Math 549 – HW 3 Solutions

Problem 2-7:

Let $N \geq 1$. We will show $\dim C^\infty(M) \geq N$. Let $p_1, \ldots, p_N \in M$ be distinct points and let $U_i \ni p_i$ be pairwise disjoint open neighborhoods. For $1 \leq i \leq N$, let φ_i be a bump function with $\varphi_i(p_i) = 1$ and $\varphi_i = 0$ outside U_i. We claim that $\{\varphi_i\}_{1 \leq i \leq N}$ is a linearly independent set. Indeed, suppose there is a linear combination

$$\sum_{1 \leq i \leq N} c_i f_i = 0.$$

By evaluating on p_j for some $1 \leq j \leq N$ and using $f_i(p_j) = \delta_{ij}$, we find that $c_j = 0$, as desired. □

Problem 2-10:

(a) This is straightforward.

(b) If F is smooth, then for any $f \in C^\infty(N)$, we have that $F^*(f) = f \circ F$ is a composition of smooth maps, hence smooth.

Conversely, suppose $F^*(C^\infty(N)) \subseteq C^\infty(M)$. Let $p \in M$ and (U, φ) and (V, ψ) be charts around $p \in M$ and $F(p) \in N$, respectively. By shrinking U if necessary we can assume $F(U) \subseteq V$. To prove F is smooth at p, it suffices to prove $\psi \circ F \circ \varphi^{-1}$ is smooth as a map between open subsets of Euclidean spaces. □

Let $\psi_i (1 \leq i \leq n)$ be the components of ψ. By the extension lemma (i.e. multiplying by a bump function and extending by 0), we can choose $W \subseteq V$ that is an open neighborhood of p such that $\psi_i|_W$ extend to smooth functions $\tilde{\psi}_i \in C^\infty(N)$. By assumption, $\tilde{\psi}_j \circ F : M \to \mathbb{R}$ is smooth, and hence the composition $\tilde{\psi}_j \circ F \circ \varphi^{-1}$ is also smooth. Shrinking U further if necessary we can assume $F(U) \subseteq W$, so that the components of $\psi \circ F \circ \varphi^{-1}$ are precisely $\tilde{\psi}_j \circ F \circ \varphi^{-1}$. Since $\psi \circ F \circ \varphi^{-1}$ is a map between open sets of Euclidean spaces whose component functions are smooth, it is itself smooth. □

(c) If F is a diffeomorphism, it is straightforward to check that $(F^{-1})^*$ is the inverse of F^*, so F^* is an isomorphism.

Conversely, suppose F^* is an isomorphism. By the previous part, F is smooth. It remains to argue that $(F^{-1}$ is smooth.

As mentioned above, it is straightforward to check that $(F^*)^{-1} = (F^{-1})^*$ (on the algebras of continuous functions). Since F^* restricts to an isomorphism $C^\infty(N) \to C^\infty(M)$, its inverse is an isomorphism in the reverse direction, i.e. $(F^{-1})^*$ is an isomorphism $C^\infty(M) \to C^\infty(N)$. Then again by the previous part F^{-1} is smooth. □

Problem 2-14:

By Theorem 2.29, there exists smooth functions $f, g : M \to \mathbb{R}$ such that $f, g \geq 0$ and f vanishes precisely on A and g vanishes precisely on B (see below how to find such a function). Consider now $\varphi := f/(f+g)$. It is clear that φ is well-defined and $0 \leq \varphi \leq 1$ and φ vanishes precisely on A and $\varphi(x) = 1$ iff $g(x) = 0$ iff $x \in B$.
Sketch of a proof of existence of $f \geq 0$ that vanishes precisely on A: Let $U_n \supseteq A$ be a decreasing sequence of open neighborhoods with $\cap_n U_n = A$. We can choose a bump function $f_n \geq 0$ such that $f_n > 0$ outside $\overline{U}_n$ and $f_n = 0$ on A. The idea is to consider a function of the form $f = \sum_{n \geq 1} a_n f_n$ for suitable choice of scalars a_n. To ensure the series and all the series of partial derivatives converge, choose a_n as follows: Let $\{B_k\}_{k \geq 1}$ be a locally finite (countable) cover of M by regular coordinate balls.
For $n \geq 1$, let b_n be the maximum of the absolute value of a partial derivative of f_n on $B_1 \cup \cdots \cup B_n$ (where these partial derivatives are computed in the local coordinates on the B_k). Set $a_n := \exp(-\max\{n, b_n\})$. By considering cases where there is a partial derivative that attains a value larger than n or not, it is easy to show that the maximal size of the absolute value of any partial derivative of $a_n f_n$ on $B_1 \cup \cdots \cup B_n$ is at most 2^{-n}.
Now let $p \in M$. Choose k such that $p \in B_k$ and let φ_k denote the corresponding chart. Then it is easy to verify that for any finite index set I, the series $\sum_n a_n \partial^I (f_n \circ \varphi_k^{-1})(\varphi_k(p))$ is absolutely convergent (since for $n \gg 1$, the summand is bounded above by 2^{-n}). From this we conclude that all partials of $f := \sum_n a_n f_n$ exist. □

