^0$ is a cocompact lattice $\implies [G'_n : \Gamma_0] < \infty$.
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- Contradiction!