Math 549 – HW 1 Solutions

Problem 1-1: Note that on any interval contained in one of the two horizontal lines, the quotient map is a homeomorphism. So M is locally Euclidean. To show M is second countable, note that a basis is given by the images of balls with rational nonzero centers and rational radii, together with the balls with rational radii centered at the two "origins". You should check this is a basis.
M is not Hausdorff since for any open neighborhoods $U_\pm$ containing the two "origins" $[(0, \pm 1)]$, their pre-images in X will be open neighborhoods $V_\pm$ of $(0, \pm 1)$ that are saturated with respect to the equivalence relation. Let $\varepsilon > 0$ be small such that $(\varepsilon, 1) \in V_+$. Then $(\varepsilon, -1) \in V_+$ as well (since V_+ consists of equivalence classes). For ε small enough, we have $(\varepsilon, -1) \in V_-$ as well. This shows $U_+ \cap U_- \neq \varnothing$. □

Problem 1-3: To show second countable implies σ-compact: Since X is locally Euclidean, every $x \in X$ has an open neighborhood U_x homeomorphic to an open ball via a chart φ_x, say of radius 1. Let V_x be the pre-image of the ball of radius 1/2 (so $\overline{V_x}$ is compact). Then $\{V_x\}_x$ is an open cover, so by Proposition A.16, it has a countable subcover $\{V_{x_n}\}_{n \geq 1}$. Then

$$X = \bigcup_{n \geq 1} \overline{V_{x_n}}$$

is a countable union of compact subsets.

To show σ-compact implies second countable: Write X as a countable union of compact subspaces $K_n, n \geq 1$. Let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be an atlas for X. For each n, we can choose a finite subcollection $A_n \subseteq A$ such that $\{U_\alpha\}_{\alpha \in A_n}$ is a cover of K_n. Set $A' := \cup_n A_n$, so A' is countable. A countable base for the topology of X is given by pre-images (under φ_α) of balls with rational centers and rational radii in $\varphi_\alpha(U_\alpha)$ for $\alpha \in A'$. You should check this is actually a base. □

Problem 1-6: Note that F_s has inverse $F_{1/s}$. Since both are continuous, we see F_s is a homeomorphism. For $0 < s < 1$ however, F_s is not smooth at the origin (e.g. because $F_s(te_i) = t^s e_i$, so it does not even have partial derivatives at the origin).
Now define smooth structures on the ball via the charts (B, F_s) for $s > 0$. These are not smoothly compatible since the transition functions are $F_s \circ F_t^{-1} = F_s \circ F_{1/t}$ and $F_t \circ F_s^{-1} = F_t \circ F_{1/s}$. Again on a coordinate axis these transition functions raise to the power s/t and t/s. If $s < t$ then the former transition function is not smooth and if $s > t$ then the latter is not smooth.
For a general manifold M, the idea is to change the smooth structure around a single point $p \in M$ using the above on a chart around p. Let $\mathcal{A}$ be a maximal smooth atlas and choose a chart $(U, \varphi) \in \mathcal{A}$ around p such that $\varphi(U) = B^n$ and $\varphi(p) = 0$. Now define a new atlas $\mathcal{A}_s$ by manually removing p from all charts of $\mathcal{A}$ and then adding in $(U, F_s \circ \varphi)$, i.e. set $\mathcal{A}_s := \{(V \backslash \{p\}, \psi|_{V \backslash p}) \mid (V, \psi) \in \mathcal{A}\} \cup \{(U, F_s \circ \varphi)\}$.
Now one should check $\mathcal{A}_s$ is a smooth atlas (the key point is that F_s and F_s^{-1} are smooth everywhere except at the origin, and no chart other than (U, φ) contains $p = \varphi^{-1}(0)$). Further one checks that $\mathcal{A}_s$ and $\mathcal{A}_t$ are not smoothly compatible for $s \neq t$ because $F_s \circ \varphi \circ (F_t \circ \varphi)^{-1} = F_s \circ F_{1/t}$ is not smooth at 0. □

Problem 1-7:

(a) We just do the computation for σ. The line from N to x is parameterized by $t \mapsto N + t(N - x)$. The last coordinate is given by $1 + t(1 - x_{n+1})$ and hence vanishes for $t = t_0 := 1/(1 - x_{n+1})$. Reading off the first n coordinates with $t = t_0$ gives $-t_0 x = (1 - x_{n+1})^{-1} x$, as desired.