Problem 3-2: For $1 \leq j \leq k$, let $\iota_j : M_j \hookrightarrow M$ denote the inclusion $\iota_j(x) = (p_1, \ldots, p_{j-1}, x, p_{j+1}, \ldots, p_k)$. Write $\beta := \oplus_j D_{p_j}\iota_j$. Then $\pi_j \circ \iota_j = \mathrm{id}_{M_j}$, so $D_p\pi_j \circ D_{p_j}\iota_j = \mathrm{Id}$ by the chain rule. Therefore

$$\begin{aligned}\alpha \circ \beta &= \left(\bigoplus_j D_p\pi_j\right) \circ \left(\bigoplus_j D_{p_j}\iota_j\right) \\ &= \left(\bigoplus_j D_p\pi_j \circ D_{p_j}\iota_j\right) \\ &= \bigoplus_j \mathrm{Id},\end{aligned}$$

so α is surjective and β is injective. Since both are linear maps and $\dim(\prod_j M_j) = \sum_j \dim(M_j)$ coincides with $\dim(\oplus_j T_{p_j}M_j)$, both are isomorphisms.

Problem 3-3:
Let $\pi_M : M \times N \to M$ and $\pi_N : M \times N \to N$ be the projections, so $D\pi_M : T(M \times N) \to TM$ and $D\pi_N : T(M \times N) \to TN$ are smooth maps. Consider $D\pi_M \times D\pi_N : T(M \times N) \to TM \times TN$. It is smooth since its component functions are smooth. One checks this is a bijection: First of all, if the images of (p, v) and (q, w) coincide, then $\pi_M(p) = \pi_M(q)$ and $\pi_N(p) = \pi_N(q)$, so $p = q \in M \times N$. So we have to show $v = w \in T_p(M \times N)$. To show this, just note that $D_p\pi_M \times D_p\pi_N : T_p(M \times N) \to T_{\pi_M(p)}M \times T_{\pi_N(p)}N$ is exactly the isomorphism from the previous problem. Surjectivity follows by a similar argument, again the key point being the map on each tangent space is an isomorphism.
It remains to check smoothness of the inverse. Let (U, φ) and (V, ψ) be charts on M and N respectively, and $(U \times V, \varphi \times \psi)$ is a chart on $M \times N$. Then $(TU, D\varphi)$ and $(TV, D\psi)$ are charts on TM and TN and hence $(TU \times TV, D\varphi \times D\psi)$ is a chart on $TM \times TN$, and $(T(U \times V), D(\varphi \times \psi))$ is a chart on $T(M \times N)$. Now compute $D\pi_M \times D\pi_N$ in these local coordinates, i.e. compute

$$(D\varphi \times D\psi) \circ (D\pi_M \times D\pi_N) \circ D(\varphi \times \psi)^{-1}.$$

Using the chain rule, it is straightforward to verify that this is just

$$D(\varphi \circ \pi_M \circ (\varphi^{-1} \times \psi^{-1})) \times D(\psi \circ \pi_N \circ (\varphi^{-1} \times \psi^{-1})).$$

Problem 3-4:
Consider the curve $\gamma(t) = e^{it}$. Note that $\gamma'(t) \neq 0$ for all t (see Problem 3-6 from HW 2). Consider now the map

$$\tilde{F} : \mathbb{R} \times \mathbb{R} \to TS^1, (t, \lambda) \mapsto \lambda\gamma'(t).$$

Note that $\tilde{F}$ descends to a map

$$F : S^1 \times \mathbb{R} = \mathbb{R}/(2\pi\mathbb{Z}) \times \mathbb{R} \longrightarrow TS^1$$

since γ is 2π-periodic. It remains to check that F is a diffeomorphism.
Injectivity: It is clear that if $\lambda\gamma'(t) = \mu\gamma'(s)$, then the basepoints of the tangent vectors coincide so $s = t$ mod $2\pi\mathbb{Z}$ and since $\gamma'(t) \neq 0$, it follows that $\lambda = \mu$.
Surjectivity: For $(p, v) \in S^1$, we can choose t, λ such that $\gamma(t) = p$ and $\lambda\gamma'(t) = v$ (since T_pS^1 is one-dimensional and $\gamma'(t) \neq 0$.
Smoothness of F and F^{-1}: Let (U, φ) be an angle chart on S^1 and let $\varphi \times \mathrm{id}$ and $D\varphi$ be the corresponding charts on $S^1 \times \mathbb{R}$ and TS^1. Then it is easy to see that in these coordinates, F (and hence also F^{-1}) is the identity map, and $F(U \times \mathbb{R}) = TU$.