

# Math 549 – HW 11 Solutions

April 17, 2026

## Problem 15-2

Define  $\varepsilon : M \rightarrow \{\pm 1\}$  where  $\varepsilon(x) = \pm 1$  according to whether  $D_x F$  is orientation-preserving or reversing. Since determinant of  $D_x F$  is continuous in  $x$  (on a chart), its sign (i.e.  $\varepsilon$ ) is locally constant. Since  $M$  is connected,  $\varepsilon$  is then constant.  $\square$

## Problem 15-3

Consider the map  $F = -\text{Id}$  as suggested by the hint. Since  $F$  is linear on  $\mathbb{R}^{n+1}$ , its derivative is itself, and the determinant is  $(-1)^{n+1}$ . Therefore  $F$  is orientation-preserving on  $\mathbb{R}^{n+1}$  if and only if  $n$  is odd. Note that  $F_*$  maps an outward-pointing vector field to another outward-pointing vector field, and therefore  $\alpha = F|_{\partial B^n}$  is orientation-preserving if and only if  $F$  is.  $\square$

## Problem 15-4

*Proof 1:* Let  $x \in M$  and consider the set

$$\begin{aligned} T^+ &:= \{t \in \mathbb{R} \mid \theta_t \text{ is orientation-preserving at } x\} \\ &= \{t \in \mathbb{R} \mid \text{sign det}(D_x \theta_t) > 0\}. \end{aligned}$$

Since  $\theta_0 = \text{id}_M$ , we have  $0 \in T^+$  so  $T^+$  is nonempty. Since  $t \mapsto \text{sign det}(D_x \theta_t)$  is locally constant, the set  $T^+$  is open and closed. Therefore  $T^+ = \mathbb{R}$ .  $\square$

*Proof 2:* Note that for any  $t \in \mathbb{R}$ , we have  $\varphi_t = \varphi_{t/2} \circ \varphi_{t/2}$ . If  $\varphi_{t/2}$  is orientation-preserving, then  $\varphi_t$  is the composition of orientation-preserving maps and hence orientation-preserving. On the other hand, if  $\varphi_{t/2}$  is orientation-reversing, then  $\varphi_t$  is the composition of two orientation-reversing maps and hence again orientation-preserving.  $\square$

## Problem 15-5

Let  $(U, \varphi)$  be a chart on  $M$  and  $(TU, D\varphi)$  be an associated chart on  $TM$ . Then locally we have  $TU \cong T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n$ , and therefore  $T(TU) \cong T\mathbb{R}^n \times T\mathbb{R}^n$ . We now want to give a local orientation on  $TU$  that is independent of the chart chosen. You can do this either with frames or orientation forms (it's essentially the same proof).

Let  $e$  be any local frame on  $T\mathbb{R}^n$ , then  $(e, e)$  is a local frame on  $T\mathbb{R}^n \times T\mathbb{R}^n$ . Note that its orientation is independent of the choice of  $e$ . (Alternatively you may write this with orientation forms: Let  $\omega$  be an orientation form on  $T\mathbb{R}^n$ , and let  $\pi_i : T\mathbb{R}^n \cong \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  denote the two projections. Then  $\eta := \pi_1^* \omega \wedge \pi_2^* \omega$  defines the orientation on  $T\mathbb{R}^n \times T\mathbb{R}^n$ .)

We claim this orientation is independent of the chart  $(U, \varphi)$  and therefore defines a global orientation. Indeed, if  $(V, \psi)$  is another chart, then the transition function on  $T(U \cap V)$  is given by  $D(\psi \circ \varphi^{-1})$ . Its derivative on a tangent space  $T_x \mathbb{R}^n \times T_v \mathbb{R}^n$  is given by a block upper triangular matrix where the two diagonal blocks are  $D_x(\psi \circ \varphi^{-1})$ . Therefore its determinant is  $(\det D_x(\psi \circ \varphi^{-1}))^2 > 0$ , so the orientation is consistent across  $U \cup V$ .

**Problem 15-10**

For  $x \in X$ , we have a linear isomorphism  $D_x F : T_x X \xrightarrow{\cong} T_{F(x)} M$ . Let  $\varepsilon_x$  denote the orientation on  $T_{F(x)} M$  induced by this isomorphism and the given orientation on  $T_x X$ . Then set

$$\hat{F} : X \rightarrow \widehat{M}, \quad x \mapsto (F(x), \varepsilon_x).$$

Then it is clear that  $\hat{\pi} \circ \hat{F} = F$ . □

**Problem 15-13**

(a) Assume  $M$  is orientable. Let  $\{(U_\alpha, \varphi_\alpha)\}_\alpha$  be an atlas by oriented charts  $\varphi_\alpha : U_\alpha \rightarrow \mathbb{R}$ . Let  $\{f_\alpha\}_\alpha$  be a partition of unity subordinate to this atlas. Then  $X := \sum_\alpha f_\alpha \varphi_\alpha^* \partial_x$  is a nonvanishing vector field on  $M$ : The point is that any point  $p$ , the vectors  $f_\alpha(x) \varphi_\alpha^*(\partial_x)(p)$  all point in the same direction in  $T_x M \cong \mathbb{R}$ , so there is no cancellation and the sum is nonzero.

The integral curves of  $X$  are immersed 1-dimensional submanifolds, and therefore open. In particular, the complement of a single integral curve is the union of the other integral curves, hence a union of open sets, hence open. So we see that any integral curve is also closed. By connectedness of  $M$ , there is only one integral curve. It is either a circle or  $\mathbb{R}$ , according to whether the orbit is periodic or not.

(b) The universal cover  $\widetilde{M}$  of  $M$  is orientable because it is simply-connected, and by (a), must therefore be diffeomorphic to  $\mathbb{R}$ . Consider the group of deck transformations  $G$  of  $\widetilde{M} \rightarrow M$ . It suffices to prove  $G$  is orientation-preserving, for then  $M$  itself is also orientable. To show this, note that any orientation-reversing diffeomorphism of  $\mathbb{R}$  has a fixed point (by 15-12), but  $G$  acts freely (since it is a group of deck transformations), so none of the elements of  $G$  are orientation-reversing. □

**Problem 16-3(a)**

Let  $U_1, \dots, U_N \subseteq M$  denote an open cover by evenly-covered sets. Let  $\widetilde{U}_i^j, 1 \leq j \leq k$  denote the sheets over  $U_i, 1 \leq i \leq N$ . Let  $\{f_i\}_{1 \leq i \leq N}$  denote a partition of unity subordinate to the open cover  $\{U_1, \dots, U_N\}$ . Then

$$\begin{aligned} \int_E \pi^* \omega &= \int_E \pi^* \left( \sum_i f_i \omega \right) \\ &= \sum_i \int_E (f_i \circ \pi) \pi^* \omega. \end{aligned}$$

Since  $f_i \circ \pi$  is supported on  $\pi^{-1}(U_i) = \sqcup_j \widetilde{U}_i^j$  and integrals are additive on disjoint unions, this is then equal to

$$\begin{aligned} \sum_{i,j} \int_{\widetilde{U}_i^j} (f_i \circ \pi) \pi^* \omega &= \sum_{i,j} \int_{U_i} f_i \omega && \text{(change of variables by } \pi|_{\widetilde{U}_i^j}) \\ &= \sum_{i,j} \int_M f_i \omega && \text{(supp}(f_i) \subseteq U_i) \\ &= \sum_j \int_M \omega && (\sum_i f_i = 1) \\ &= k \int_M \omega. \end{aligned}$$