Math 549 – HW 12 Solutions

April 29, 2026

Problem 16-2

Solution by a symmetry argument: Observe that ω and T^2 are invariant under the reflection $(w, x, y, z) \mapsto (w, x, -y, z)$, which is orientation-reversing, so $\int_{T^2} \omega = -\int_{T^2} \omega$ so the integral must be 0.

Solution by a computation: Consider polar coordinates (r_1, θ_1) and (r_2, θ_2) on $\mathbb{R}^4$. In these coordinates, we have on T^2 (note that $r_1 = r_2 = 1$ and hence $dr_1 = dr_2 = 0$:

$$\omega = \sin^2(\theta_1)\cos(\theta_2)\sin^2(\theta_2)d\theta_1 \wedge d\theta_2.$$

So

$$\int_{T^2} \omega = \left(\int_0^{2\pi} \sin^2\theta_1 d\theta_1\right)\left(\int_0^{2\pi} \sin^2\theta_2 \cos(\theta_2) d\theta_2\right).$$

You can either compute this integral directly or observe the second integral vanishes (again for symmetry reasons) so $\int_{T^2} \omega = 0$.

Problem 16-4

We argue by contradiction so suppose that $r : M \to \partial M$ is a retraction (i.e. $r|_{\partial M} = \mathrm{id}_{\partial M}$). By Whitney approximation, we can assume that r is smooth. Now let ω be an orientation form on ∂M. Then $d\omega = 0$ since ω has top degree. Consider now the form $r^*\omega$ on M. Note that it is given by ω on ∂M, so that by using Stokes' theorem, we have

$$\int_{\partial M} \omega = \int_{\partial M} r^*\omega = \int_M dr^*\omega = \int_M r^*d\omega = 0,$$

where we used that d commutes with pull-back along smooth maps. □

Problem 16-5

Let ω be an orientation form on N. So $\int_M F^*\omega = \pm \int_N \omega$ according to whether F is orientation-preserving or reversing, and similarly for G. Therefore it suffices to prove $\int_M F^*\omega = \int_M G^*\omega$.

Suppose $H : M \times I \to N$ is a homotopy from F to G. Then we have

$$\int_M G^*\omega - \int_M F^*\omega = \int_{\partial(M\times I)} H^*\omega = \int_{M\times I} dH^*\omega = \int_{M\times I} H^*d\omega = 0$$

where in the last equality we used that $d\omega = 0$ since ω is a top degree form. □

Problem 16-9

(a) Let $x \in S^{n-1}$ and $v_1, \dots, v_{n-1} \in T_xS^{n-1}$ be a basis. We need to check that $\omega(v_1, \dots, v_{n-1}) \neq 0$.

Note that $\nu(y) = y$ is the outward pointing vector field on the sphere, and in particular $\nu(x) = x$, so $(x, v_1, \dots, v_{n-1})$ is a basis of $T_x\mathbb{R}^n \cong \mathbb{R}^n$. Since $\|x\| = 1$, we have

$$\omega(v_1, \dots, v_{n-1}) = \sum_{i=1}^n (-1)^{i-1}x^i(dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots dx^n)(v_1, \dots, v_{n-1}) = \sum_{i=1}^n (-1)^{i-1}x^i \det(v_j^k)_{k\neq i},$$

where $(v_k^i)_{k\neq i}$ denotes the matrix whose columns are $v_1, \dots, v_{n-1}$ with the ith row removed. Now recognize that the expression on the right-hand side is the column expansion along the first column of the matrix with columns $x, v_1, \dots, v_{n-1}$. This determinant is nonzero since $x, v_1, \dots, v_{n-1}$ form a basis of $\mathbb{R}^n$. □

(b) Note that $d(|x|^{-n}) = -n|x|^{-(n+1)} \sum_i x^i dx^i$. Inserting this, one readily sees $d\omega = 0$. To show it is not exact, note that by (a), $\int_{S^{n-1}} \omega \neq 0$, whereas for any exact form, the integral over S^{n-1} vanishes by Stokes' theorem. □

Problem 17-1

It suffices to prove that if $d\omega$ is exact, then so is $d\omega \wedge \eta$ for any closed form η. In fact we have

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d\eta = d\omega \wedge \eta,$$

where we used that η is closed. □

(The exercise seems to assume it, but we should also show that the wedge product of closed forms is closed: Since $d\omega = d\eta = 0$, we have $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{|\omega|}\omega \wedge d\eta = 0$.)

Problem 17-4

We will take the approach of tracing through the proof of the Poincaré lemma: Consider the homotopy $H : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n, (x,t) \mapsto tx$ from the identity map to the zero map. We need to verify that $h(\omega) = \eta$, where h is the chain homotopy defined in the proof of the Poincaré lemma. For $x \in \mathbb{R}^n$ and $v_1, \dots, v_{k-1} \in T_x\mathbb{R}^n$, we have

$$\begin{aligned} h(\omega)_x(v_1, \dots, v_{k-1}) &= \int_0^1 (\iota_{\partial_t} H^*\omega)_{x,t}(v_1, \dots, v_{k-1}) dt \\ &= \int_0^1 (H^*\omega)_{x,t}(\partial_t, v_1, \dots, v_{k-1}) dt \\ &= \int_0^1 \omega_{H(x,t)}(H_*\partial_t, H_*v_1, \dots, H_*v_{k-1}) dt. \end{aligned}$$

Note that $H(x,t) = tx$ and $H_*v_i = tv_i$ and $H_*\partial_t(x,t) = \frac{d}{ds}|_{s=t} H(x,s) = x$. Inserting all of this, we obtain

$$h(\omega)_x(v_1, \dots, v_{k-1}) = \int_0^1 t^{p-1}\omega_{tx}(x, v_1, \dots, v_{k-1}).$$

Writing this out with $\omega = \sum_I \omega_I dx^I$, we obtain the expression η (when evaluating on the first vector, use $dx^i(x) = x^i$ for any i). □