

# Math 549 – HW 10 Solutions

April 8, 2026

## Problem 11-15

Define  $\omega$  as in the hint (so that  $\int_\gamma X \cdot ds = \int_\gamma \omega$ ) and write  $X = \sum_i X^i \frac{\partial}{\partial x^i}$ . Note that  $\omega = \sum_i X^i dx^i$ .

(a) We have that  $X$  is conservative if and only if  $\omega$  is conservative if and only if  $\omega$  is exact.

Now for  $f \in C^\infty(U)$ , we have  $\omega = df = \sum_i \frac{\partial f}{\partial x^i} dx^i$ , which holds if and only if  $X^i = \frac{\partial f}{\partial x^i}$  (for all  $i$ ). The latter just means that  $X = \nabla f$ .  $\square$

(b) If  $X$  is conservative, then  $\omega$  is conservative and hence closed, so  $\frac{\partial \omega_i}{\partial x^j} = \frac{\partial \omega_j}{\partial x^i}$  for all  $i, j$ . Using that  $\omega_i = X^i$ , we see that the component functions of  $\text{curl}(X)$  identically vanish.  $\square$

(c) We saw above that  $\text{curl}(X) = 0$  if and only if  $\omega$  is closed. For  $U = \mathbb{R}^n$ ,  $\omega$  is closed if and only if it is exact, and by (a),  $\omega$  is exact if and only if  $X$  is conservative.  $\square$

## Problem 11-17

If  $\omega$  is exact, then it is conservative so  $\int_{\gamma_j} \omega = 0$  for all  $j$ . Now let us prove the other direction, so assume  $\int_{\gamma_j} \omega = 0$  for all  $j$  and  $\omega$  is closed. Then  $\pi^* \omega$  is also closed (because being closed is a local condition and  $\pi$  is a local diffeomorphism) and by the Poincaré lemma, it is exact. So  $\pi^* \omega = d\tilde{f}$  for some function  $\tilde{f} : \mathbb{R}^n \rightarrow \mathbb{R}$ . We claim that  $\tilde{f}$  descends to a function  $f$  on the torus (i.e.  $\tilde{f}$  is invariant under integer translations), which will then satisfy  $df = \omega$ . To prove the invariance, the key point is that  $d\tilde{f} = \pi^* \omega$  is invariant under integer translations (since  $\pi \circ \tau = \pi$  for any integer translation  $\tau$ ). One can use this in several ways: Let  $v \in \mathbb{Z}^n$  and let  $\tau$  denote translation by  $v$ .

*Proof 1 of invariance:* Using the proof that an exact covector field is conservative, we explicitly have  $\tilde{f}(x) = \int_{c_x} \pi^* \omega$  for any piecewise smooth path  $c_x$  from 0 to  $x$ . In particular, we can use the concatenation of a path from 0 to  $v$  and  $\tau \circ c_x$  for the path from 0 to  $x+v$  that is used to compute  $\tilde{f}(x+v)$ . By invariance of  $\pi^* \omega$  and change of variables, we have

$$\int_{\tau \circ c_x} \pi^* \omega = \int_{c_x} \tau^* \pi^* \omega = \int_{c_x} \pi^* \omega = \tilde{f}(x).$$

So  $\tilde{f}(x+v) = \tilde{f}(x) + \tilde{f}(v)$ . Writing  $v = \sum_i v^i e_i$ , by the same token we have  $\tilde{f}(v) = \sum_i v^i \tilde{f}(e_i)$ . But  $\tilde{f}(e_i) = \int_{\gamma_i} \pi^* \omega = 0$  by assumption. We conclude that  $\tilde{f}(x+v) = \tilde{f}(x)$ .

*Proof 2 of invariance:* Alternatively, one may consider the function  $g(x) := \tilde{f}(x+v) - \tilde{f}(x)$ . Note that  $dg = 0$ . Indeed  $g = \tau^* \tilde{f} - \tilde{f}$  so

$$dg = d(\tau^* \tilde{f}) - d\tilde{f} = \tau^* d\tilde{f} - d\tilde{f} = d\tilde{f} - d\tilde{f} = 0.$$

Therefore  $g$  is constant. Further  $g(0) = \tilde{f}(v) - \tilde{f}(0) = \tilde{f}(v)$ . By changing  $\tilde{f}$  by a constant, we can assume  $\tilde{f}(0) = 0$ . Then one argues  $\tilde{f}(v) = 0$  using  $\int_{\gamma_i} \omega = 0$  as above.  $\square$

**Problem 14-1**

If there is a linear relation, say  $\omega^i = \sum_{j \neq i} c_j \omega^j$  for some  $i$  and constants  $c_j$ , then

$$\omega^1 \wedge \cdots \wedge \omega^k = \sum_{j \neq i} c_j \omega^1 \wedge \cdots \wedge \omega^j \wedge \cdots \wedge \omega^k$$

where the middle term is in the  $i$ th spot. Using  $\omega^j \wedge \omega^j = 0$  (by graded commutativity and using these are degree 1), we see the wedge product vanishes.

