Math 549 – HW 2 Solutions

Problem 2-2: Let $p \in N$ and write $F(p) = q = (q_1, \dots, q_k)$. For every i, let (U_i, φ_i) be a chart around q_i. Then $(\prod_i U_i, \prod_i \varphi_i)$ is a chart around q. Also let (V, ψ) be a chart around p in M.
Note that

$$(\prod_i \varphi_i) \circ F \circ \psi^{-1} = \prod_i (\varphi_i \circ F_i \circ \psi^{-1}).$$

Since a map $\mathbb{R}^n \to \mathbb{R}^m$ is smooth if and only if all of its component functions are smooth, we see that F is smooth if and only if F_i is smooth for all i. □

Problem 2-3: The general method is to show that for every point $p \in M$, we can find a chart (U, φ) around p and (V, ψ) around $F(p)$ such that the local coordinate representation $\psi \circ F \circ \varphi^{-1}$ is smooth. We do not need to check $F(U) \subseteq V$ because all the maps here are continuous.

(a) For $z \in S^1$, consider the chart $\varphi_z : S^1 \backslash \{p\} \to I \subseteq \mathbb{R}, e^{i\theta} \mapsto \theta$ where I is an open interval of length 2π. Note that $p_n(e^{i\theta}) = e^{in\theta}$, so in these local coordinates, p_n is represented by $\theta \mapsto n\theta$ and hence is smooth. Note that here we use the same chart φ_z on both the domain and the range.This is allowed because for any $z_0 \in S^1$, we can find $z_1 \in S^1$ such that $z_0, p_n(z_0) \neq z_1$, so that we can check smoothness at z_0 by using the chart associated to $z = z_1$.

(b) With respect to standard charts $(U_i^{\pm}, \varphi_i)$ where $U_i^{\pm}$ are hemispheres, we have $\varphi_i^{\pm} \circ \alpha \circ (\varphi_i^{\mp})^{-1}(x) = -x$, which is smooth. One can check this explicitly using the formulas for $\varphi_i^{\pm}$.

(c) One can certainly check this explicitly but a small trick is to instead consider the extension $\widetilde{F} : \mathbb{C}^2 \backslash \{0\} \to \mathbb{R}^3$ given by the same formula. This map is clearly smooth (because all the components are polynomials). In addition the normalization map $\nu : \mathbb{C}^2 \backslash \{0\} \to S^3$ given by $\nu(x) := \frac{1}{||x||} x$ is smooth. Now we have $F = \widetilde{F} \circ \nu$ is a composition of smooth functions, hence smooth.

Problem 3-1: We showed in class already that the derivative of any constant function is 0, so it remains to prove the other implication:
Without loss of generality assume M is connected. Fix a basepoint $p_0 \in M$ and set $A := \{p \in M \mid F(p) = F(p_0)\}$. Then A is nonempty (because $p_0 \in A$) and closed (because $A = F^{-1}(F(p_0))$). Therefore it suffices to show A is open, for then by connectedness, we have $A = M$ so that $F(p) = F(p_0)$ for all p so F is constant.
Suppose that $p \in A$ and let (U, φ) be a chart near p and (V, ψ) a chart near $F(p)$. Then the coordinate representation

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

has vanishing derivative (e.g. you can see this either by using the chain rule). Since the result of the problem is true for maps between open sets of Euclidean spaces (i.e. whenever such a map has vanishing derivative, it is constant), we see that $\psi \circ F \circ \varphi^{-1}$ is constant. Since ψ and φ^{-1} are invertible, F is constant also. □

Problem 3-6: For smoothness, you can use for example the standard (identity) chart on $\mathbb{R}$ and the hemisphere charts on S^3. The local coordinate representation will be a map $\mathbb{R} \to \mathbb{R}^3$ whose components are trigonometric functions (there are a few cases to consider, I won't write them all out here).
To prove the velocity is nowhere vanishing: Let $\iota : S^3 \to \mathbb{C}^2$ denote the inclusion. Note that $\iota \circ \gamma_z : \mathbb{R} \to \mathbb{C}^2$ has nowhere vanishing velocity (just by calculus). But this suffices, because if γ_z had vanishing velocity at say $t = t_0$, then $(\iota \circ \gamma_z)'(t_0) = D\iota(\gamma_z'(t_0)) = 0$ as well, which is a contradiction. □

Problem 3-8: We first show it is well-defined. Suppose that $[\gamma_1] = [\gamma_2]$. We need to show that as derivations $\gamma_1'(0) = \gamma_2'(0)$. So let $f : M \to \mathbb{R}$ be any smooth function. Then by definition, for $i = 1, 2$, we have

$$\gamma_i'(0)(f) = \frac{d}{dt}|_{t=0} f(\gamma_i(t)).$$

Since $[\gamma_1] = [\gamma_2]$, the right-hand sides coincide for $i = 1, 2$ for any function on a neighborhood of p, and in particular for f. We conclude the map is well-defined.
As mentioned on the problem set, the surjectivity of the map is exactly Proposition 3.23. I won't write the entire proof here but the gist is to a take a chart near p, and then every derivation $\alpha \in T_pM$ is given by a

directional derivative D_v for some $v \in \mathbb{R}^n$ (in the local coordinates of the chart), and v is the velocity vector of the curve $t \mapsto tv$ (in these coordinates).
We prove it is injective: Suppose $\gamma_1'(0) = \gamma_2'(0)$. Let f be a smooth function defined on an open neighborhood U of p. Then $\gamma_1'(0) = \gamma_2'(0) \in T_pU$ as well by locality of tangent spaces, and hence $\gamma_1'(0)(f) = \gamma_2'(0)(f)$. As in the proof of the well-definedness, these expressions are just $(f \circ \gamma_i)'(0)$. So we conclude $(f \circ \gamma_1)'(0) = (f \circ \gamma_2)'(0)$. Since f was arbitrary, we see $[\gamma_1] = [\gamma_2]$. □