Math 549 – HW 13 Solutions

April 30, 2026

Problem 17-8

(a) There is a 1-1 correspondence between orientations of M and $\Omega^n(M)$ modulo positive functions (i.e. equivalence classes of orientation forms). On the other hand, note that an orientation on $H^n(M) \cong \mathbb{R}$ consists of all positive scalar multiples of a nonzero element. Via the integration isomorphism $H^n(M) \to \mathbb{R}$, we see that two orientation forms are equivalent modulo a positive function if and only if their cohomology classes differ by a positive scalar, i.e. if they determine the same orientation on $H^n(M)$. □

(b) Let ω be an orientation form on N (that induces the given orientation on N). Then F^* is orientation preserving on cohomology if and only if $\int_M F^*\omega > 0$. But $\int_M F^*\omega = \pm \int_N \omega$ with sign depending on whether F is orientation-preserving or reversing. Since $\int_N \omega > 0$, we see F^* is orientation-preserving on cohomology if and only if F is orientation-preserving. □

Problem 17-10

Let d be the degree of p. Then a generic point has d pre-images. Note that p is orientation-preserving since its derivative is multiplication by $p'(z) = f(z) + ig(z)$ where f, g are real. Taking the basis $\partial_x, i\partial_x$, the matrix of multiplication by $p'(z)$ is

$$\begin{pmatrix} f(z) & -g(z) \\ g(z) & f(z) \end{pmatrix}$$

which has determinant $f^2 + g^2 \geq 0$ (and if $p'(z) \neq 0$, then the determinant is strictly positive). Therefore the local degree of p at each of the d pre-images of a regular value is 1, so that $\deg(p) = d$. □

Problem 17-12

Suppose that F is not surjective. Then choose any $y \in N$ outside the image. Then $F^{-1}(y) = \varnothing$, so that $\deg(F) = 0$.

For the map that is surjective but has degree 0, consider a map $S^1 \to S^1$ as follows: On the upper half, apply the map $z \mapsto z^2$ (i.e. run around the circle counterclockwise). Then on the bottom half, apply the map $z \mapsto z^{-2}$ (i.e. run around the circle clcokwise). It is easy to check using a regular value (or by taking the angle form, pulling it back, and integrating by splitting up the integral over two regions) that the degree is 0. This map is not smooth at the two points where the upper and lower semicircles meet, but you can reparametrize it so that it is smooth.

You can similarly construct maps $S^n \to S^n$ (collapse the boundary of the upper hemisphere to a point so that it's a sphere, and then use a map $S^n \to S^n$ of degree d that fixes the distinguished point that is the image of the boundary of the upper hemisphere, and similarly on the lower hemisphere but use a map of degree $-d$).

Problem 17-13

f is the identity map so it has degree 1. g is an orientation-preserving diffeomorphism so it also has degree 1. However on $\pi_1 T^2 \cong \mathbb{Z}^2$, the map induced by g is nontrivial. It is the anti-diagonal matrix with entries $-1, 1$.