(b) It is not hard to verify $\sigma \circ \sigma^{-1} = \mathrm{id}$ and $\sigma^{-1} \circ \sigma = \mathrm{id}$ (for both, you need to use the identity $\|x\|^2 = 1 - x_{n+1}^2$).

(c) It is not hard to compute $\widetilde{\sigma} \circ \sigma^{-1}(u) = \|u\|^{-2} u$ and a similar formula for $\sigma \circ \widetilde{\sigma}^{-1}$. The image of σ of the overlap of the two charts (i.e. of the complement of the two poles) is $\mathbb{R}^n \backslash 0$. Since the transition function is rational with its only pole at the origin, the transition function is smooth on its domain. So the stereographic coordinates define a smooth structure.

(d) We need to verify that the stereographic coordinates are smoothly compatible with the coordinates from the example given by $\varphi_\pm^i(x) = (x_1, \ldots, \widehat{x_i}, \ldots, x_n)$. We have

$$\varphi_\pm^i \circ \sigma^{-1}(u) = (1 + \|u\|^2)^{-1}(2u_1, \ldots, \widehat{2u_i}, \ldots, 2u_n, \|u\|^2 - 1).$$

This map is rational with no poles so it is smooth. For the inverse we compute

$$\sigma \circ (\varphi_{\pm}^{i})^{-1}(u) = (1-u_n)^{-1}(u_1, \ldots, \sqrt{1-\|u\|^2}, \ldots, u_{n-1})$$

which is smooth on the unit ball (which is the image of $\varphi_{\pm}^{i}$). Similarly, the transition functions between $\varphi_{\pm}^{i}$ and $\widetilde{\sigma}$ are smooth.

Problem 1-9: Hausdorffness: The base of the quotient topology on $\mathbb{C}P^n$ is given by images of open cones in $\mathbb{C}^{n+1}\backslash\{0\}$. Given two complex lines ℓ_1, ℓ_2, we can take disjoint small open cones around each which give disjoint open neighborhoods of $\ell_1, \ell_2 \in \mathbb{C}P^n$.
The definition of the charts is exactly the same as for $\mathbb{R}P^n$ (except we are now working with complex numbers). In particular, $\mathbb{C}P^n$ is locally Euclidean and the transition functions are smooth.
A point-set topology argument shows that $\mathbb{C}P^n \cong S^{2n+1}/S^1$. This proves compactness. Using Problem 1-3, this also shows second countability.

Problem 1-11: Here is a solution that does not use the hint (but there are many ways do this problem). We extend the stereographic projection to the interior of the ball, enlarging its codomain to $\mathbb{H}^n$, so that it maps the interior of the ball into the open upper half space $\{x_{n+1} > 0\}$, e.g. consider

$$\varphi : \overline{B^n}\backslash\{N\} \to \mathbb{H}^n$$

defined by $\varphi(x) = (\sigma(x/\|x\|), 1-\|x\|^2)$. Here the first part refers to the first $n-1$ coordinates and the second part is the last (i.e. nth) coordinate. We define a similar map $\widetilde{\varphi}$ using stereographic projection from the south pole. For the transition functions we have

$$\widetilde{\varphi} \circ \varphi^{-1}(u) = (\widetilde{\sigma} \circ \sigma^{-1}(u_1, \ldots, u_{n-1}), u_n).$$

This clearly extends to a smooth function on an open neighborhood of $\mathbb{H}^n$ (the last coordinate extends by the same expression, and the first n coordinates extend with the same expression as well, see Problem 1-7(c) for a exact formula.
To check that on the interior this gives the same as the standard smooth structure on B^n, we use the identity chart on B^n, and note that φ and φ^{-1} are smooth on the interior of their domains when these are viewed as subsets of $\mathbb{R}^n$.