

Math 549 – HW 4 Solutions

Problem 4-2 :

Solution 1: By composing with the inclusion $\mathbb{H}^n \hookrightarrow \mathbb{R}^n$, we obtain a map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ whose derivative at p is an isomorphism. By the inverse function theorem, it is locally a diffeomorphism at p , in particular its image contains an open neighborhood of $F(p)$. On the other hand, the image is contained in \mathbb{H}^n . This proves that $F(p)$ lies in the interior of \mathbb{H}^n .

Solution 2: We argue by contradiction so suppose $F(p) \in \partial N$. Let $V \ni F(p)$ be a chart and let ν be the outward pointing unit normal to ∂N on V (i.e. in local coordinates where $V \cap N \cong \mathbb{H}^n$, we have $\nu = -e_n$). Without loss of generality, we have $F(p) = 0$ in these local coordinates.

Let $c : I \rightarrow M$ be a curve with $c(0) = p$ and $\dot{c}(0) = D_p F^{-1}(\nu(p))$. Then in the above local coordinates on V , we have

$$\begin{aligned} F(c(t)) &= (F \circ c)'(0)t + O(t^2) \\ &= D_p F(\dot{c}(0))t + O(t^2) \\ &= -te_n + O(t^2). \end{aligned}$$

Therefore for $0 < t \ll 1$, we have $F(c(t)) \notin \mathbb{H}^n$, which is a contradiction. \square

Problem 4-4:

There are many arguments here (some of which give better ‘rates of equidistribution’ than others). Here is one (without obtaining a rate):

It suffices to show that for any $(x, y) \in \mathbb{R}^2$ and $\varepsilon > 0$, there exist $t \in \mathbb{R}$ and $(m, n) \in \mathbb{Z}^2$ such that

$$\|(t, \alpha t) - (x + m, y + n)\| < \varepsilon.$$

We will take $t = x + m$ so that the problem becomes to show that there exist $m, n \in \mathbb{Z}$ such that

$$|\alpha m - n + \alpha(x - y)| < \varepsilon.$$

It therefore suffices to show that $\mathbb{Z}[\alpha] \subseteq \mathbb{R}$ is dense. By Dirichlet’s Approximation Theorem (see 4.21), for any $N \geq 1$, we can find $\tau \in \mathbb{Z}[\alpha]$ with $|\tau| < 1/N$. It follows that $\mathbb{Z}[\alpha]$ is dense: Indeed, if $z \in \mathbb{R}$ and $N \geq 1$, then at least one of the sequence $\{k\tau\}_{k \in \mathbb{Z}}$ approximates z to within $1/N$. \square

Problem 4-5(a), submersivity:

Let $p \in \mathbb{C}P^n$, and let $U_i \ni p$ be a standard affine chart containing $p = (p_1, \dots, p_{n+1})$. Consider the map $\sigma : U_i \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ given by $\sigma[z_0 : \dots : 1 : \dots : z_n] = p_i(z_0, \dots, 1, \dots, z_n)$. Then $p \in \text{im}(\sigma)$ and $\pi \circ \sigma = \text{id}$ (i.e. σ is a local section of π on U_i) and hence π is a submersion on $\pi^{-1}(U_i)$, in particular at p .

Problem 4-6:

We argue by contradiction so suppose there is such a submersion. By the local form for submersions, a submersion is locally surjective. In particular, its image is open. By compactness of M , the image is also closed. Therefore the image is both open and closed and by connectedness of \mathbb{R}^k , the submersion must be surjective. On the other hand by compactness of M , the image is compact. This is a contradiction.

Problem 4-9:

Suppose \mathcal{A}, \mathcal{B} are two smooth structures with respect to which π is a smooth covering. Let $(U, \varphi) \in \mathcal{A}$ and $(V, \psi) \in \mathcal{B}$. We have to verify these charts are smoothly compatible. If $U \cap V = \emptyset$ we are done. If not, let $p \in U \cap V$. By shrinking U and V , we may assume they (and hence also $U \cap V$) are sheets of evenly covered neighborhoods of $\pi(p)$. Write $W := U \cap V$, and let $W_{\mathcal{A}}$ and $W_{\mathcal{B}}$, denote W with the corresponding smooth structures.

The smooth covering property implies that π is a diffeomorphism from both $W_{\mathcal{A}}$ and $W_{\mathcal{B}}$ to $\pi(W) \subseteq M$. Therefore the composition

$$W_{\mathcal{A}} \xrightarrow{\pi} \pi(W) \xrightarrow{\pi^{-1}} W_{\mathcal{B}}$$

is a diffeomorphism. On the other hand, this composition is the identity map. Since the identity map is a diffeomorphism, the two charts are smoothly compatible (indeed, $\psi \circ \text{id} \circ \varphi^{-1}$ is smooth). \square

Problem 4-13:

To show F descends, it suffices to show that $F(x) = F(-x)$ for all $x \in S^2$, and this is straightforward. Let $\bar{F} : \mathbb{R}P^2 \rightarrow \mathbb{R}^4$ denote the induced map. To prove it is an embedding, it suffices to prove it is an injective proper immersion. The properness is immediate since $\mathbb{R}P^2$ is compact.

Injectivity: Suppose $[x : y : z], [u : v : w] \in \mathbb{R}P^2$ with $\bar{F}[x : y : z] = \bar{F}[u : v : w]$ and $(x, y, z), (u, v, w) \in S^2$. Then $x^2 - y^2 = u^2 - v^2$ and $xy = uv$ and $xz = uw$ and $yz = vw$. It is easy to manipulate these equations to conclude that $(x, y, z) = \pm(u, v, w)$, so $[x : y : z] = [u : v : w]$, as desired.

Immersivity: Let $\pi : S^2 \rightarrow \mathbb{R}P^2$ denote the covering map. Since $F = \bar{F} \circ \pi$, it suffices to prove F is an immersion. There is any number of ways this computation can be done. E.g. an ad hoc argument is that at any point $p = (x_0, y_0, z_0) \in S^2$, consider two coordinates that are not ± 1 , say $y_0, z_0 \neq \pm 1$. Now consider the circles through p with $y = y_0$ and $z = z_0$ constant, respectively. Since the argument is similar for each, let us just consider the circle with $z = z_0$ constant. Then the last two components of F are the map $(x, y, z_0) \mapsto z_0(x, y)$ which is just a multiple of the standard embedding $S^1 \hookrightarrow \mathbb{R}^2$, and is therefore an immersion. Considering F restricted to these circles (which are not tangent at p) shows the image of $D_p F$ has rank 2 and is therefore injective.

Another method is as follows: Let $\iota : S^2 \rightarrow \mathbb{R}^3$ denote the natural inclusion and let $\tilde{F} : \mathbb{R}^3 \rightarrow \mathbb{R}^4$ denote the extension of F by the same polynomials in x, y, z . One can explicitly compute $D\tilde{F}$ (as a matrix of partials) and verify that at any point $p \in S^2$, we have $\ker D_p \tilde{F} \cap \text{im} D_p \iota = \{0\}$. (Note that $\text{im} D_p \iota = p^\perp$.) \square