On the other hand, if  $\omega^1, \dots, \omega^k$  are linearly independent, then extend them to a basis  $\omega^1, \dots, \omega^n$  of  $V^*$ , and  $\omega^I, |I| = k$ , gives a basis of  $\Lambda^k V^*$ , so letting  $I = (1, \dots, k)$ , we see  $\omega^1 \wedge \cdots \wedge \omega^k \neq 0$ .  $\square$

**Problem 14-5**

Extend  $\omega^1, \dots, \omega^k$  to a local coframe  $\omega^1, \dots, \omega^n$ , and write  $\alpha^i = \sum_j a_{ij} \omega^j$  for some functions  $a_{ij}$ . Then

$$0 = \sum_{i=1}^k \alpha^i \wedge \omega^i = \sum_{i=1}^k \sum_{j=1}^n a_{ij} \omega^j \wedge \omega^i.$$

Note that  $\omega^j \wedge \omega^i, j > i$ , form a local frame for  $\Omega^2(M)$ , and if  $j > k$ , it appears only once in the above sum with coefficient  $a_{ij}$ , so we must have that  $a_{ij} = 0$ , as desired.  $\square$

**Problem 14-6**

(a) We have  $dx = \sin \phi \cos \theta d\rho + \rho \cos \phi \cos \theta d\phi - \rho \sin \phi \sin \theta d\theta$  and similarly for  $dy$  and  $dz$ . Then one has, for example,

$$dx \wedge dy = (\sin \phi \cos \theta d\rho + \rho \cos \phi \cos \theta d\phi - \rho \sin \phi \sin \theta d\theta) \wedge (\sin \phi \sin \theta d\rho + \rho \cos \phi \sin \theta d\phi + \rho \sin \phi \cos \theta d\theta)$$

Expanding this out using bilinearity of the wedge product and using  $d\rho \wedge d\rho = d\theta \wedge d\theta = d\phi \wedge d\phi = 0$ , you get an expression for  $dx \wedge dy$ . You can similarly work out  $dy \wedge dz$  and  $dz \wedge dx$ . Inserting these and the expressions for  $x, y, z$  in the formula for  $\omega$ , you will eventually find  $\rho^3 d\varphi \wedge d\theta$ .

(b) In Cartesian coordinates we have  $d\omega = 3dx \wedge dy \wedge dz$ . Inserting the expressions for  $dx, dy, dz$  in spherical coordinates (found at the beginning of (a)), we find  $\omega = 3\rho^2 d\rho \wedge d\varphi \wedge d\theta$ . (Alternatively, you can take the answer to (a) and apply  $d$ .)

(c) We use the computation from (a) with  $\rho = 1$  and find

$$i^* \omega = d\varphi \wedge d\theta.$$

(d) We have  $i^* \omega(\partial_\varphi, \partial_\theta) = 1$  (identically) so  $i^* \omega \neq 0$  on the domain of definition of spherical coordinates (i.e. everywhere except the north and south pole  $z = \pm 1$ ). To verify  $i^* \omega \neq 0$  at the poles as well, note that  $\omega$  is symmetric in  $x, y, z$ , so we can just do the same computation but with the poles being  $x = \pm 1$ .

**Problem 14-7(a)**

From 11-7(a), we know that  $F^* \omega = ste^t dt$ . Hence

$$d(F^* \omega) = te^t ds \wedge dt.$$

On the other hand,  $d\omega = dx \wedge dy$ , so  $F^* \omega = (F^* dx) \wedge (F^* dy)$ . Similar to the above one computes  $F^* dx = tds$  and  $F^* dy = e^t dt$ , so

$$F^* dx \wedge F^* dy = (tds) \wedge (e^t dt) = te^t ds \wedge dt.$$

$\square$

**Problem 14-9**

If  $\omega = \pi^*\eta$  then for any  $v \in \ker \pi_*$ , we have

$$\iota_v \omega = \iota_v \pi^* \eta = \pi^* \pi^* (\iota_{\pi_* v} \eta) = 0$$

because  $\pi_* v = 0$ .

Conversely, suppose  $\iota_v \omega = 0$  for all  $v \in \ker \pi_*$ . It is easy to check that  $\omega(v_1, \dots, v_k)$  only depends on  $\pi_* v_1, \dots, \pi_* v_k$ , i.e. “ $\omega$  descends to a  $k$ -form on the vector bundle  $TM/\ker(\pi)$ ” (you can make sense of this as a vector bundle, but we won’t do that here, and we just take this sentence to mean the aforementioned independence).

Now let  $\{U_\alpha\}_\alpha$  be an open cover of  $M$  such that on  $\pi(U_\alpha)$ , we have local sections  $\sigma_\alpha : \pi(U_\alpha) \rightarrow U_\alpha$  of  $\pi$ . Let  $\{f_\alpha\}_\alpha$  be a partition of unity subordinate to this open cover and set

$$\eta := \sum_\alpha \sigma_\alpha^* (f_\alpha \omega).$$

Then  $\pi^* \eta = \sum_\alpha \pi^* \sigma_\alpha^* (f_\alpha \omega)$ . Then using the above property that “ $\omega$  descends to a form on  $TM/\ker(\pi)$ ”, it is not hard to check that  $\pi^* \eta(v_1, \dots, v_k) = \omega(v_1, \dots, v_k)$  for any  $(v_1, \dots, v_k) \in (TM)^{\oplus k}$ .  $\square$