Math 549 – HW 5 Solutions

Problem 5-1

Note that $\Phi(x, y, s, t) = (0, 1)$ if and only if $y = -x^2$ and $x^4 + s^2 + t^2 = 1$. Computing the matrix of partials of Φ we obtain

$$D\Phi = \begin{pmatrix} 2x & 1 & 0 & 0 \\ 2x & 2y+1 & 2s & 2t \end{pmatrix}.$$

It is easy to do an analysis by cases to check this has rank 2. E.g. if $x = 0$ then $y = 0$ as well and at least one of s, t is nonzero, so evaluating on e_2 shows that $e_1 + e_2$ is in the image and evaluating on e_3 or e_4 shows that e_2 is in the image. On the other hand if $x \neq 0$, then $DF(e_1) = 2x(e_1 + e_2)$ and $DF(e_2) = e_1 + (1 - 2x^2)e_2$ are linearly independent. So either way $D\Phi$ has rank 2.
To see $\Phi^{-1}(0, 1) \cong S^2$, first note that the projection $(x, y, s, t) \mapsto (x, s, t)$ is a diffeomorphism when restricted to $\Phi^{-1}(0, 1)$ (because $y = -x^2$). Write M for the image. Now note that for fixed $x_0 \in [-1, 1]$, we have $M \cap \{x = x_0\}$ is given by the equation $s^2 + t^2 = 1 - x_0^4$ and hence is a circle of radius $(1 - x_0^4)^{1/2}$. Rescaling this circle so that instead it has radius $(1 - x_0^2)^{1/2}$ gives the desired diffeomorphism. I.e. writing $r(x) := (1 - x^2)^{1/2}/(1 - x^4)^{1/2}$, consider the map $F : M \to S^2$ given by $F(x, s, t,) = (x, r(x)s, r(x)t)$. You can check that r is smooth and positive, so both F and F^{-1} are smooth. $\square$

Problem 5-5

It suffices to note that $\gamma(\mathbb{R})$ (with the subspace topology) is not locally Euclidean. Indeed, the intersection of $\gamma(\mathbb{R})$ and any sufficiently small open set in T^2 is a countable disjoint union of line segments, which is not connected. $\square$

Problem 5-6

Consider the function $N : TM \to \mathbb{R}$ given by $N(v) = \|v\|^2$. As shown in lecture, DN has constant rank 1 away from the zero section (even if you consider its restriction to a tangent space $T_pM\backslash\{0\}$), so 1 is a regular value of N, so $UM = N^{-1}(1)$ is an embedded codimension 1 submanifold of TM.
To prove UM is embedded in $T\mathbb{R}^n$, it suffices to prove TM is embedded (because the composition of embeddings is an embedding). To show TM is embedded, consider a slice chart (U, φ) for $M \subseteq \mathbb{R}^n$, say $\varphi(M) = \mathbb{R}^m \times \{0\} \subseteq \mathbb{R}^n$.. Then

$$D\varphi(TM) = T\mathbb{R}^m \times T\{0\} = \mathbb{R}^{2m} \times \{0\} \subseteq \mathbb{R}^{2n}$$

So $D\varphi$ is a slice chart for $TM \subseteq T\mathbb{R}^n$. $\square$

Problem 5-10

For $a \in \mathbb{R}$, consider $F_a : \mathbb{R}^2 \to \mathbb{R}$ given by $F_a(x, y) = y^2 - x(x - 1)(x - a)$ with derivative

$$DF = \begin{pmatrix} -3x^2 + 2(a+1)x + a & 2y \end{pmatrix}.$$

Let $(x_0, y_0) \in M_a$. We check whether $D_{(x_0,y_0)}F_a$ is surjective (which in this case is equivalent to nonzero). If $y_0 \neq 0$ this is immediate. If $y_0 = 0$ then $x_0(x_0 - 1)(x_0 - a) = 0$ so $x_0 = 0$ or 1 or a. If $x = a$ then $D_{(x_0,y_0)}F \neq 0$ unless $a = 0$ or 1, so this case reduces to the previous two. So we conclude that if $a \neq 0, 1$, then 0 is a regular value of F_a so M_a is an embedded submanifold.
For $a = 0, 1$one needs to inspect by hand the behavior of M_a at the problematic points $(0, 0)$ (if $a = 0$) and $(1, 0)$ (if $a = 1$) . For the former, it is easy to see that $(0, 0)$ is isolated in M_0. So M_0 is not embedded or immersed since its connected components have different dimensions (however, each connected component is an embedded submanifold).
For $a = 1$, we note that $x(x - 1)^2 > 0$ as long as $x > 0$. Therefore $y^2 = x(x - 1)^2$ has two solutions $y_\pm(x)$ and the intersection of M_1 with a small neighborhood of $(1, 0)$ is the union of the graphs of $y_\pm(x)$, and in

particular M_1 is not locally Euclidean. However, M_1 is the image of an immersion of the line (but not an injective immersion).

Problem 5-23

We will construct slice charts for $S \subseteq M$. Let $p \in S$. If $p \notin \partial M$, then there exists an open neighborhood $U \ni p$ disjoint from ∂M, so that the result follows from applying the regular value theorem for manifolds without boundary to the restriction $F|_U$.
Now assume $p \in \partial M$ and let $\varphi : U \to \mathbb{H}^n$ be a chart around p with $\varphi(p) = 0$. By definition, there exists $V \subseteq \mathbb{R}^n$ open and a smooth map $\widetilde{F} : V \to N$ extending $F \circ \varphi^{-1}$ on $V \cap \mathbb{H}^n$. Since surjectivity of the derivative is an open condition, by possibly shrinking V further, we can assume that $\widetilde{F}$ is a submersion. In particular c is a regular value of $\widetilde{F}$, so $\widetilde{F}^{-1}(c)$ is an embedded submanifold of $\mathbb{R}^n$.
Now choose local coordinates (i.e. apply some diffeomorphism ψ) on $\mathbb{R}^n$ such that $\widetilde{F}$ is given by projection onto the first r coordinates: Explicitly $\psi(x) = (F_1(x), \dots, F_r(x), x_{r+1}, \dots, x_n)$. Note that the nth coordinate (used to define $\mathbb{H}^n = \{x_n > 0\}$) is not projected onto, because otherwise c is not a regular value of $F|_{\partial M}$).
Hence $\psi(\mathbb{H}^n) = \mathbb{H}^n$.
Now we see that in these coordinates, $F^{-1}(c) = \mathbb{H}^{n-r}$. □

Problem 6-1

(This solution does not follow the hint.) Reduce to the case of charts and then observe that every map $F : \mathbb{R}^m \to \mathbb{R}^n$ can be extended to a map $\widetilde{F} : \mathbb{R}^n \to \mathbb{R}^n$ (e.g. by precomposition with projection to the first m coordinates). Then note that $\mathbb{R}^m \subseteq \mathbb{R}^n$ is null, so $F(\mathbb{R}^m) = \widetilde{F}(\mathbb{R}^m \times \{0\})$ is null by Proposition 6.5. □